EECS 120 Signals & Systems — University of California, Berkeley: Fall 2005
Gastpar — September 28, 2005

# Handout 2: Linear Algebra and Fourier

## 1 Basic Concepts of Linear Algebra

### 1.1 Definitions of vectors and matrices

Formally, a vector is an object in a vector space; which is a set that is closed under vector addition and scalar multiplication, and possesses certain properties. Informally, we can think of a vector as a one-dimensional array of scalars. In this course, we will mostly be interested in the vector space of length $n$ complex numbers, denoted by $\mathbb{C}^n$. A vector $\mathbf{v}$ in $\mathbb{C}^n$ can be written as

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad (1)$$

where $v_i \in \mathbb{C} \qquad i = 1, 2, \ldots, n$

A matrix is a two-dimensional, rectangular array of scalars. In this course, we will mostly be interested in the set of $m$-by-$n$ matrices of complex numbers, denoted by $\mathbb{C}^{mxn}$. A matrix $A$ in $\mathbb{C}^{mxn}$ can be written as

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{pmatrix} \qquad (2)$$

where $A_{ij} \in \mathbb{C} \qquad i = 1, 2, \ldots, m \qquad j = 1, 2, \ldots, n$

### 1.2 Vector and matrix operations

**Vector Norm**: The norm of a vector $\mathbf{v}$, denoted by $\|\mathbf{v}\|$, is defined as

$$\|\mathbf{v}\| = \left( \sum_{i=1}^{n} |v_i|^2 \right)^{1/2}$$

**Inner product**: The inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{C}^n$, denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i^* y_i$$

where $x_i^*$ is the complex conjugate of $x_i$. Note that two vectors must be in the same vector space for the inner product to be defined. Also, it should be clear that $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$

If $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, then we say that $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal*.

**Matrix multiplication**: If $A \in \mathbb{C}^{mxn}$ and $B \in \mathbb{C}^{nxp}$, then $C = A \cdot B$ is a matrix in $\mathbb{C}^{mxp}$ with elements given by the equation

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

The matrix multiplication $A \cdot B$ is only defined when the number of columns in $A$ is equal to the number of rows in $B$.

A matrix can also be thought of as a transformation between two vector spaces. If $\mathbf{x}$ is a vector in $\mathbb{C}^n$ and $A$ is a matrix in $\mathbb{C}^{mxn}$, then $y = A \cdot x$ is a vector in $\mathbb{C}^m$.

**Determinant**: The determinant of a square matrix $A$, denoted by $|A|$, is a scalar that is computed in the following manner. If $A$ is a $1$-by-$1$ matrix, then $|A| = A_{11}$. If $A$ is $2$-by $2$, then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For larger square matrices, the determinant is computed recursively, as we will see in class.

**Hermitian transpose**: The Hermitian transpose of $A$ is denoted by $A^*$. The element in row $i$, column $j$ of $A^*$ is equal to the complex conjugate of the element in row $j$, column $i$ of $A$, i.e., $(A^*)_{ij} = (A_{ji})^*$. The matrix $H$ is called *Hermitian* if $H^* = H$. (Note that many texts also use the notation $A^H$ to denote the Hermitian transpose of the matrix $A$.)

## 1.3 Unitary and inverse matrices

**Identity matrix**: The $n$-by-$n$ identity matrix $I_n$ is equal to $1$ on its main diagonal, and $0$ everywhere else.

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{3}$$

**Unitary matrices**: An $n$-by-$n$ matrix $A$ is unitary if $A^* \cdot A = I_n$ and $A \cdot A^* = I_n$. Note that because matrix multiplication is not commutative, both conditions must be checked.

**Matrix Inverse**: For an $n$-by-$n$ square matrix $A$, the inverse matrix is the matrix $A^{-1}$ such that $A \cdot A^{-1} = I_n$ and $A^{-1} \cdot A = I_n$. The inverse of $A$ exists if and only if $|A| \neq 0$.

## 1.4 Basis vectors

A set of $n$ linearly independent vectors in $\mathbb{C}^n$ is referred to as a basis. If we label the vectors in the basis set as $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then any vector $\mathbf{y}$ in $\mathbb{C}^n$ can be written as

$$\mathbf{y} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$$

# 2 The DTFS as a matrix multiplication

## 2.1 Three Definitions of the DTFS (or DFT)

The definition of the DFT (or DTFS) in our textbook is

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \quad \text{and} \quad x[n] = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}. \tag{4}$$

In MATLAB, the definition is

$$X_M[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \quad \text{and} \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_M[k]e^{j\frac{2\pi k}{N}n}. \tag{5}$$

Apparently, people cannot agree as to where the factor $\frac{1}{N}$ should go. In order to gain geometric insight, the best definition is actually the *unitary form of the DFT*, where the factor $\frac{1}{N}$ is fairly split between the Fourier transform and its inverse:

$$X_u[k] = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \quad \text{and} \quad x[n] = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} X_u[k]e^{j\frac{2\pi k}{N}n}. \tag{6}$$

## 2.2 Signals as vectors

For a periodic signal, let us define the vector containing one period of the signal simply as

$$\mathbf{x} = \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[N-1] \end{pmatrix} \tag{7}$$

## 2.3 The DTFS as a matrix multiplication

We define the "Fourier matrix" as follows:

$$F = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j2\frac{2\pi}{N}} & e^{-j3\frac{2\pi}{N}} & \dots & e^{-j(N-1)\frac{2\pi}{N}} \\ 1 & e^{-j2\frac{2\pi}{N}} & e^{-j4\frac{2\pi}{N}} & e^{-j6\frac{2\pi}{N}} & \dots & e^{-j2(N-1)\frac{2\pi}{N}} \\ 1 & e^{-j3\frac{2\pi}{N}} & e^{-j6\frac{2\pi}{N}} & e^{-j9\frac{2\pi}{N}} & \dots & e^{-j3(N-1)\frac{2\pi}{N}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j(N-1)\frac{2\pi}{N}} & e^{-j2(N-1)\frac{2\pi}{N}} & e^{-j3(N-1)\frac{2\pi}{N}} & \dots & e^{-j(N-1)(N-1)\frac{2\pi}{N}} \end{pmatrix} \tag{8}$$

In terms of this matrix, we can easily express our three DTFS, as follows:

$$\mathbf{X} = \frac{1}{\sqrt{N}}F\mathbf{x} \tag{9}$$

$$\mathbf{X}_M = \sqrt{N}F\mathbf{x} \tag{10}$$

$$\mathbf{X}_u = F\mathbf{x} \tag{11}$$

The key property of the matrix $F$ is that it is *unitary*. Therefore, it is entirely straightforward to find the matrix that characterizes the inverse DTFS: it is simply the Hermitian transpose of the Fourier matrix, as follows:

$$\mathbf{x} \quad = \quad F^{*}\mathbf{X}_{u}. \tag{12}$$

It is entirely straightforward to prove this: simply note that

$$\mathbf{x} \quad = \quad F^{*}\mathbf{X}_{u} = F^{*}\left(F\mathbf{x}\right) = F^{*}F\mathbf{x} = \mathbf{x}, \tag{13}$$

where the last step follows precisely because $F$ is unitary, which means that $F^{H}F = I$.

# 3 Advanced Concepts of Linear Algebra

## 3.1 Eigenvalues and eigenvectors

Let $A$ be a square matrix in $\mathbb{C}^{nxn}$. If $\mathbf{x}$ is a non-zero vector in $\mathbb{C}^{n}$ and $\lambda$ is a complex scalar that satisfy this equation

$$A \cdot \mathbf{x} = \lambda\mathbf{x}$$

then $\lambda$ is referred to as an *eigenvalue*, and $\mathbf{x}$ is referred to as an *eigenvector*. The matrix $A$ will have at least one and at most $n$ unique eigenvalues.