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## Handout 3: Bode Diagrams

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This handout is a complement to the textbook, OWN Chapter 9 (Section 9.4 in particular); see also Section 6.2.3, pages 436-439.

Given a transfer function  $H(s)$ , the goal is to sketch the frequency response of the system in a quick and easy manner (specifically, we will plot both the magnitude response  $|H(j\omega)|$  and the phase response  $\arg H(j\omega)$ ). At first, this may seem unnecessary - any pocket calculator (cell phone? iPod?) can plot functions! However, we hope that you will agree that the Bode plot method described below is much more insightful. For example, think about slightly varying some of the parameters in  $H(s)$ : what happens to the frequency response? It is much harder to gain insight into this using only your pocket calculator...

### Step 1: Rewrite the transfer function

From the differential equation, one generally obtains a transfer function of the form

$$H(s) = \frac{\sum_{r=0}^R b_r s^r}{\sum_{p=0}^P a_p s^p}. \quad (1)$$

From general facts about polynomials, this can be rewritten as

$$H(s) = M \frac{\prod_{r=1}^R \left(1 + \frac{s}{\beta_r}\right)}{\prod_{p=1}^P \left(1 + \frac{s}{\alpha_p}\right)}, \quad (2)$$

where  $M$  is some (complex-valued) constant. Notice that this implies that the zeros of  $H(s)$  are  $-\beta_r$ , and the poles are  $-\alpha_p$ .

Hence, the frequency response of the system at hand can be written as

$$H(j\omega) = M \frac{\prod_{r=1}^R \left(1 + \frac{j\omega}{\beta_r}\right)}{\prod_{p=1}^P \left(1 + \frac{j\omega}{\alpha_p}\right)}. \quad (3)$$

### Step 2: For the magnitude response, switch to decibels (dB)

The key trick of the Bode diagram is to consider the frequency response in terms of decibels, that is, to consider the quantity

$$|H(j\omega)|_{dB} \stackrel{def}{=} 20 \log_{10} |H(j\omega)|. \quad (4)$$

Hence, we find

$$|H(j\omega)|_{dB} = 20 \log_{10} \left| M \frac{\prod_{r=1}^R \left(1 + \frac{j\omega}{\beta_r}\right)}{\prod_{p=1}^P \left(1 + \frac{j\omega}{\alpha_p}\right)} \right|. \quad (5)$$

The magnitude of a product is the product of the magnitudes, hence

$$|H(j\omega)|_{dB} = 20 \log_{10} \left( |M| \frac{\prod_{r=1}^R \left|1 + \frac{j\omega}{\beta_r}\right|}{\prod_{p=1}^P \left|1 + \frac{j\omega}{\alpha_p}\right|} \right). \quad (6)$$

But by the standard properties of the logarithm (that is,  $\log(ab) = \log(a) + \log(b)$  and  $\log(a/b) = \log(a) - \log(b)$ ), this can be rewritten as

$$|H(j\omega)|_{dB} = 20 \log_{10} |M| + \sum_{r=1}^R 20 \log_{10} \left|1 + \frac{j\omega}{\beta_r}\right| + \sum_{p=1}^P \left( -20 \log_{10} \left|1 + \frac{j\omega}{\alpha_p}\right| \right). \quad (7)$$

Hence, to sketch the magnitude response in decibels, we can simply *add up* many plots, namely, one plot for each zero and one plot for each pole.

### Step 3a: The Plot for Real-valued Poles

Suppose that  $\alpha_p$  is real-valued. The corresponding term that we want to sketch is

$$-20 \log_{10} \left| 1 + \frac{j\omega}{\alpha_p} \right|, \quad (8)$$

which can be approximated for two different ranges of  $\omega$ .

1. Low frequencies:  $\omega \ll |\alpha_p|$ . In that case,  $|\omega/\alpha_p| \ll 1$ , and hence,

$$-20 \log_{10} \left| 1 + \frac{j\omega}{\alpha_p} \right| \approx 0. \quad (9)$$

2. High frequencies:  $\omega \gg |\alpha_p|$ . In that case,  $|\omega/\alpha_p| \gg 1$ , and hence,

$$-20 \log_{10} \left| 1 + \frac{j\omega}{\alpha_p} \right| \approx -20 \log_{10} \left| \frac{j\omega}{\alpha_p} \right| \quad (10)$$

$$= -20 \log_{10} \left| \frac{\omega}{\alpha_p} \right| \quad (11)$$

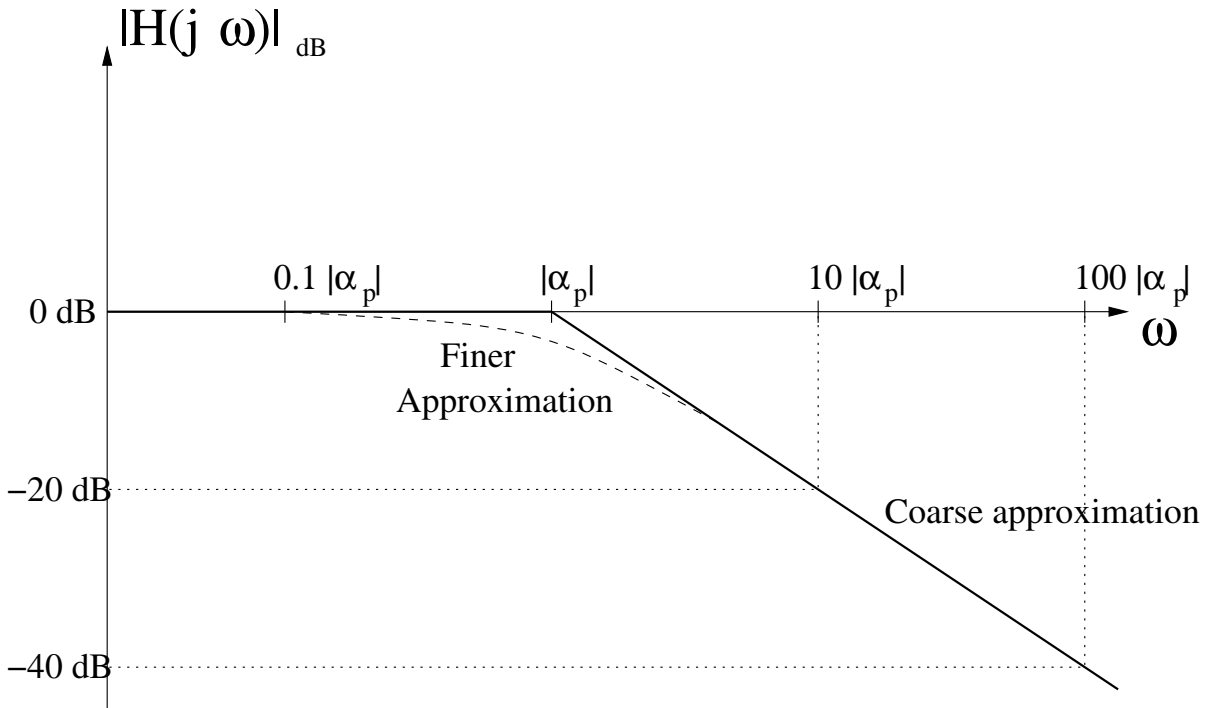


Figure 1: Bode diagram for a real-valued pole  $\alpha_p$ .

### Step 3b: The Plot for Real-valued Zeros

Suppose that  $\beta_r$  is real-valued. The corresponding term that we want to sketch is

$$20 \log_{10} \left| 1 + \frac{j\omega}{\beta_r} \right|, \quad (12)$$

which can be approximated for two different ranges of  $\omega$ .

1. Low frequencies:  $\omega \ll |\beta_r|$ . In that case,  $|\omega/\beta_r| \ll 1$ , and hence,

$$20 \log_{10} \left| 1 + \frac{j\omega}{\beta_r} \right| \approx 0. \quad (13)$$

2. High frequencies:  $\omega \gg |\beta_r|$ . In that case,  $|\omega/\beta_r| \gg 1$ , and hence,

$$20 \log_{10} \left| 1 + \frac{j\omega}{\beta_r} \right| \approx 20 \log_{10} \left| \frac{j\omega}{\beta_r} \right| \quad (14)$$

$$= 20 \log_{10} \left| \frac{\omega}{\beta_r} \right| \quad (15)$$

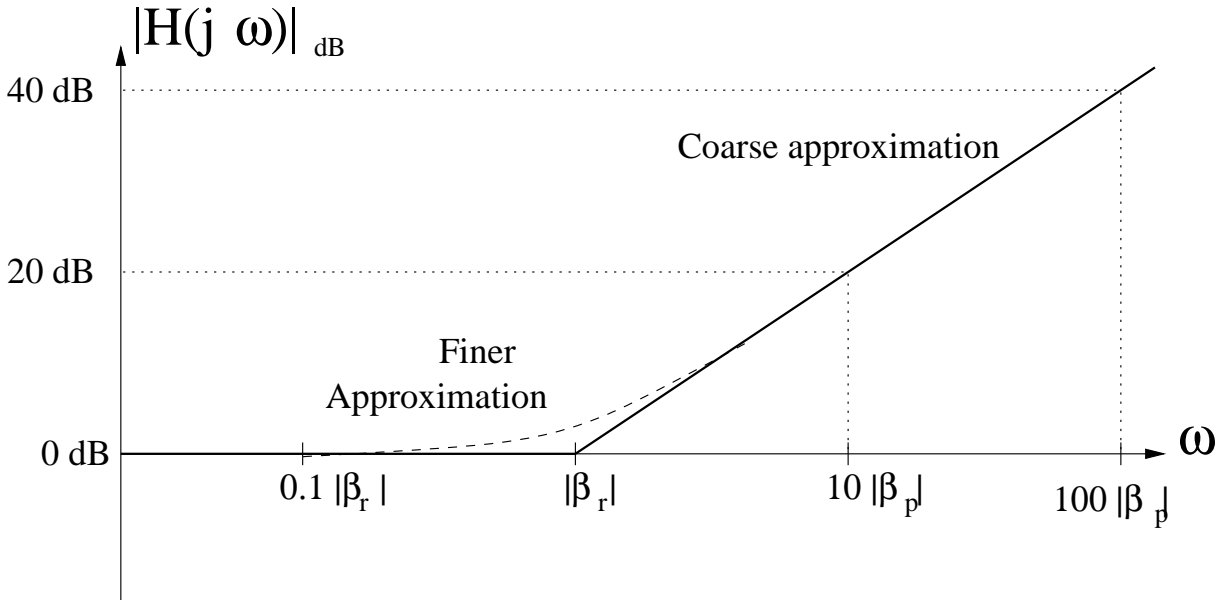


Figure 2: Bode diagram for a real-valued zero  $\beta_r$ .

### Step 3c: The Plot for Complex-valued Poles

We only consider systems that have a real-valued impulse response. Therefore, *any* complex-valued pole must show up as a *pair of complex-conjugate* poles. The way to handle this situation is by leaving the two complex-conjugate poles together as a second-order term.

For such a complex-conjugate "double" pole, first bring it into the following "standard" shape:

$$Q(s) = \frac{1}{1 + 2\gamma(s/\omega_n) + (s/\omega_n)^2}, \quad (16)$$

hence the two (complex-conjugate) poles sit at locations

$$-\gamma\omega_n \pm j\omega_n\sqrt{1 - \gamma^2}. \quad (17)$$

The corresponding term that we want to sketch is

$$-20 \log_{10} \left| 1 + 2\gamma \frac{j\omega}{\omega_n} + \left( \frac{j\omega}{\omega_n} \right)^2 \right|, \quad (18)$$

which can be approximated for two different ranges of  $\omega$ .

1. Low frequencies:  $\omega \ll |\omega_n|$ . In that case,  $|\omega/\omega_n| \ll 1$ , and hence,

$$-20 \log_{10} \left| 1 + 2\gamma \frac{j\omega}{\omega_n} + \left( \frac{j\omega}{\omega_n} \right)^2 \right| \approx 0. \quad (19)$$

2. High frequencies:  $\omega \gg |\omega_n|$ . In that case,  $|\omega/\omega_n| \gg 1$ , and hence,

$$-20 \log_{10} \left| 1 + 2\gamma \frac{j\omega}{\omega_n} + \left( \frac{j\omega}{\omega_n} \right)^2 \right| \approx -20 \log_{10} \left| \left( \frac{j\omega}{\omega_n} \right)^2 \right| \quad (20)$$

$$= -40 \log_{10} \left| \frac{\omega}{\omega_n} \right| \quad (21)$$

**Note:** The treatment of complex-conjugate pairs of zeros is along the exact same lines.

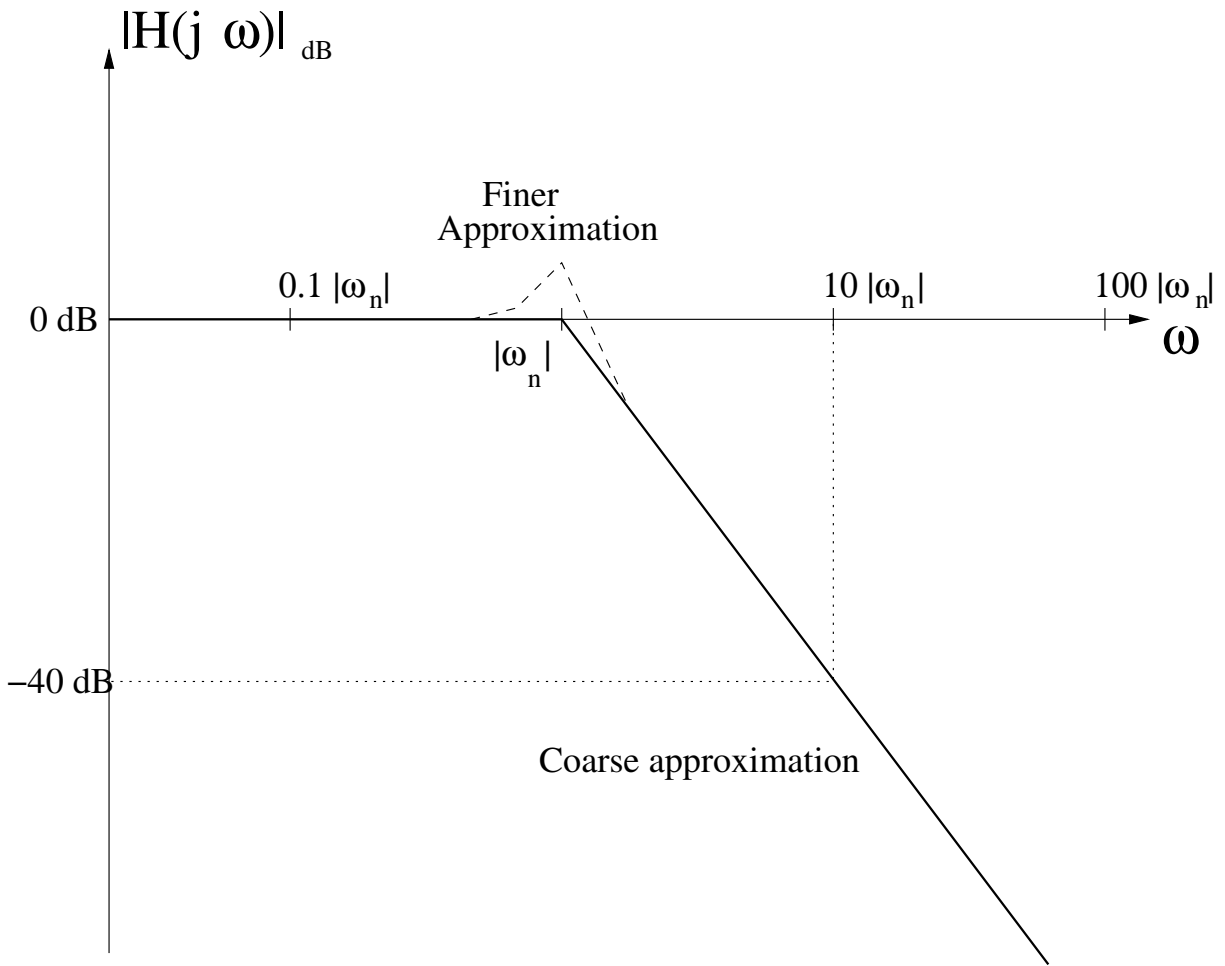


Figure 3: Bode diagram for a pair of complex-conjugate poles.

## Step 4: The Phase is Easy

For the phase plot, *no decibels are necessary*. We just plot the phase as before, except that we use a logarithmic scale for the frequency axis.

$$\arg(H(j\omega)) = \arg\left(M \frac{\prod_{r=1}^R \left(1 + \frac{j\omega}{\beta_r}\right)}{\prod_{p=1}^P \left(1 + \frac{j\omega}{\alpha_p}\right)}\right) \quad (22)$$

$$= \arg(M) + \sum_{r=1}^R \arg\left(1 + \frac{j\omega}{\beta_r}\right) + \sum_{p=1}^P \left(-\arg\left(1 + \frac{j\omega}{\alpha_p}\right)\right) \quad (23)$$

1. Real-valued poles  $\alpha_p$ :

$$\left(-\arg\left(1 + \frac{j\omega}{\alpha_p}\right)\right). \quad (24)$$

Again, split into two ranges of  $\omega$ .

(a) Low frequencies:  $\omega \ll |\alpha_p|$ . In that case,  $|\omega/\alpha_p| \ll 1$ , and hence,

$$-\arg\left(1 + \frac{j\omega}{\alpha_p}\right) \approx 0. \quad (25)$$

(b) High frequencies:  $\omega \gg |\alpha_p|$ . In that case,  $|\omega/\alpha_p| \gg 1$ , and hence,

$$-\arg\left(1 + \frac{j\omega}{\alpha_p}\right) \approx -\arg\left(\frac{j\omega}{\alpha_p}\right) = -\frac{\pi}{2} \quad (26)$$

2. Real-valued zero  $\beta_r$ :

$$\arg\left(1 + \frac{j\omega}{\beta_r}\right). \quad (27)$$

Again, split into two ranges of  $\omega$ .

(a) Low frequencies:  $\omega \ll |\beta_r|$ . In that case,  $|\omega/\beta_r| \ll 1$ , and hence,

$$\arg\left(1 + \frac{j\omega}{\beta_r}\right) \approx 0. \quad (28)$$

(b) High frequencies:  $\omega \gg |\beta_r|$ . In that case,  $|\omega/\beta_r| \gg 1$ , and hence,

$$\arg\left(1 + \frac{j\omega}{\beta_r}\right) \approx \arg\left(\frac{j\omega}{\beta_r}\right) = \frac{\pi}{2} \quad (29)$$

3. Complex-conjugate pairs of poles: See Homework 9...

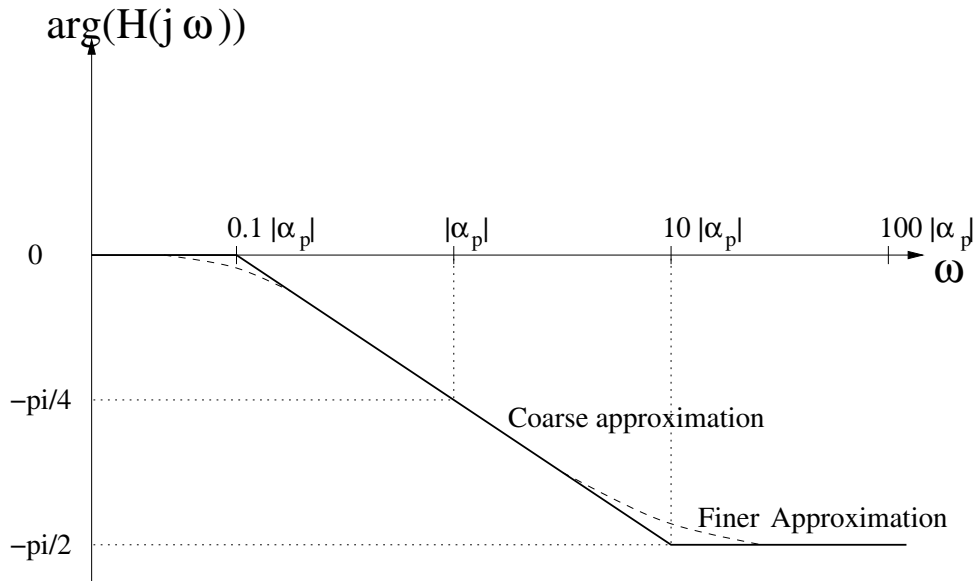


Figure 4: Sketch of the phase response of a single real-valued pole at  $\alpha_p$  with a logarithmic frequency scale.

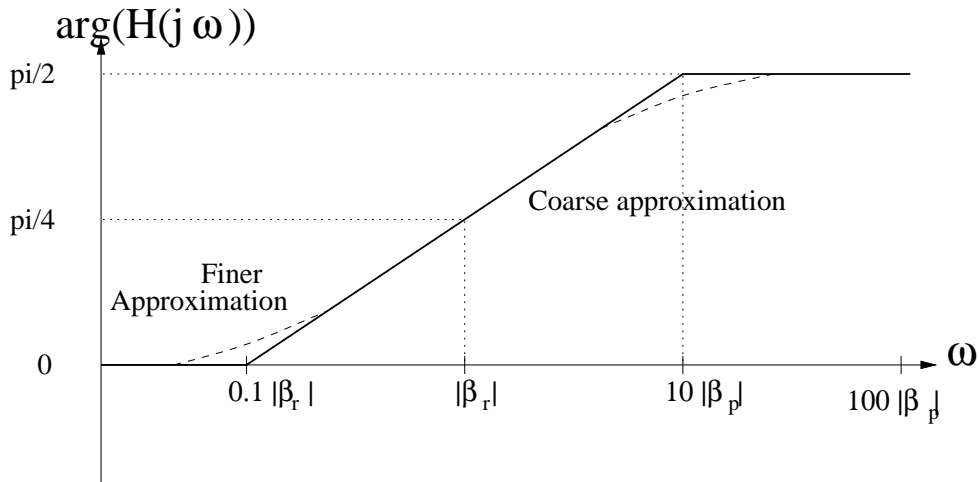


Figure 5: Sketch of the phase response of a single real-valued zero at  $\beta_r$  with a logarithmic frequency scale.