Homework 10 Solutions
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Problem 1 *(Inverse Laplace.)*

OWN 9.22 (c)

$$X(s) = \frac{s+1}{(s+1)^2+9}, \Re e\{s\} < -1$$

Since $X(s)$ is left sided, from Table 9.2 in OWN:

$$x(t) = -[e^{-t}\cos 3t]u(-t)$$

OWN 9.22 (e)

$$X(s) = \frac{s+1}{s^2+5s+6}, -3 < \Re e\{s\} < -2$$

$$\Rightarrow X(s) = \frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}$$

$$x(t) = 2e^{-3t}u(t) + e^{-2t}u(-t)$$

OWN 9.22 (g)

$$X(s) = \frac{s^2-s+1}{(s+1)^2}, \Re e\{s\} > -1$$

$$\Rightarrow X(s) = \frac{s^2+2s+1-3s}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2}$$

$$\Rightarrow x(t) = \delta(t) - 3\frac{d}{dt}\{te^{-t}u(t)\}$$

$$x(t) = \delta(t) - 3e^{-t}u(t) + 3te^{-t}u(t)$$

Problem 2 *(Region of convergence.)*

OWN 9.23. The four pole-zero plots shown may have the following possible ROCs:

- Top Left (TL): $\Re\{s\} < -2$ or $-2 < \Re\{s\} < 2$ or $\Re\{s\} > 2$
- Top Right (TR): $\Re\{s\} < -2$ or $\Re\{s\} > -2$
- Bottom Left (BL): $\Re\{s\} < 2$ or $\Re\{s\} > 2$

- Bottom Right (BR): Entire s-plane

Also, suppose that the signal $x(t)$ has Laplace transform $X(s)$ with ROC R

- (1)

  If $x(t)e^{-3t}$ is absolutely integrable, then the ROC of $x(t)$ must contain the line $\Re\{s\} = 3$. This can be seen by looking at the definition of the ROC in equation 9.36 in OWN. The ROC R of $x(t)$ is:

  (TL) $\Re\{s\} > 2$, (TR) $\Re\{s\} > -2$, (BL) $\Re\{s\} > 2$, (BR) Entire s-plane

- (2)

  We know from Table 9.2 that $e^{-t}u(t)$ has Laplace transform $\frac{1}{s+1}$ with ROC $\Re\{s\} > -1$. From Table 9.1, the Laplace transform of $x(t) \star [e^{-t}u(t)]$ is

$$\frac{X(s)}{s+1}$$

  with ROC $R_2 = R \cap [\Re\{s\} > -1]$. If $x(t) \star [e^{-t}u(t)]$ is absolutely integrable, then R_2 must include the $j\omega$ axis, which means that R must include the $j\omega$ axis. The ROC R is:

  (TL) $-2 < \Re\{s\} < 2$, (TR) $\Re\{s\} > -2$, (BL) $\Re\{s\} < 2$, (BR) Entire s-plane

- (3)

  If $x(t) = 0$ for $t > 1$, then $x(t)$ is left-sided or finite duration. This implies that if $\Re\{s\} = \sigma_1$ is in the ROC, then all values of s for which $\Re\{s\} < \sigma_1$ will also be in the ROC. The ROC R is:

  (TL) $\Re\{s\} < -2$, (TR) $\Re\{s\} < -2$, (BL) $\Re\{s\} < 2$, (BR) Entire s-plane

- (4)

  If $x(t) = 0$ for $t < -1$, then $x(t)$ is right-sided or finite duration. This implies that if $\Re\{s\} = \sigma_1$ is in the ROC, then all values of s for which $\Re\{s\} > \sigma_1$ will also be in the ROC. The ROC R is:

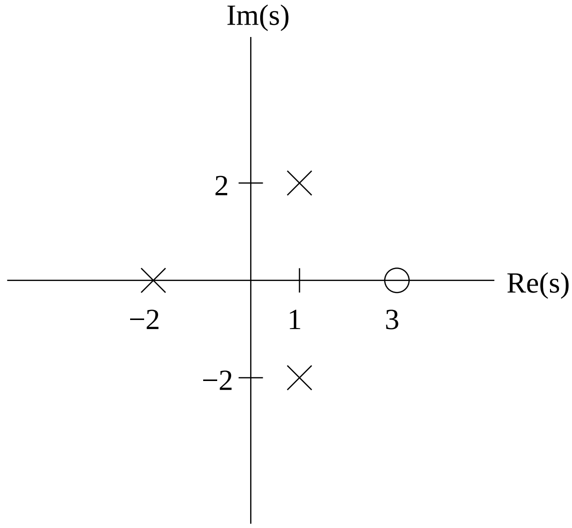
  (TL) $\Re\{s\} > 2$, (TR) $\Re\{s\} > -2$, (BL) $\Re\{s\} > 2$, (BR) Entire s-plane

Problem 3 *(An LTI system.)*

- (a)

$$H(s) = \frac{5(s-3)}{(s+2)(s^2-2s+5)}$$

  There is a zero at $s = 3$, and poles at $s = -2$ and $s = 1 \pm j2$. The pole/zero diagram is shown in the following figure.



- (b)

  Because $H(s) = Y(s)/X(s)$, we can write $Y(s)(s+2)(s^2-2s+5) = 5(s-3)X(s)$, which can be expanded as

  $$s^3Y(s) + sY(s) + 10Y(s) = 5sX(s) - 15X(s)$$

  The time-domain differential equation is given by

  $$\frac{d^3y(t)}{dt^3} + \frac{dy(t)}{dt} + 10y(t) = 5\frac{dx(t)}{dt} - 15x(t)$$

- (c)

  First we find the partial fraction expansion of the transfer function.

  $$H(s) = \frac{5s-15}{s^3+s+10} = \frac{5(s-3)}{(s+2)(s^2-2s+5)}$$
  $$= \frac{A}{s+2} + \frac{Bs+C}{s^2-2s+5}$$
  $$A = \frac{-25}{13}, B = \frac{25}{13}, C = \frac{-35}{13}$$
  $$\Rightarrow H(s) = \frac{1}{13}[\frac{-25}{s+2} + \frac{25s-35}{(s-1)^2+2^2}]$$
  $$\Rightarrow H(s) = \frac{1}{13}[\frac{-25}{s+2} + \frac{25(s-1)}{(s-1)^2+2^2} - \frac{10}{(s-1)^2+2^2}]$$

  If the system is causal, then the ROC is the right half plane $\Re\{s\} > 1$. Using the transform pairs in Table 9.2, we see that the impulse response is given by

  $$h(t) = -\frac{25}{13}e^{-2t}u(t) + \frac{25}{13}e^t\cos(2t)u(t) - \frac{5}{13}e^t\sin(2t)u(t)$$

  The resulting system is not stable, since the ROC does not contain the $j\omega$ axis.

- (d)

  If the system is stable, then the ROC must contain the $j\omega$ axis, hence the ROC is $-2 < \Re\{s\} < 1$. To find the inverse transform, we first combine transform pair 11 in Table 9.2 with the time scaling property in Table 9.1, with $a = -1$, to see that the function $\cos(\omega_0 t)u(-t)$ has Laplace transform

  $$\frac{(-s)}{(-s)^2 + \omega_0^2} = \frac{-s}{s^2 + \omega_0^2}$$

  and ROC $\Re\{s\} < 0$. Then, applying the shifting in the s-domain property from Table 9.1, with $s_0 = 1$, we see that the function $e^t \cos(\omega_0 t)u(-t)$ has Laplace transform

  $$\frac{-(s-1)}{(s-1)^2 + \omega_0^2}$$

  with ROC $\Re\{s\} < 1$. Using similar reasoning, we see that the function $-e^t \sin(\omega_0 t)u(-t)$ has Laplace transform

  $$\frac{\omega_0}{(s-1)^2 + \omega_0^2}$$

  with ROC $\Re\{s\} < 1$. Therefore, the impulse response of the system is given by

  $$h(t) = -\frac{25}{13}e^{-2t}u(t) - \frac{25}{13}e^t \cos(2t)u(-t) + \frac{5}{13}e^t \sin(2t)u(-t)$$

  This is clearly not a causal system.

Problem 4*(System analysis.)*

OWN 9.32

If $x(t) = e^{2t}$ produces $y(t) = (1/6)e^{2t}$, then $H(2) = 1/6$. By taking the Laplace transform of both sides of the differential equation, we find that

$$H(s) = \frac{\frac{1}{s+4} + b\frac{1}{s}}{s+2} = \frac{s + b(s+4)}{s(s+4)(s+2)}$$

Substituting in $H(2) = 1/6$, we find that $b = 1$. Therefore,

$$H(s) = \frac{2(s+2)}{s(s+4)(s+2)} = \frac{2}{s(s+4)}$$

Problem 5*(A simple fact about Laplace transforms.)*

- OWN 9.41(a)

  The Laplace transform of a signal $y(t) = x(-t)$ is given by

  $$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} x(-t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{st}dt \\ &= X(-s) \end{aligned}$$

If $x(t) = x(-t)$, then $X(s) = X(-s)$

- OWN 9.41(c), Plot (a)

  For a signal to be even, it must be either two-sided or finite duration. Thus, if $X(s)$ has poles, the ROC must be a strip in the s-plane.

  For plot (a), we have

$$X(s) = \frac{As}{(s+1)(s-1)}$$

  Therefore

$$X(-s) = \frac{-As}{(s-1)(s+1)} = -X(s)$$

  This means that this plot **cannot** correspond to an even function $x(t)$. (It actually corresponds to an odd function.)

- OWN 9.41(c), Plot (b)

  We see that the ROC cannot be chosen to be a strip in the s-plane. Therefore, the plot **cannot** correspond to an even function.

- OWN 9.41(c), Plot (c)

  We have

$$X(s) = \frac{A(s-j)(s+j)}{(s-1)(s+1)} = \frac{A(s^2+1)}{s^2-1}$$

  Thus

$$X(-s) = \frac{A(s^2+1)}{s^2-1} = X(s)$$

  This means that $x(t)$ **can** be an even function, as long as the ROC is chosen to be $-1 < \Re\{s\} < 1$

- OWN 9.41(c), Plot (c)

  We see that the ROC cannot be chosen to be a strip in the s-plane. Therefore, the plot **cannot** correspond to an even function.

Problem 6 *(Deconvolution.)*

OWN 9.47

$$y(t) = e^{-2t}u(t) \Leftrightarrow Y(s) = \frac{1}{s+2}, \Re e\{s\} > -2$$

$$H(s) = \frac{s-1}{s+1}, \Re e\{s\} > -1$$

(a)

both $y(t)$ and $h(t)$ are both causal and stable.

$$\Rightarrow X(s) = \frac{Y(s)}{H(s)} = \frac{(s+1)}{(s+2)(s-1)} = \frac{1}{3(s+2)} + \frac{2}{3(s-1)}$$

$X(s)$ has two poles located at $s = 1$ and $s = -2$. Clearly, there are three possible choices for $x(t)$ depending on the ROC.

$$\Rightarrow X_1(s) = \frac{1}{3(s+2)} + \frac{2}{3(s-1)}, \Re e\{s\} > 1$$

$$\Rightarrow x_1(t) = \frac{1}{3}e^{-2t}u(t) + \frac{2}{3}e^{t}u(t)$$

$$\Rightarrow X_2(s) = \frac{1}{3(s+2)} + \frac{2}{3(s-1)}, -2 < \Re e\{s\} < 1$$

$$\Rightarrow x_2(t) = \frac{1}{3}e^{-2t}u(t) - \frac{2}{3}e^{t}u(-t)$$

$$\Rightarrow X_3(s) = \frac{1}{3(s+2)} + \frac{2}{3(s-1)}, \Re e\{s\} < -2$$

$$\Rightarrow x_3(t) = -\frac{1}{3}e^{-2t}u(-t) - \frac{2}{3}e^{t}u(-t)$$

However, $x_3(t)$ is not a possible input since its ROC does not overlap with the ROC of $y(t)$. Therefore, only $x_1(t)$ and $x_2(t)$ can generate $y(t)$ at the output.

(b)

If we know that $x(t)$ is stable, then we have only one option: $x_2(t)$.

(c)

If both the input $y(t)$ and the filter $g(t)$ are stable, then $x(t) = y(t) * g(t)$ is also stable. Therefore, the output must be $x_2(t)$. Also, since $x_2(t)$ is a left-sided signal, $g(t)$ cannot be causal.

$$\Rightarrow G(s) = \mathcal{LT}\{g(t)\} = \frac{X(s)}{Y(s)} = \frac{1}{3} + \frac{2(s+2)}{3(s-1)}$$

Since $G(s)$ has one pole at $s = 1$, the ROC must be $\Re e\{s\} < 1$ (for stability).

$$\Rightarrow g(t) = \frac{1}{3}\delta(t) + \frac{2}{3}(-2e^{t}u(-t) + \frac{d}{dt}\{-e^{t}u(-t)\}$$

$$= \frac{1}{3}\delta(t) - 2e^{t}u(-t) + \frac{2}{3}\delta(t) = \delta(t) - 2e^{t}u(-t)$$

$$\Rightarrow g(t) * y(t) = e^{-2t}u(t) + \int_{-\infty}^{\infty} -2e^{-2\tau}u(\tau)e^{t-\tau}u(\tau - t)d\tau$$

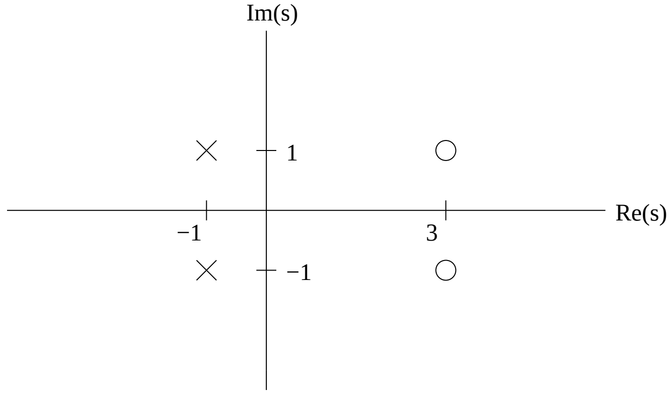
$$\int_{-\infty}^{\infty} -2e^{-2\tau}u(\tau)e^{t-\tau}u(\tau-t)d\tau = -2e^{t}\int_{0}^{\infty} e^{-3\tau}u(\tau-t)d\tau = \begin{cases} -2e^{t}\int_{t}^{\infty} e^{-3\tau}d\tau = \frac{-2}{3}e^{-2t} & \text{for } t > 0 \\ -2e^{t}\int_{0}^{\infty} e^{-3\tau}d\tau = \frac{-2}{3}e^{t} & \text{for } t \leqslant 0 \end{cases}$$

$$= \frac{-2}{3}(e^{-2t}u(t) + e^{t}u(-t))$$

$$\Rightarrow g(t) * y(t) = e^{-2t}u(t) - \frac{2}{3}(e^{-2t}u(t) + e^{t}u(-t)) = \frac{1}{3}e^{-2t}u(t) - \frac{2}{3}e^{t}u(-t) = x_2(t)$$

Problems 7/8 *(Stability and Causality.)*

Since $h(t)$ is real, its poles and zeros must occur in complex conjugate pairs. Therefore, the known poles and zeros of $H(s)$ are shown in the following figure.



Since $H(s)$ has exactly 2 zeros at infinity, $H(s)$ has *at least* two more unknown finite poles. If $H(s)$ has more than 4 poles, then it will have a zero at some finite location for every additional pole. Furthermore, because $h(t)$ is causal and stable, all poles of $H(s)$ must lie in the left half of the s-plane and the ROC must contain the $j\omega$ axis.

- OWN 9.51(a)

  This is **true**. Consider $g(t) = h(t)e^{-3t}$. The Laplace transform $G(s) = H(s+3)$ will have a ROC that is equal to the ROC of $H(s)$ shifted to the left by 3. Clearly, the ROC of $G(s)$ will still contain the $j\omega$ axis.

- OWN 9.51(b)

  There is **insufficient information**. As stated before, $H(s)$ has at least two more unknown poles. We do not know where the rightmost pole is, so we cannot determine the ROC of $H(s)$.

- OWN 9.51(c)

  This is **true**. Since $H(s)$ is rational, it may be expressed as the ratio of two polynomials in s. Furthermore, since $h(t)$ is real, the coefficients of these polynomials will be real. We can write

  $$\frac{Y(s)}{X(s)} = H(s) = \frac{P(s)}{Q(s)}$$

  where $P(s)$ and $Q(s)$ are polynomials in s. The differential equation relating $y(t)$ and $x(t)$ is found by taking the inverse Laplace transform of $Y(s)Q(s) = X(s)P(s)$. Clearly, this differential equation will have only real coefficients.

- OWN 9.51(d)

  This is **false**. Because $H(s)$ has two zeros at $s = \infty$, the limit of $H(s)$ as s goes to ∞ will be 0.

- OWN 9.51(e)

  This is **true**. As stated at the beginning of the problem, $H(s)$ has two zeros at infinity and at least two finite zeros, so $H(s)$ must have at least four poles.

- OWN 9.51(f)

  There is **insufficient information**. As stated earlier, $H(s)$ may have other zeros, including zeros on the $j\omega$ axis.

- OWN 9.51(g)

  This is **false**. We know that

$$e^{3t}\sin t = \frac{1}{j2}e^{(3+j)t} = \frac{1}{j2}e^{(3-j)t}$$

Both $e^{(3+j)t}$ and $e^{(3-j)t}$ are eigenfunctions of the LTI system. The response of the system to these two inputs will be $H(3+j)e^{(3+j)t}$ and $H(3-j)e^{(3-j)t}$, respectively. Because $H(s)$ has zeros at $3 \pm j$, the output of the system to the two exponentials has to be zero. Therefore, the response of the system to the input $e^{3t}\sin t$ will be zero.

Problems 9 *(LT Properties.)*

Let $z(t)$ be the unit step response of $h(t)$.

$$\Rightarrow z(t) = u(t) * h(t)$$

$$\Rightarrow Z(s) = \mathcal{LT}\{z(t)\} = \frac{H(s)}{s}$$

The steady state value $z(\infty) = \frac{1}{3}$. From Table 9.1 in OWN:

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} sZ(s) = \lim_{s\to 0} H(s)$$

$$H(0) = \frac{1}{3} \quad \text{by continuity}$$

Let $y(t) = x(t) * h(t)$, where $x(t) = e^t u(t)$.

$$Y(s) = \mathcal{LT}\{y(t)\} = H(s)X(s) = \frac{H(s)}{s-1}$$

Since $y(t)$ is causal and absolute.y integrable (stable). The ROC of $y(t)$ must contain the imaginary axis. Therefore, there cannot be a pole at $s = 1$. Therefore, $H(s)$ has to contain a zero at $s = 1$.

Let $w(t) = \frac{d^2h(t)}{dt^2} + 5\frac{dh(t)}{dt} + 6h(t)$. Since $w(t)$ is of finite duration, the ROC must span the entire s-plane.

$$W(s) = \mathcal{LT}\{w(t)\} = s^2H(s) + 5sH(s) + 6H(s) = (s+2)(s+3)H(s)$$

Since $W(s)$ cannot contain any poles, $H(s)$ can have at most two poles at $s = -2$ and $s = -3$. However, we are also given that the number of zeros in $H(s)$ is one less the number of poles. Therefore, $H(s)$ has exactly one zero at $s = 1$ and two poles at $s = -2$ and $s = -3$.

$$H(s) = \frac{K(s-1)}{(s+2)(s+3)}$$

$$\Rightarrow H(0) = \frac{-K}{6} = \frac{1}{3} \Rightarrow K = -2$$

$$\Rightarrow H(s) = \frac{-2(s-1)}{(s+2)(s+3)}, \Re e\{s\} > -2$$