Final Exam Review Problems

Problem 1 *(Sampling.)*

A real-valued data signal $x(t)$ is known to be band-limited, i.e., $X(j\omega) = 0$, for $|\omega| > W$.

(a) Suppose the signal $x(t)$ is sampled non-uniformly using the impulse train $q_1(t)$ shown in Figure 1. Show that the spectrum of the sampled signal $y_1(t) = x(t)q_1(t)$ is

$$Y_1(j\omega) \quad = \quad \frac{1}{2T}\sum_{k=-\infty}^{\infty}(1 + e^{-j\frac{\pi k}{3}})X(j(\omega - \frac{\pi k}{T})). \tag{1}$$

Carefully justify every step in your derivation, including references to results from the tables.

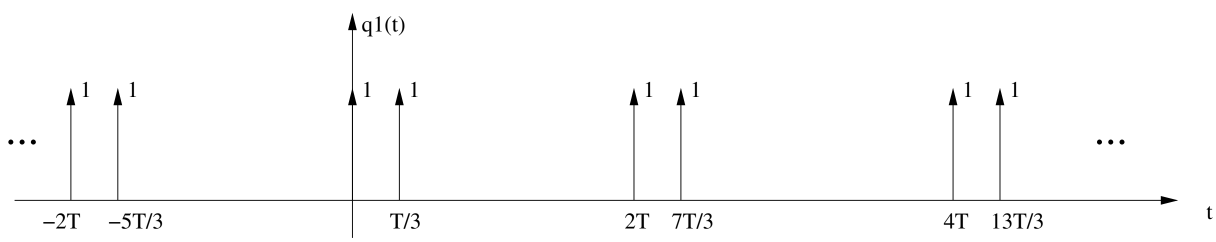


Figure 1: The sampling impulse train $q_1(t)$, where $T = \pi/W$.

(b) Is it possible to low-pass filter the signal $y_1(t)$ to get back the signal $x(t)$?

Answer: yes / no. (circle one)

Explanation: *(Hint: Pick an example band-limited spectrum for $x(t)$, and sketch the resulting spectrum of $y_1(t)$. Based on your plot, explain.)*

(c) Suppose the signal is sampled non-uniformly using the impulse train $q_2(t)$ shown in Figure 2. The impulses are in the same locations as in Figure 1, but they have different weights a and b (both real numbers). Find an expression for the spectrum $Y_2(j\omega)$ of the sampled signal $y_2(t) = x(t)q_2(t)$.

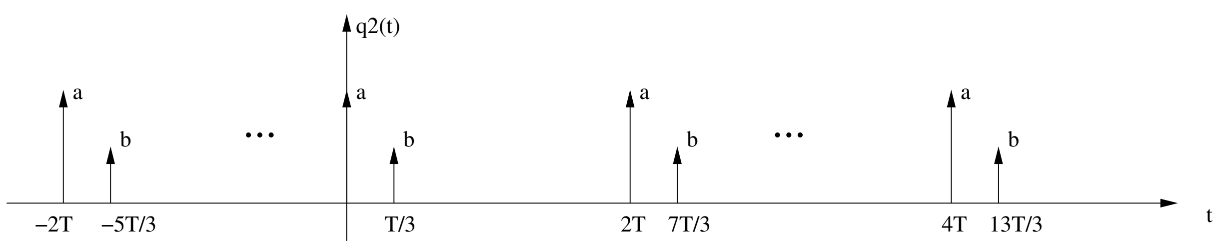


Figure 2: The sampling impulse train $q_2(t)$, where $T = \pi/W$.

(d) For $|\omega| < W$, write out the spectrum

$$Y(j\omega) \quad = \quad \begin{cases} Y_1(j\omega) + jY_2(j\omega), & \text{if } \omega \geq 0, \\ Y_1(j\omega) - jY_2(j\omega), & \text{if } \omega < 0. \end{cases} \tag{2}$$

Select the real numbers a and b such that for $0 < \omega < W$, it is true that $Y(j\omega) = \frac{1}{2T}(2+j(a+b))X(j\omega)$. *Hint:* A complex number is zero if and only if its real part is zero *and* its imaginary part is zero.

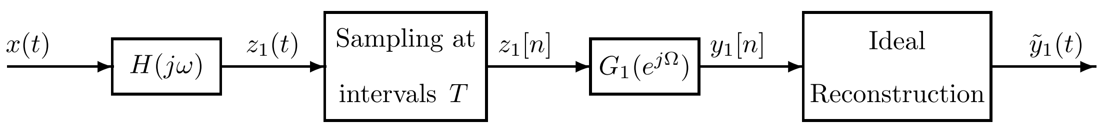
(e) It can also be shown that with the choice of a and b as in Part *(d)*, it is true that $Y(j\omega) = \frac{1}{2T}(2-j(a+b))X(j\omega)$ for $-W < \omega < 0$. Hence, from $Y(j\omega)$, one can determine $X(j\omega)$ using a simple filter. This means that non-uniform sampling using the sampling intervals shown in Figure 1 (and in Figure 2) permits to perfectly reconstruct $x(t)$ from samples. Give an intuitive explanation why this makes sense.

Problem 2 *(Discrete-time Processing of Continuous-time Signals.)*

We would like to construct a system D that implements a derivative, that is, for an input $x(t)$, the system should give an output $y(t)$ given by

$$y(t) \quad = \quad D\{x(t)\} = \frac{dx(t)}{dt}. \tag{3}$$

(a) It is suggested to use the following system:



where

$$H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \le \pi/T \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

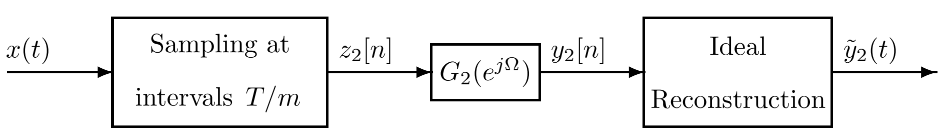
and

$$G_1(e^{j\Omega}) \quad = \quad j\frac{\Omega}{T}, \text{ for } |\Omega| \le \pi. \tag{5}$$

This system does not exactly implement the desired system D. Instead, it produces an output $\tilde{y}_1(t)$ which is, in general, not equal to the desired output $y(t)$.

Determine and sketch the overall frequency response (magnitude and phase) of the system with input $x(t)$ and output $\tilde{y}_1(t)$.

(b) A second system is also proposed as follows



Note that in the second system, the Ideal Reconstruction block works with the sampler that samples at intervals of T/m, so it is not necessarily identical to the Ideal Reconstruction block in the first system.

Considering that we would like to implement a first derivative with a digital system and given the sampler in the second system, what should we choose for the filter $G_2(e^{j\Omega})$?

(c) If $m = 4$ and the input signal $x(t)$ has a Fourier transform

$$X(j\omega) \quad = \quad \begin{cases} 10, & \text{for } \; 0 \le |\omega| \le \frac{\pi}{T} \\ 1, & \text{for } \; \frac{\pi}{T} \le |\omega| \le \frac{5\pi}{T} \\ 0, & \text{elsewhere} \end{cases}, \tag{6}$$

Determine the error between the desired signal, $y(t)$, and the outputs of the two systems, $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$, given by

$$E_i = \int_{-\infty}^{\infty} |y(t) - \tilde{y}_i(t)|^2 dt, \qquad i = 1, 2 \tag{7}$$

If $T = 10^{-3}$, which of the systems is a better choice to implement the first derivative?

Problem 3 *(PAM with fading.)*

Consider a PAM system as discussed in class: A discrete-time data signal $s[n]$ is used to produce the following modulated signal:

$$x(t) = \sum_{n=-\infty}^{\infty} s[n] p(t - nT), \tag{8}$$

where $p(t)$ is a box function that starts at time zero and ends at time $\Delta < T$, of height one.

The signal $x(t)$ is passed through a fading communication channel with frequency response

$$H(j\omega) = \frac{1}{(e^{a\Delta} - 1)(1 + j\omega/a)}, \tag{9}$$

where a is *positive and real-valued.* Call the channel output signal $y(t)$.

(a) For $a = T$ and

$$s[n] = \begin{cases} 1, & n = 0, 2 \\ 2, & n = 1, 3, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

sketch the signal $y(t)$ at the receiver. Carefully label the time axis. *Hint:* Consider a single pulse first, and use linearity.

(b) The channel output signal $y(t)$ is sampled at times nT to yield the signal $y[n]$. Write $y[n]$ in terms of $s[n]$ (for general $s[n]$, not merely for the $s[n]$ of Part *(a)*). Can the relationship between $s[n]$ and $y[n]$ be understood as a discrete-time linear time-invariant (LTI) system? If so, determine its impulse response $h[n]$.

(c) Find a discrete-time LTI system with impulse response $g[n]$ and frequency response $G(e^{j\omega})$ that takes the received signal $y[n]$ and outputs the original transmitted signal $s[n]$.

(d) Generally, at the receiver, you do not exactly know the constant a that governs the channel behavior, but you can estimate it, giving you some $\hat{a}$ which is hopefully close to the true a. Moreover, since the decoder that you designed in Part *(c)* will operate in digital logic, you have to round $\hat{a}$ to the closest number $\tilde{a}$ that can be represented on your DSP chip. But if the decoder of Part *(c)* uses $\tilde{a}$ instead of a, it will recover a signal $\tilde{s}[n]$ that is not exactly equal to $s[n]$.

For the special case where $s[n] = 1$, for all n, find the error, defined as

$$E = \sum_{n=-\infty}^{\infty} |s[n] - \tilde{s}[n]|^2, \tag{11}$$

as a function of the a and $\tilde{a}$.

Problem 4 *(Continuous-time LTI systems.)*

Consider the function

$$G(s) \quad = \quad \frac{1-(s/10)^6}{1-s^6}. \tag{12}$$

(a) Draw the pole-zero diagram for $G(s)$.

(b) There are multiple linear time-invariant systems whose transfer function $H(s)$ satisfies $H(s)H(-s) = G(s)$. Among them, determine the one $H(s)$ that

1. corresponds to a stable and causal system
2. with real-valued impulse response function,
3. and for which there exists an inverse system that is also stable and causal.

Each requirement in this list corresponds to a requirement on the pole/zero plot of $H(s)$. Explicitly state these three requirements.

(c) Determine the differential equation describing the system that has the transfer function that you have found in Part *(b)*.

(d) Draw the Bode diagram for the magnitude of the frequency response, $|H(j\omega)|$, for the system described by the differential equation that you have found in Part *(c)*. Describe in words the operation performed by this system.

(e) How many different transfer functions $H(s)$ can be identified that satisfy $H(s)H(-s) = G(s)$ and represent stable systems (not necessarily with a real-valued impulse response)? Justify your answer.

Problem 5 *(Discrete-time LTI systems.)*

(i) The causal discrete-time LTI system G has transfer function

$$G(z) \quad = \quad \frac{\left(1-\frac{1}{16}z^4\right)\left(1-az^4\right)\left(1-bz^4\right)}{z^9\left(1-\frac{1}{8}z^3\right)}, \tag{13}$$

where a and b are *complex* numbers.

- Determine the poles and zeros of the system $G(z)$, and sketch them in the z plane. *Hint:* Write a and b in polar form, i.e., as $xe^{-j\alpha}$, where x is real-valued and positive, and $0 \leq \alpha < 2\pi$.
- Choose a and b such that the system with transfer function $G(z)$ is stable.
- For your choice of a and b, can the system be implemented as a real-valued LTI system, i.e., only involving multiplication by *real* numbers, delay elements, and summations? Give a brief argument.

(ii) The causal discrete-time LTI system G has transfer function

$$G(z) \quad = \quad \frac{1}{d+\frac{1}{2d}-z^3}, \tag{14}$$

where d is a *real* number. (This is a simple version of the $G(z)$ of Part *(i)*.)

- For $d = 3/4$, show that the system is unstable.
- Draw a block diagram of the system, using only multiplication by real-valued constants, delay elements, and summations. Use as few delay elements as possible.

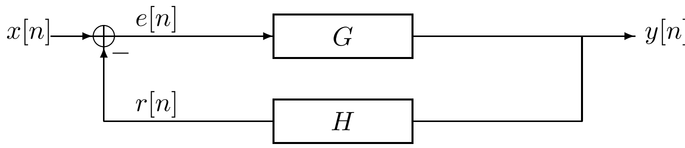


Figure 3: Feedback system.

(iii) As we have seen in class, the feedback system of Figure 3 has the following transfer function:

$$T(z) \quad = \quad \frac{G(z)}{1+G(z)H(z)}. \tag{15}$$

We take $G(z)$ as in Part *(ii)*, and $H(z)$ as follows:

$$G(z) = \frac{1}{d+\frac{1}{2d}-z^3}, \quad \text{and} \quad H(z) = -\frac{1}{2}z^{-3}. \tag{16}$$

- Determine the range of real numbers d such that the (causal) overall system with transfer function $T(z)$ is stable. *Hint:* Substitute $x = z^3$ and recall that $x^2 - (a+b)x + ab = (x-a)(x-b)$.
- Is the overall system now stable for $d = 3/4$?