Homework 11 Solutions

Problem 1 *(Block Diagram Representations.)*

The overall system may be treated as two feedback systems of the form shown in Figure 9.31 (on page 708 of OWN) connected in parallel. (Note that in this problem there is no minus sign on one of the inputs to the adders.) By repeating the analysis in Equations 9.159 - 9.163 in OWN, we find that the transfer function of the upper feedback system is

$$H_1(s) = \frac{1/s}{1+(1/s)(5)} = \frac{1}{s+5}$$

Similarly, the transfer function of the lower feedback system is

$$H_2(s) = \frac{3/s}{1+(3/s)(2)} = \frac{3}{s+6}$$

The transfer function of the overall system is given by

$$H(s) = H_1(s) + H_2(s) = \frac{4s+21}{s^2+11s+30}$$

Because $H(s) = Y(s)/X(s)$, we can write

$$Y(s)[s^2+11s+30] = X(s)[4s+21]$$

Taking the inverse Laplace transform of this equation, we obtain

$$\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 30y(t) = 4\frac{dx(t)}{dt} + 21x(t)$$

Problem 2

(Unilateral Laplace Transform.)

- (a) Labeling the voltage across the inductor as $v_L(t)$ and the voltage across the resistor as $v_R(t)$, we use Kirchoff's loop law to find that

  $$v_i(t) = v_R(t) + v_L(t) + v_o(t)$$

  Using the fundamental equation for a capacitor, the current in the circuit is given by $i(t) = C\frac{dv_o(t)}{dt}$. Furthermore, the voltage across the resistor is $v_R(t) = R \cdot i(t) = RC\frac{dv_o(t)}{dt}$ and the voltage across the inductor is given by $v_L(t) = L\frac{di(t)}{dt} = LC\frac{d^2v_o(t)}{dt^2}$. Combining these equations, we obtain

$$v_i(t) = LC\frac{d^2v_o(t)}{dt^2} + RC\frac{dv_o(t)}{dt} + v_o(t)$$

which can be rewritten as

$$\frac{d^2v_o(t)}{dt^2} + \frac{R}{L}\frac{dv_o(t)}{dt} + \frac{1}{LC}v_o(t) = \frac{1}{LC}v_i(t)$$

Substituting in the values of R, L, and C,

$$\frac{d^2v_o(t)}{dt^2} + 3\frac{dv_o(t)}{dt} + 2v_o(t) = 2v_i(t)$$

- (b)

  In order to take the unilateral Laplace transform of this differential equation, we need to have initial conditions at $t = 0^-$. Because the voltage across a capacitor cannot change instantaneously, the initial condition $v_o(0^-) = v_o(0^+)$. Similarly, since the current through an inductor cannot change instantaneously, $i(0^-) = i(0^+)$. Using the fact that $i(t) = C\frac{dv_o(t)}{dt}$, it follows that the initial condition

$$\left.\frac{dv_o(t)}{dt}\right|_{t=0^-} = \left.\frac{dv_o(t)}{dt}\right|_{t=0^+}$$

  Now, we can take the unilateral Laplace transform of the differential equation in part (a).

$$s^2\mathcal{V}_o(s) - sv_o(0^-) - v_o'(0^-) + 3s\mathcal{V}_o(s) - 3v_o(0^-) + 2\mathcal{V}_o(s) = 2\mathcal{V}_i(S)$$

  Because $v_i(t) = e^{-3t}u(t)$ is equal to 0 for $t < 0^-$, the unilateral Laplace transform of $v_i(t)$ is identical to the bilateral Laplace transform

$$\mathcal{V}_i(s) = \frac{1}{s+3} \qquad Re\{s\} > -3$$

  Substituting the expression for $\mathcal{V}_i(s)$ and the initial conditions

$$\mathcal{V}_o(s)[s^2 + 3s + 2] = s + 2 + 3 + \frac{2}{s+3} \tag{1}$$
$$\mathcal{V}_o(s)[(s+2)(s+1)] = s + 5 + \frac{2}{s+3} \tag{2}$$
$$\mathcal{V}_o(s)[(s+2)(s+1)] = \frac{(s+5)(s+3)+2}{s+3} \tag{3}$$
$$\mathcal{V}_o(s) = \frac{s^2 + 8s + 17}{(s+1)(s+2)(s+3)} \tag{4}$$

  By taking a partial fraction expansion of this expression, we obtain

$$\frac{s^2 + 8s + 17}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \tag{5}$$
$$s^2 + 8s + 17 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2) \tag{6}$$
$$s^2 + 8s + 17 = A(s^2 + 5s + 6) + B(s^2 + 4s + 3) + C(s^2 + 3s + 2) \tag{7}$$

Equating the coefficients of identical powers of s on both sides of the equation gives the system

$$\begin{aligned} 1 &= A+B+C && (8) \\ 8 &= 5A+4B+3C && (9) \\ 17 &= 6A+3B+2C && (10) \\ & && (11) \end{aligned}$$

We can solve this system numerically to find that $A=5$, $B=-5$, and $C=1$. Taking the inverse unilateral Laplace transform (and knowing that ROC of a unilateral Laplace transform must be a right half plane) we find that

$$v_o(t) = 5e^{-t}u(t) - 5e^{-2t}u(t) + e^{-3t}u(t)$$

Problem 3

(Pole/Zero Plots)

- (a)

  (3). Note that

$$\frac{|j\omega - a|}{|j\omega + a|} = \frac{\sqrt{\omega^2 + (-a)^2}}{\sqrt{\omega^2 + a^2}} = 1$$

- (b)

  (4). $|H(j\omega)|$ must be zero at $\omega = 0$. Since the number of poles and the number of zeros are equal, the limit of $|H(j\omega)|$ as ω approaches ∞ is non-zero and finite.

- (c)

  (5). $|H(j\omega)|$ must have two symmetric peaks because of the two poles.

- (d)

  (1). $|H(j\omega)|$ must be zero at $\omega = 0$. Since there are two poles and only one zero, the limit of $|H(j\omega)|$ as ω approaches ∞ is equal to 0.

- (e)

  (2). $|H(j\omega)|$ should approach 0 at $\omega = 0$. It will not equal 0, because the zeros are not on the $j\omega$ axis. Also, because there are two zeros and no poles, $|H(j\omega)|$ should increase with $|\omega|$.

Problem 4

(A simple feedback control system)

- (a)

$$\begin{aligned} Y(s) &= \frac{s+2}{s-1}F(s)E(s) && (12) \\ E(s) &= X(s) - Y(s) && (13) \end{aligned}$$

Substituting the second equation into the first, we see that

$$Y(s) = \frac{s+2}{s-1}F(s)\left(X(s) - Y(s)\right)$$

$$T(s) = \frac{Y(s)}{X(s)} = \frac{\frac{s+2}{s-1}F(s)}{1 + \frac{s+2}{s-1}F(s)}$$

Further, we observe that

$$E(s) = X(s) - Y(s) = X(s) - T(s)X(s) = (1 - T(s))\,X(s)$$

- (b)
  The overall transfer function of the feedback system is

$$T(s) = \frac{\frac{s+2}{s-1}K}{1 + \frac{s+2}{s-1}K} = \frac{K(s+2)}{(K+1)s + (2K-1)}$$

where K is a real number which represents an adjustable gain in the system. The root locus is the path in the complex plane of the poles of $T(s)$ as K is varied. $T(s)$ has a zero at $s = -2$, and a pole at $s = -\frac{2K-1}{K+1}$. When $K = 0$, the pole is located at $s = 1$. For K positive, as $K \to \infty$, the pole moves left to $s \to -2$. For K negative, as $K \to -1$, the pole moves right to $s \to \infty$. As K is varied from -1 to $-\infty$, the pole moves right from $-\infty$ to $s \to -2$.

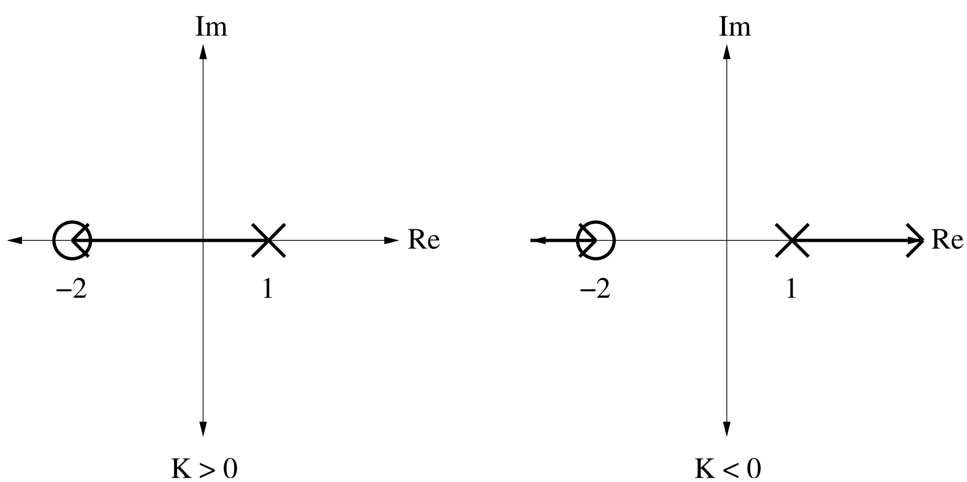

Figure 1: Root Locus for Problem 4

Since the system is known to be causal, the ROC is a right-half plane, to the right of the rightmost pole. The system is stable iff the ROC includes the $j\omega$-axis. Therefore the system is stable iff all the poles of $T(s)$ lie in the left-half plane, which is true when $K < -1$ or $K > \frac{1}{2}$.

- (c) To find the asymptotic value of the error $e(t)$ as $t \to \infty$, we use the final-value theorem (see OWN Table 9.1). For the given input $x(t) = u(t)$, we first check the conditions. Since $x(t) = 0$ for $t < 0$ and $T(s)$ is causal, $y(t) = 0$ for $t < 0$, and thus $e(t) = x(t) - y(t) = 0$ for $t < 0$. Since the system is stable, the output is bounded for a bounded input, and thus $\exists M$ such that $|e(t)| < M$ for all t. The final-value theorem then says that $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$. We find in OWN Table 9.2 that the Laplace transform of $x(t) = u(t)$ is $X(s) = \frac{1}{s}$.

$$
\begin{aligned}
E(s) &= (1-T(s))X(s) = (1-T(s))\frac{1}{s} \\
sE(s) &= 1-T(s) = \frac{s-1}{(K+1)s+(2K-1)} \\
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} \frac{s-1}{(K+1)s+(2K-1)} \\
&= -\frac{1}{2K-1}
\end{aligned}
$$

Therefore the error signal can be made close to zero by choosing K to be very large.

Problem 5 *(Bode Plots)*

- (a)

  The Bode plot of the magnitude frequency response of system $H(s)$ is defined as

$$
\begin{aligned}
H(s) &= \frac{1}{1+s/10} \\
|H(j\omega)|_{dB} &\stackrel{def}{=} 20\log_{10}|H(j\omega)| \\
&= -20\log_{10}\left|1+\frac{j\omega}{10}\right|
\end{aligned}
$$

  For low frequencies $\omega << 10$, the magnitude frequency response can be approximated as

$$-20\log_{10}\left|1+\frac{j\omega}{10}\right| \approx 0$$

  For high frequencies $\omega >> 10$, the magnitude frequency response can be approximated as

$$-20\log_{10}\left|1+\frac{j\omega}{10}\right| \approx -20\log_{10}\left|\frac{j\omega}{10}\right| = -20\log_{10}\left|\frac{\omega}{10}\right|$$

  The phase response of $H(s)$

$$\arg(H(j\omega)) = -\arg(1+\frac{j\omega}{10})$$

  can also be approximated for low frequencies $\omega << 10$ as

$$-\arg(1+\frac{j\omega}{10}) \approx 0$$

  and for high frequencies $\omega >> 10$ as

$$-\arg(1+\frac{j\omega}{10}) \approx -\arg(\frac{j\omega}{10}) = -\frac{\pi}{2}$$

  The Bode plots of the magnitude frequency response and phase response are shown in Figure 2.

- (b)

The Bode plot of the magnitude frequency response of system $H(s)$ is

$$\begin{aligned} H(s) &= \frac{1}{1+s/20+(s/10)^2} \\ |H(j\omega)|_{dB} &= -20\log_{10}\left|1+\frac{1}{2}\frac{j\omega}{10}+\left(\frac{j\omega}{10}\right)^2\right| \end{aligned}$$

For low frequencies $\omega << 10$, (ie $|\omega/10| << 1$), this can be approximated by

$$-20\log_{10}\left|1+\frac{1}{2}\frac{j\omega}{10}+\left(\frac{j\omega}{10}\right)^2\right| \approx 0$$

For high frequencies $\omega >> 10$, (ie $|\omega/10| >> 1$), this can approximated by

$$-20\log_{10}\left|1+\frac{1}{2}\frac{j\omega}{10}+\left(\frac{j\omega}{10}\right)^2\right| \approx -20\log_{10}\left|\left(\frac{j\omega}{10}\right)^2\right| = -40\log_{10}\left|\frac{\omega}{10}\right|$$

The phase response of $H(s)$ is

$$-\arg\left(1+\frac{1}{2}\frac{j\omega}{10}+\left(\frac{j\omega}{10}\right)^2\right)$$

For low frequencies $\omega << 10$, (ie $|\omega/10| << 1$), the phase response can be approximated as

$$-\arg\left(1+\frac{1}{2}\frac{j\omega}{10}+\left(\frac{j\omega}{10}\right)^2\right) \approx 0$$

For high frequencies $\omega >> 10$, (ie $|\omega/10| >> 1$), the phase response can be approximated as

$$\begin{aligned} -\arg\left(1+\frac{1}{2}\frac{j\omega}{10}+\left(\frac{j\omega}{10}\right)^2\right) &\approx -\arg\left(\left(\frac{j\omega}{10}\right)^2\right) \\ &= -\arg\left(-\left(\frac{\omega}{10}\right)^2\right) \\ &= -\pi \end{aligned}$$

The Bode plots of the magnitude frequency response and phase response are shown in Figure 3.

- (c)

  The Bode plot of the magnitude frequency response of the system $H(s)$ is

$$\begin{aligned} H(s) &= \frac{(s+1)(s+1000)}{(s+10)(s+100)} \\ &= \frac{(1+s)(1+\frac{s}{1000})}{(1+\frac{s}{10})(1+\frac{s}{100})} \\ |H(j\omega)|_{dB} &= 20\log_{10}|H(j\omega)| \\ &= 20\log_{10}|1+j\omega| + 20\log_{10}\left|1+\frac{j\omega}{1000}\right| - 20\log_{10}\left|1+\frac{j\omega}{10}\right| - 20\log_{10}\left|1+\frac{j\omega}{100}\right| \end{aligned}$$

For low frequencies, we can approximate each term of the magnitude frequency response

$$\begin{aligned}
\omega << 1 \quad &\Rightarrow \quad 20\log_{10}|1+j\omega| \approx 0 \\
\omega << 1000 \quad &\Rightarrow \quad 20\log_{10}\left|1+\frac{j\omega}{1000}\right| \approx 0 \\
\omega << 10 \quad &\Rightarrow \quad -20\log_{10}\left|1+\frac{j\omega}{10}\right| \approx 0 \\
\omega << 100 \quad &\Rightarrow \quad -20\log_{10}\left|1+\frac{j\omega}{100}\right| \approx 0
\end{aligned}$$

For high frequencies, we can approximate each term of the magnitude frequency response

$$\begin{aligned}
\omega >> 1 \quad &\Rightarrow \quad 20\log_{10}|1+j\omega| \approx 20\log_{10}|\omega| \\
\omega >> 1000 \quad &\Rightarrow \quad 20\log_{10}\left|1+\frac{j\omega}{1000}\right| \approx 20\log_{10}\left|\frac{\omega}{1000}\right| \\
\omega >> 10 \quad &\Rightarrow \quad -20\log_{10}\left|1+\frac{j\omega}{10}\right| \approx -20\log_{10}\left|\frac{\omega}{10}\right| \\
\omega >> 100 \quad &\Rightarrow \quad -20\log_{10}\left|1+\frac{j\omega}{100}\right| \approx -20\log_{10}\left|\frac{\omega}{100}\right|
\end{aligned}$$

The overall magnitude frequency response Bode plot is found by summing these terms. Thus the system $H(s)$ has the approximate frequency response of a bandpass filter. The Bode plot of the magnitude frequency response is shown in Figure 4.

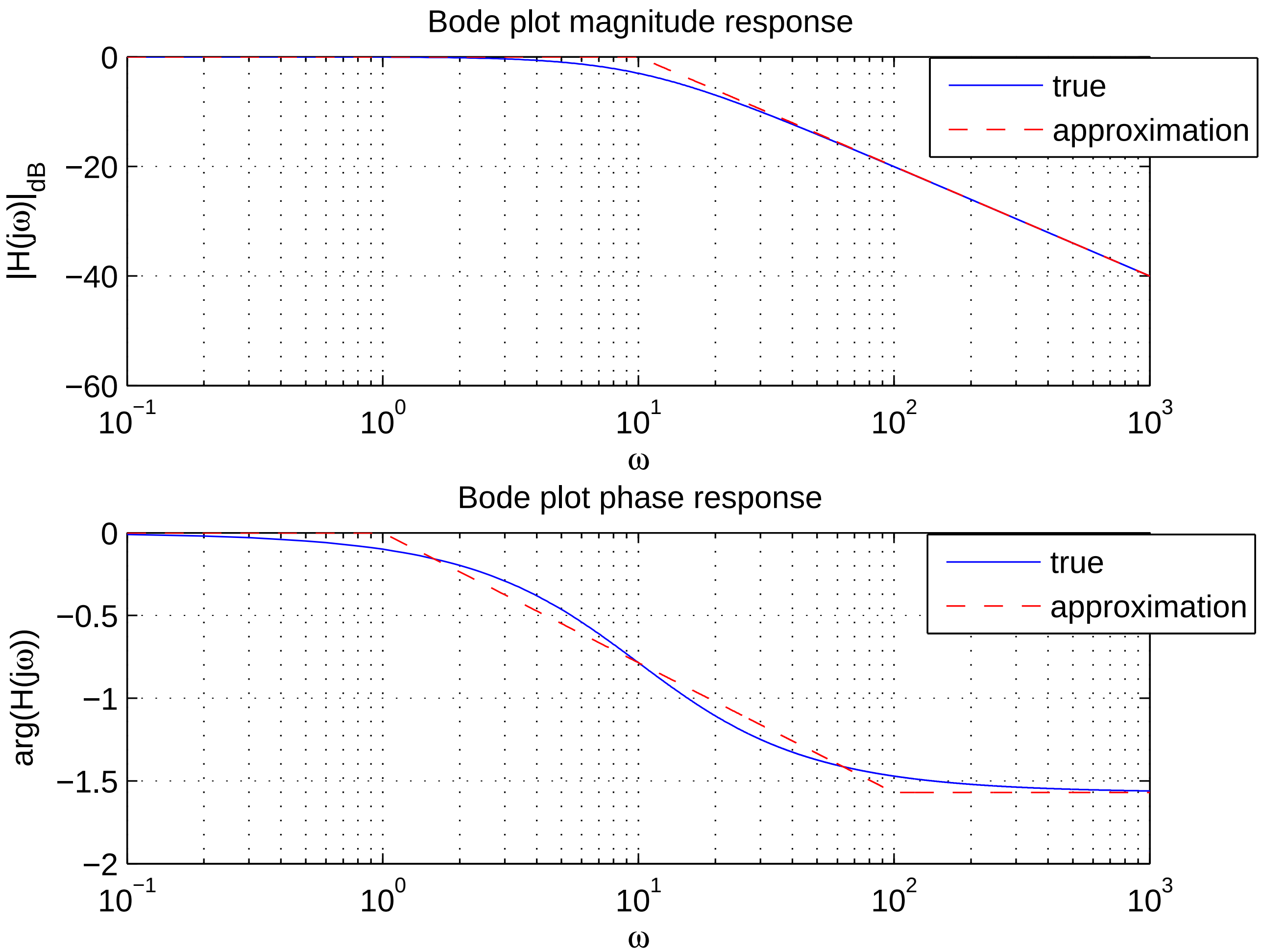

Figure 2: Problem 5 (a)

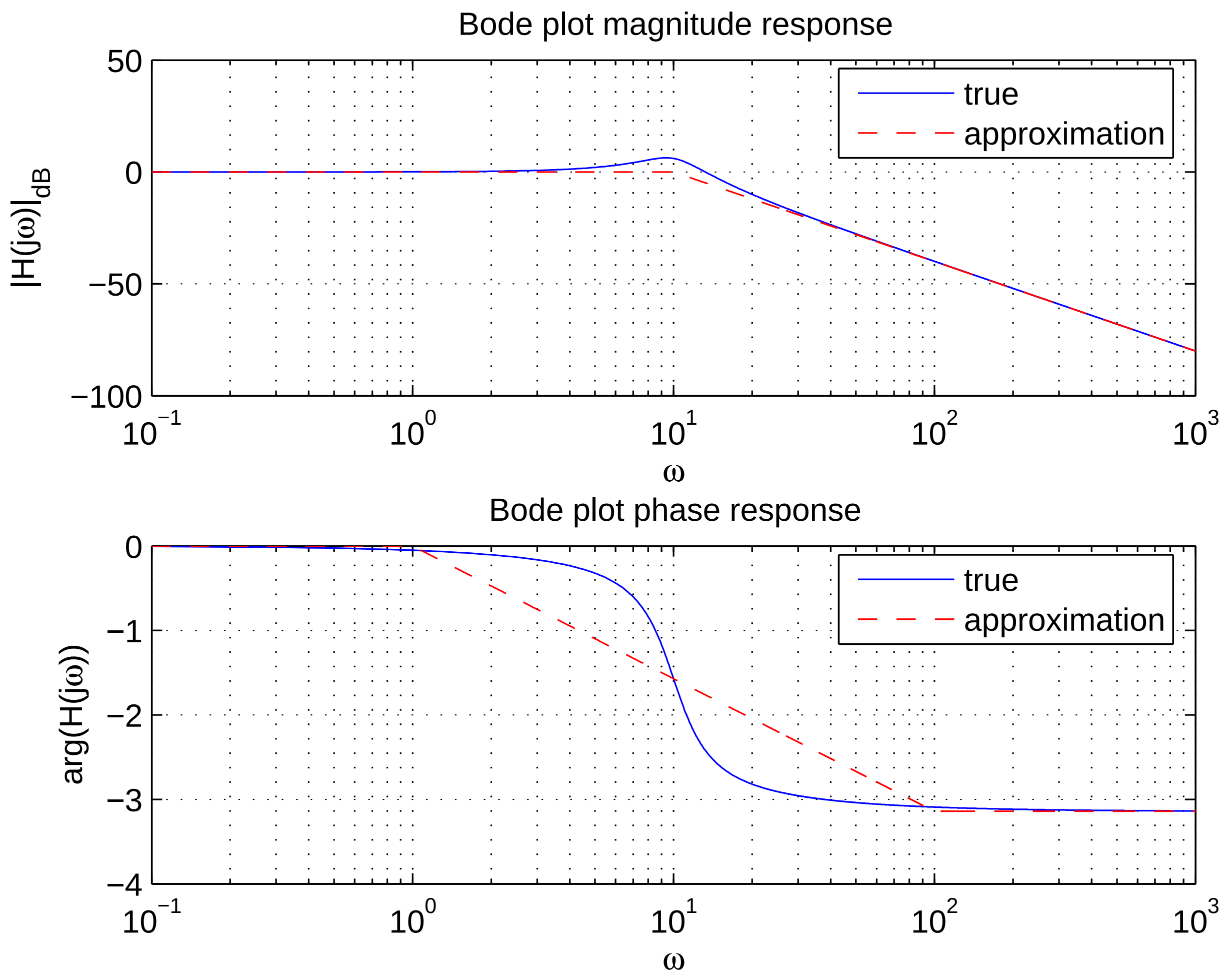

Figure 3: Problem 5 (b)

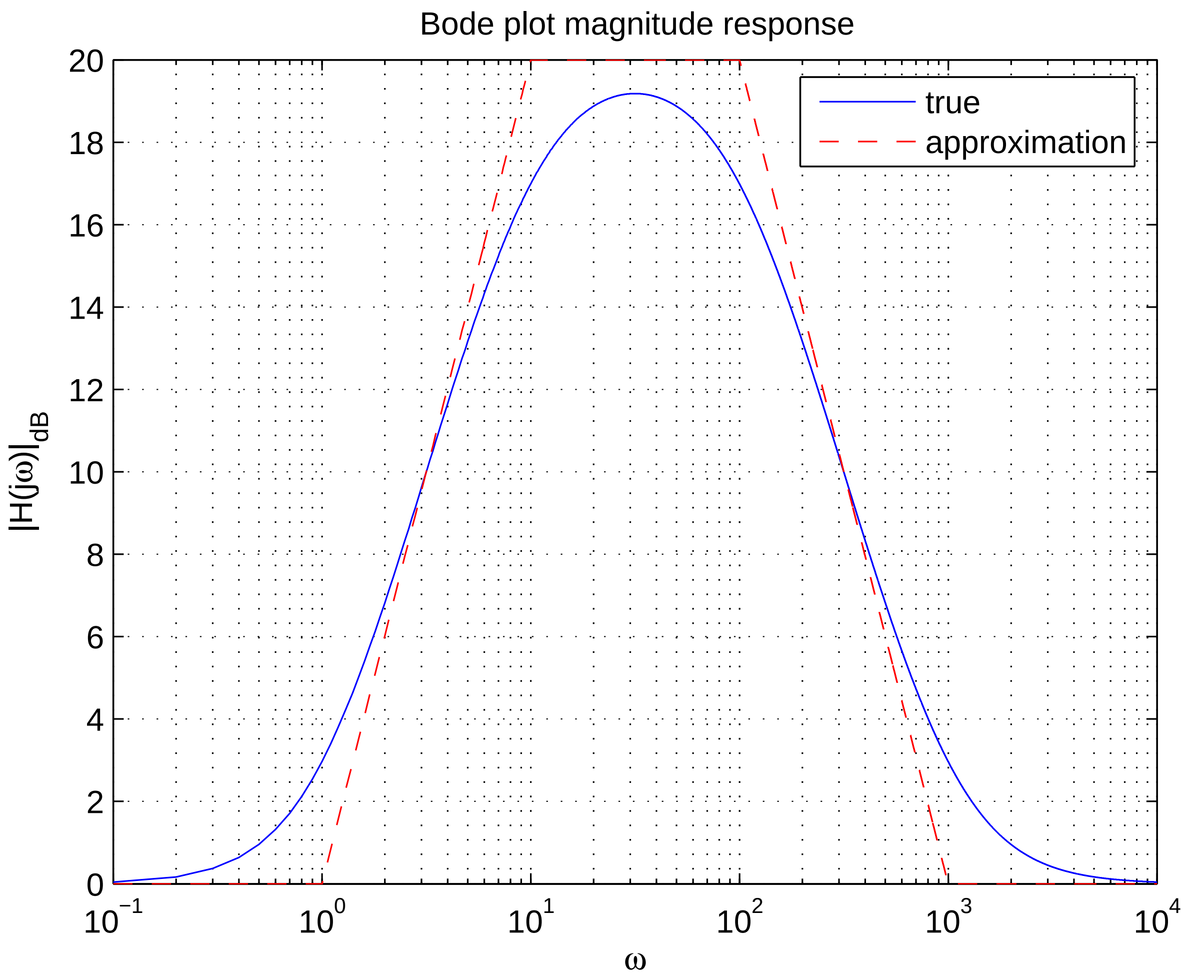

Figure 4: Problem 5 (c)