EECS 120 Signals & Systems | University of California, Berkeley: Fall 2006
Ramchandran | December 4, 2006

Homework 12 Solutions

Problem 1 *(z-Transform Basics)*

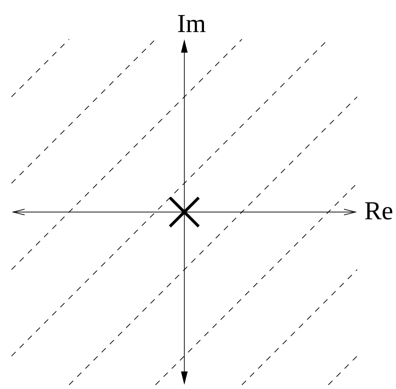
(a)

OWN 10.21 (b)

The z-transform of the discrete-time signal $x[n] = \delta[n-5]$ can be found in OWN Table 10.2 as

$$X(z) = z^{-5}, \qquad \text{with ROC all } z \text{ except } 0.$$

Notice that $X(z)$ has 5 poles at $z = 0$. Recall that the discrete-time Fourier transform of $x[n]$ is $X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$. Thus the Fourier transform of $x[n]$ exists because the ROC of $X(z)$ includes the unit circle $z = e^{j\omega}$.

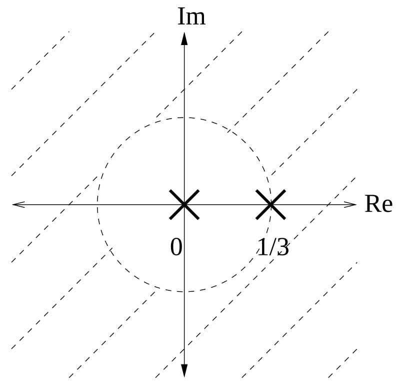


OWN 10.21 (h)

We find the z-transform of $x[n] = \left(\frac{1}{3}\right)^{n-2} u[n-2]$ by first defining $y[n] = \left(\frac{1}{3}\right)^{n} u[n]$. According to OWN Table 10.2, the z-transform of $y[n]$ is $Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$ with ROC $|z| > \frac{1}{3}$. Then by the time shifting property of the z-transform, given in OWN Table 10.1, the z-transform of $x[n] = y[n-2]$ is

$$X(z) = \frac{z^{-2}}{1-\frac{1}{3}z^{-1}} = \frac{1}{z\left(z-\frac{1}{3}\right)} \qquad \text{with ROC } |z| > \frac{1}{3}.$$

$X(z)$ has poles at $z = 0$ and $z = \frac{1}{3}$. Since the ROC of $X(z)$ includes the unit circle, the Fourier transform exists.



(b)

OWN 10.22 (b)

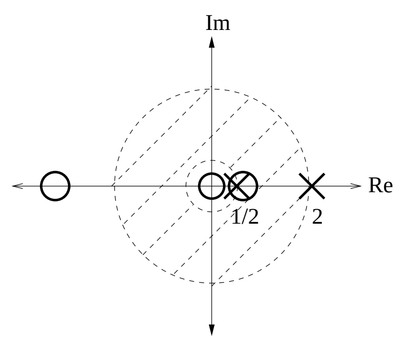
We can rewrite $x[n]$ as

$$\begin{aligned} x[n] &= n\left(\frac{1}{2}\right)^{|n|} \\ &= n\left(\frac{1}{2}\right)^{n} u[n] + n\left(\frac{1}{2}\right)^{-n} u[-n-1] \\ &= n\left(\frac{1}{2}\right)^{n} u[n] + n\,(2)^{n}\, u[-n-1]. \end{aligned}$$

In OWN Table 10.2, we find the z-transform of $n\left(\frac{1}{2}\right)^n u[n]$ is $\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$, with ROC $|z| > \frac{1}{2}$. Also in Table 10.2, we find the z-transform of $n2^n u[-n-1]$ is $-\frac{2z^{-1}}{(1-2z^{-1})^2}$, with ROC $|z| < 2$. Thus by linearity (see OWN Table 10.1),

$$\begin{aligned} X(z) &= \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} - \frac{2z^{-1}}{\left(1-2z^{-1}\right)^2} \\ &= -\frac{3}{2}\frac{z(z+1)(z-1)}{(z-\frac{1}{2})^2(z-2)^2} \end{aligned}$$

with ROC $\frac{1}{2} < |z| < 2$. Since the ROC includes the unit circle, the Fourier transform of $x[n]$ exists.



OWN 10.22 (d)

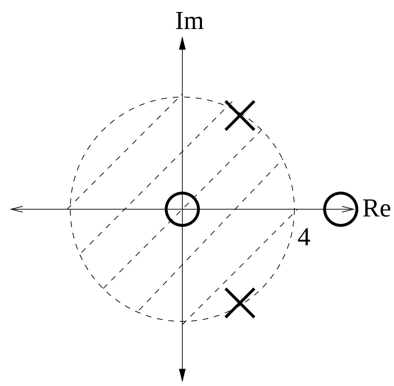
We can rewrite $x[n]$ as

$$\begin{aligned} x[n] &= 4^n \cos(\frac{2\pi}{6}n + \frac{\pi}{4})u[-n-1] \\ &= \frac{1}{2}\left(e^{j\left(\frac{2\pi}{6}n+\frac{\pi}{4}\right)} + e^{-j\left(\frac{2\pi}{6}n+\frac{\pi}{4}\right)}\right)4^n u[-n-1] \\ &= \frac{1}{2}e^{j\frac{\pi}{4}}e^{j\left(\frac{2\pi}{6}n\right)}4^n u[-n-1] + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j\left(\frac{2\pi}{6}n\right)}4^n u[-n-1]. \end{aligned}$$

In OWN Table 10.2, we find the z-transform of $y[n] = 4^n u[-n-1]$ is $Y(z) = -\frac{1}{1-4z^{-1}}$, with ROC $|z| < 4$. The scaling in the z-domain property, given in OWN Table 10.1, states that the z-transform of $e^{j\omega_0 n}y[n]$ is $Y(e^{-j\omega_0}z)$. Therefore, by linearity, the z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= -\frac{\frac{1}{2}e^{j\pi/4}}{1-4e^{j2\pi/6}z^{-1}} - \frac{\frac{1}{2}e^{-j\pi/4}}{1-4e^{-j2\pi/6}z^{-1}} \\ &= -\frac{z(\cos(\frac{\pi}{4})z - 4\cos(\frac{\pi}{12}))}{(z-4e^{j2\pi/6})(z-4e^{-j2\pi/6})} \end{aligned}$$

with ROC $|z| < 4$. Since the ROC includes the unit circle, the Fourier transform of $x[n]$ exists.



Problem 2 *(Inverse z-Transform)*

(a)

OWN 10.23 (i)

By partial fraction expansion, we rewrite $X(z)$ as

$$\begin{aligned} X(z) &= \frac{1-z^{-1}}{1-\frac{1}{4}z^{-2}} \\ &= \frac{1-z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} \\ &= \frac{-\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{1+\frac{1}{2}z^{-1}} \end{aligned}$$

In OWN Table 10.2, we find the inverse z-transform of $\frac{1}{1-\alpha z^{-1}}$ with ROC $|z| > |\alpha|$ is $\alpha^n u[n]$. Thus by linearity,

$$x[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{3}{2}\left(-\frac{1}{2}\right)^n u[n].$$

OWN 10.23 (ii)

Again, we rewrite $X(z)$ using partial fraction expansion.

$$\begin{aligned} X(z) &= \frac{1-z^{-1}}{1-\frac{1}{4}z^{-2}} \\ &= \frac{-\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{1+\frac{1}{2}z^{-1}} \end{aligned}$$

Now we find in OWN Table 10.2, that the inverse z-transform of $\frac{1}{1-\alpha z^{-1}}$ with ROC $|z| < |\alpha|$ is $-\alpha^n u[-n-1]$. Thus by linearity,

$$x[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[-n-1] - \frac{3}{2}\left(-\frac{1}{2}\right)^n u[-n-1].$$

(b)

OWN 10.26 (a)

$$\begin{aligned} X(z) &= \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \\ &= \frac{z^2}{\left(z-\frac{1}{2}\right)(z-1)} \end{aligned}$$

OWN 10.26 (b)

$$\begin{aligned} X(z) &= z^2\left(\frac{-2}{z-\frac{1}{2}} + \frac{2}{z-1}\right) \\ &= 2z\left(-\frac{z}{z-\frac{1}{2}} + \frac{z}{z-1}\right) \end{aligned}$$

OWN 10.26 (c)

Since $x[n]$ is left-sided, the ROC of $X(z)$ is $|z| < \frac{1}{2}$. In OWN Table 10.2, we find the inverse z-transform of $Y(z) = \frac{1}{1-\alpha z^{-1}}$ with ROC $|z| < |\alpha|$ is $y[n] = -\alpha^n u[-n-1]$. By the time shifting property, in OWN Table 10.1, the inverse z-transform of $zY(z)$ is $y[n+1]$. Thus by linearity, the inverse z-transform of $X(z)$ is

$$x[n] = 2\left(\frac{1}{2}\right)^{n+1} u[-n-2] - 2u[-n-2].$$

Problem 3 *(Properties of the z-Transform)*

OWN 10.44 (a)

By the time shifting and linearity properties in OWN Table 10.1, the z-transform of $x_a[n] = x[n] - x[n-1]$ is

$$X_a(z) = X(z) - z^{-1}X(z) = \frac{z-1}{z}X(z)$$

with ROC R with the possible deletion of $z = 0$.

OWN 10.44 (b)

We can find the z-transform of

$$x_b[n] = \begin{cases} x[\frac{n}{2}] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

by using the time expansion property in OWN Table 10.1, as

$$X_b(z) = X(z^2) \quad \text{with ROC} \quad R^{1/2} = \{z : z^2 \in R\}.$$

Alternatively, we can find the z-transform by evaluating the definition

$$\begin{aligned} X_b(z) &= \sum_{n=-\infty}^{\infty} x_b[n] z^{-n} \\ &= \sum_{n \text{ even}} x[\frac{n}{2}] z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m] z^{-2m} \\ &= X(z^2). \end{aligned}$$

OWN 10.44 (c)

Define

$$g[n] = \frac{1}{2}(x[n] + (-1)^n x[n]).$$

Observe that $g[2n] = x[2n]$, and that $g[n] = 0$ for n odd. By the scaling in the z-domain property and the linearity property in OWN Table 10.1, the z-transform of $g[n]$ is $G(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$, with ROC R. Now we find the z-transform of $x_c[n] = x[2n]$ by evaluating the definition of the z-transform,

$$\begin{aligned} X_c(z) &= \sum_{n=-\infty}^{\infty} x_c[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[2n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} g[2n] z^{-n} \\ &= \sum_{m \text{ even}} g[m] z^{-m/2} \\ &= \sum_{m=-\infty}^{\infty} g[m] z^{-m/2} \\ &= G(z^{1/2}) \\ &= \frac{1}{2} X(z^{1/2}) + \frac{1}{2} X(-z^{1/2}) \end{aligned}$$

and the ROC is R.

Problem 4 *(Properties of the z-Transform: minimum-phase system.)*

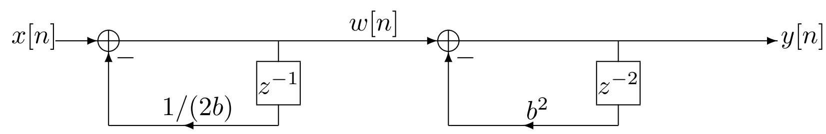
OWN 10.58

Consider a causal and stable system with system function $H(z)$. Let its inverse system have the system function $H_i(z)$. The poles of $H(z)$ are the zeros of $H_i(z)$ and the zeros of $H(z)$ are the poles of $H_i(z)$.

For $H(z)$ to correspond to a causal and stable system, all of its poles must be within the unit circle. This means all the zeros of $H_i(z)$ must be within the unit circle. Similarly, for $H_i(z)$ to correspond to a causal and stable system, all of its poles must be within the unit circle. That means all the zeros of $H(z)$ must be within the unit circle.

Therefore, all poles and zeros of a minimum-phase system must lie within the unit circle.

Problem 5 *(Discrete-time LTI system analysis.)*



To find the overall transfer function $H(z)$ of this causal LTI system, we examine the input-output relationships of the sub-systems.

$$
\begin{aligned}
W(z) &= X(z) - \frac{1}{2b} z^{-1} W(z) \\
W(z)\left(1 + \frac{1}{2b} z^{-1}\right) &= X(z) \\
\frac{W(z)}{X(z)} &= \frac{1}{1 + \frac{1}{2b} z^{-1}} \\
Y(z) &= W(z) - b^2 z^{-2} Y(z) \\
Y(z)\left(1 + b^2 z^{-2}\right) &= W(z) \\
\frac{Y(z)}{W(z)} &= \frac{1}{1 + b^2 z^{-2}} \\
H(z) &= \frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)} \\
&= \frac{1}{\left(1 + \frac{1}{2b} z^{-1}\right)\left(1 + b^2 z^{-2}\right)} \\
&= \frac{z^3}{\left(z + \frac{1}{2b}\right)(z + bj)(z - bj)}
\end{aligned}
$$

$H(z)$ has three zeros located at $z = 0$, and three poles located at $z = -\frac{1}{2b}, -bj, bj$ respectively. We know the system is causal and rational, which implies the ROC must be outside of the pole with the largest magnitude. Since the system is stable iff the ROC includes the unit circle, all the poles of $H(z)$ must be inside the unit circle.

$$
\begin{aligned}
|bj| < 1 &\Leftrightarrow |b| < 1 \\
|-bj| < 1 &\Leftrightarrow |b| < 1 \\
\left|-\frac{1}{2b}\right| < 1 &\Leftrightarrow \frac{1}{2} < |b|
\end{aligned}
$$

Therefore $H(z)$ is stable iff

$$\frac{1}{2} < |b| < 1.$$

Problem 6 *(Discrete-time LTI system.)*

OWN Problem 10.34

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

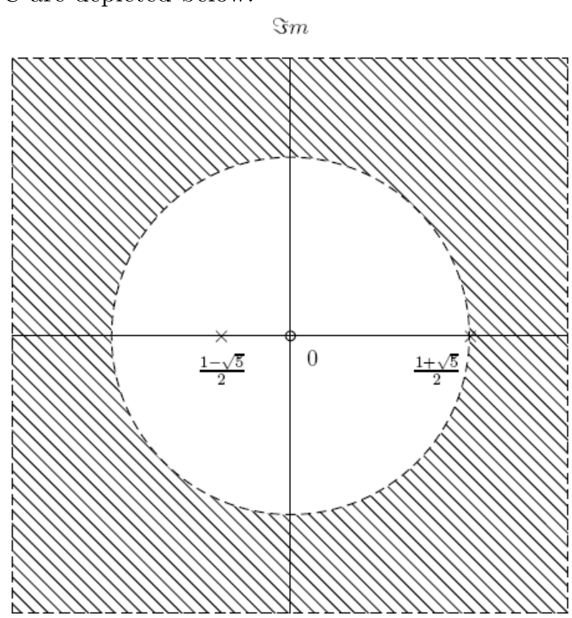
Taking the z-transform of this equation, we get

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - z^{-2}} = \frac{z}{z^2 - z - 1} \\ &= \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)} \end{aligned}$$

Therefore $H(z)$ has a zero at $z = 0$ and poles at $z = \frac{1 \pm \sqrt{5}}{2}$.

Since the system is causal, the ROC of $H(z)$ will be outside the circle containing its outermost pole. The pole-zero map and ROC are depicted below.



(b)

The partial fraction expansion of $H(z)$ is

$$H(z) = -\frac{1/\sqrt{5}}{1 - \frac{1+\sqrt{5}}{2}z^{-1}} + \frac{1/\sqrt{5}}{1 - \frac{1-\sqrt{5}}{2}z^{-1}}.$$

Therefore

$$h[n] = -\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

(c)

The system is unstable, as its ROC does not contain the unit circle. The instability is also apparent in $h[n]$, as the $-\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n]$ term will grown indefinitely as $n \to \infty$.

To make the system stable, the ROC must contain the unit circle. The ROC should then be: $\frac{\sqrt{5}-1}{2} < |z| < \frac{\sqrt{5}+1}{2}$. In this case, we get

$$h[n] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n u[-n-1] + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

Problem 7 *(Discrete-time LTI system analysis.)*

OWN Problem 10.47

(a)

From Clue 1, we know that $H(-2) = 0$. From Clue 2, we know that when

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad , \qquad |z| > \frac{1}{2}$$

then

$$Y(z) = 1 + \frac{a}{1-\frac{1}{4}z^{-1}} = \frac{1-\frac{1}{4}z^{-1}+a}{1-\frac{1}{4}z^{-1}} \quad , \qquad |z| > \frac{1}{4}$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1-\frac{1}{4}z^{-1}+a\right)\left(1-\frac{1}{2}z^{-1}\right)}{1-\frac{1}{4}z^{-1}} \quad , \qquad |z| > \frac{1}{4}$$

Substituting $z = -2$ into this equation, and using the fact that $H(-2) = 0$, we find that

$$a = -\frac{9}{8}$$

(b)

The response to the signal $x[n] = 1 = 1^n$ will be $y[n] = H(1)x[n]$.

$$y[n] = H(1) = -\frac{1}{4}$$

Problem 8 *(Unilateral Z-Transform.)*

OWN Problem 10.42 (b)

Taking the unilateral z-transform of both sides of the difference equation,

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) - \frac{1}{2}y[-1] = \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z).$$

To find the zero-input response, set $X(z) = 0$ and we get $\mathcal{Y}_{zi}(z) = 0$. Taking the inverse unilateral z-transform gives the zero-input response $y_{ni}[n] = 0$.

Now, since it is given that $x[n] = u[n]$, we have $\mathcal{X}(z) = \frac{1}{1-z^{-1}}$, with ROC $|z| > 1$.

To find the zero-state response, set $y[-1] = 0$ and we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) = \frac{1}{1-z^{-1}} - \frac{\frac{1}{2}z^{-1}}{1-z^{-1}}.$$

Therefore $\mathcal{Y}(z) = \frac{1}{1-z^{-1}}$.

The inverse unilateral z-transform gives the zero-state response $y_{zs}[n] = u[n]$.

OWN Problem 10.42 (c)

As in (b), taking the unilateral z-transform of both sides of the difference equation, we get

$$\mathcal{Y}(z) - \frac{1}{2}z^{-1}\mathcal{Y}(z) - \frac{1}{2}y[-1] = \mathcal{X}(z) - \frac{1}{2}z^{-1}\mathcal{X}(z).$$

However, given the different initial state, we get the following when we try to find the zero-input response by setting $\mathcal{X}(z) = 0$:

$$\mathcal{Y}(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}}.$$

The inverse unilateral z-transform then gives the zero-input response $y_{zi}[n] = \left(\frac{1}{2}\right)^{n+1} u[n]$.

Since the input $x[n]$ is the same as the one used in part (b), the zero-state response is still $y_{zs}[n] = u[n]$.

Comments: different initial conditions will result in different zero-input responses even though the differential equations are the same.

Problem 9 *(Fibonacci Numbers.)*

(a)

We are given that $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n \;\; \forall n \geq 1$.

We would like to find an explicit formula for F_n. We can use the unilateral Z-Transform to find it. One consequence of this method is that our function will be $0 \forall n < 0$. We would like a function that can generate F_1 onwards so let's set $x[n] = F_{n+1}$ so that $x[0] = F_1$.

We then have

$$x[n+1] = x[n] + x[n-1].$$

Taking unilateral Z-Transform we get

$$zX(z) - zx[0] = X[z] + z^{-1}X(z) + x[-1].$$

Note that $x[-1] = F_0 = 0$ since $F_0 + F_1 = F_2$ and $F_1 = F_2 = 1$. Plugging in the values for $x[-1]$ and $x[0]$, we get

$$zX(z) - z = X[z] + z^{-1}X(z).$$

$$\therefore X(z) = \frac{z}{z - 1 - z^{-1}} = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{\left(1 - \frac{1-\sqrt{5}}{2}z^{-1}\right)\left(1 - \frac{1+\sqrt{5}}{2}z^{-1}\right)} = \frac{A}{1 - \frac{1+\sqrt{5}}{2}z^{-1}} + \frac{B}{1 - \frac{1-\sqrt{5}}{2}z^{-1}}.$$

Solving for A and B, we get $A = \frac{5+\sqrt{5}}{10}$ and $B = \frac{5-\sqrt{5}}{10}$.

$$\therefore X(z) = \frac{5+\sqrt{5}}{10} \cdot \frac{1}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} + \frac{5-\sqrt{5}}{10} \cdot \frac{1}{1 - \frac{1-\sqrt{5}}{2} z^{-1}}$$

Taking unilateral inverse Z-Transform, we get

$$x[n] = \frac{5+\sqrt{5}}{10} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n u[n] + \frac{5-\sqrt{5}}{10} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n u[n].$$

By definition, $F_n = x[n-1]$.

$$\therefore F_n = \frac{5+\sqrt{5}}{10} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{5-\sqrt{5}}{10} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \quad \text{for } n \geq 1.$$

Note that there are many different possible formulas for $x[n]$ (thus also for F_n), depending on what we choose for our "starting point" at $n = 0$.

(b)

Using the formula from above, $F_{21} = 10946$.

Problem 10 *(Root locus of discrete-time systems.)*

$$T(z) = \frac{\frac{1}{z+2}}{1 + K\frac{1}{z+2}\frac{1}{z}} = \frac{z}{z^2 + 2z + K}.$$

The poles of the system are therefore the roots of $z^2 + 2z + K = 0$.

- If $K \leq 1$, the poles are at $-1 \pm \sqrt{1-K}$.
- If $K \geq 1$, the poles are at $-1 \pm j\sqrt{K-1}$.

Now let's first look at the case when $K < 0$.

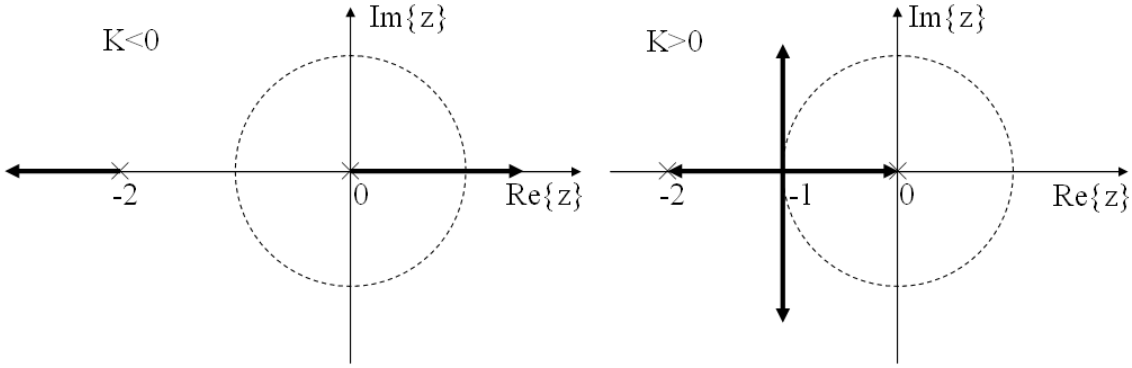
If $K < 0$, the poles are at $-1 \pm \sqrt{1-K}$.

- As K varies from 0 to $-\infty$, $-1 + \sqrt{1-K}$ will vary from $-1 + \sqrt{1-K}|_{K=0} = 0$ to $-1 + \sqrt{1-K}|_{K=-\infty} = +\infty$.
- As K varies from 0 to $-\infty$, $-1 - \sqrt{1-K}$ will vary from $-1 - \sqrt{1-K}|_{K=0} = -2$ to $-1 - \sqrt{1-K}|_{K=-\infty} = -\infty$.

If $K > 0$, there are two cases:

1. $0 < K < 1$: the poles are at $-1 \pm \sqrt{1-K}$.
   - As K varies from 0 to 1, $-1 + \sqrt{1-K}$ will vary from $-1 + \sqrt{1-K}|_{K=0} = 0$ to $-1 + \sqrt{1-K}|_{K=1} = -1$.
   - As K varies from 0 to 1, $-1 - \sqrt{1-K}$ will vary from $-1 - \sqrt{1-K}|_{K=0} = -2$ to $-1 - \sqrt{1-K}|_{K=1} = -1$.
2. $K > 1$: the poles are at $-1 \pm j\sqrt{K-1}$.

The root locus is the following:



The system is stable if both poles are inside unit circle. In this case, there are no values of K such that both poles are inside the unit circle.

Below is the answer to OWN Problem 11.25 (a) in case you are interested.

The system given is

$$G(z)H(z) = \frac{z-1}{z^2 - \frac{1}{4}} = \frac{z-1}{(z+\frac{1}{2})(z-\frac{1}{2})}.$$

Therefore the system has poles at $\pm\frac{1}{2}$ and a zero at 1.

The poles of the overall system satisfy $G(z)H(z) = -\frac{1}{K}$, which means solving for

$$\frac{z-1}{(z+\frac{1}{2})(z-\frac{1}{2})} = -\frac{1}{K}.$$

This can be rewritten as

$$z^2 + Kz - (K + \frac{1}{4}) = 0. \tag{1}$$

The roots are $\frac{-K\pm\sqrt{K^2+4K+1}}{2}$ if $K^2+4K+1 \geq 0$ and $\frac{-K\pm j\sqrt{-K^2-4K-1}}{2}$ if $K^2+4K+1 \leq 0$.

Solving $K^2+4K+1=0$, we know that $K^2+4K+1 \geq 0$ if $K \geq -2+\sqrt{3}$ or $K \leq -2-\sqrt{3}$ and $K^2+4K+1 \leq 0$ if $-2-\sqrt{3} \leq K \leq -2+\sqrt{3}$.

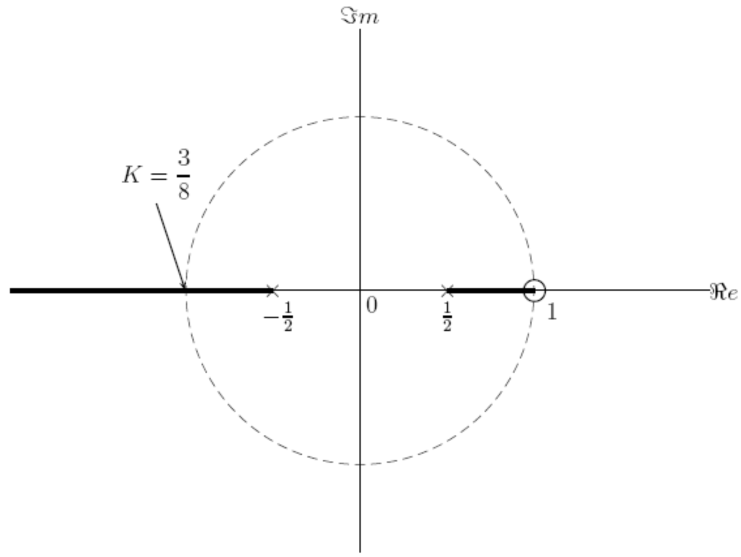
Therefore the roots are $\frac{-K\pm\sqrt{K^2+4K+1}}{2}$ if $K \geq -2+\sqrt{3}$ or $K \leq -2-\sqrt{3}$ and $\frac{-K\pm j\sqrt{-K^2-4K-1}}{2}$ if $-2-\sqrt{3} \leq K \leq -2+\sqrt{3}$.

Let's first look at when $K > 0$.

We see that $K^2+4K+1 > 0$ if $K > 0$. Therefore the roots for Equation (1) are just $\frac{-K\pm\sqrt{K^2+4K+1}}{2}$.

- As K varies from 0 to $+\infty$, $\frac{-K-\sqrt{K^2+4K+1}}{2}$ varies from $-\frac{1}{2}$ (evaluated at $K=0$ to $-\infty$ (evaluated at $K=+\infty$).
- As K varies from 0 to $+\infty$, $\frac{-K+\sqrt{K^2+4K+1}}{2}$ varies from $+\frac{1}{2}$ (evaluated at $K=0$ to 1 (evaluated at $K=+\infty$).

The root locus for $K>0$ is shown below.

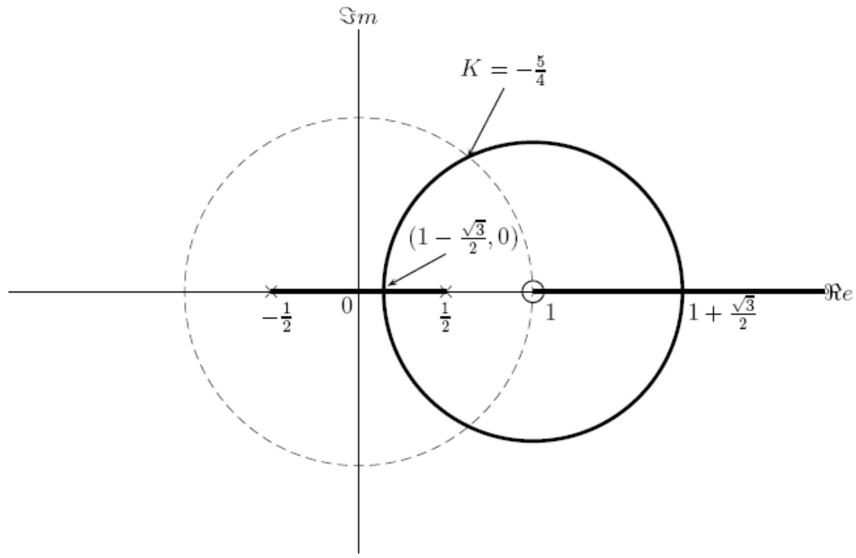


Let's now look at the when $K < 0$.

If $K < 0$, we have to consider several cases.

1. When $-2+\sqrt{3} < K < 0$: the roots for Equation (1) are $\frac{-K\pm\sqrt{K^2+4K+1}}{2}$ because $K^2+4K+1 > 0$.
    - As K varies from 0 to $-2+\sqrt{3}$, $\frac{-K+\sqrt{K^2+4K+1}}{2}$ varies from $\frac{1}{2}$ (evaluated at $K=0$) to $1-\frac{\sqrt{3}}{2}$ (evaluated at $K=-2+\sqrt{3}$).
    - As K varies from 0 to $-2+\sqrt{3}$, $\frac{-K-\sqrt{K^2+4K+1}}{2}$ varies from $-\frac{1}{2}$ (evaluated at $K=0$) to $1-\frac{\sqrt{3}}{2}$ (evaluated at $K=-2+\sqrt{3}$).
2. When $K < -2-\sqrt{3}$: the roots for Equation (1) are again $\frac{-K\pm\sqrt{K^2+4K+1}}{2}$ because $K^2+4K+1 > 0$.
    - As K varies from $-2-\sqrt{3}$ to $-\infty$, $\frac{-K+\sqrt{K^2+4K+1}}{2}$ varies from $1+\frac{\sqrt{3}}{2}$ (evaluated at $K=-2-\sqrt{3}$) to $+\infty$ (evaluated at $K=-\infty$).
    - As K varies from $-2-\sqrt{3}$ to $-\infty$, $\frac{-K-\sqrt{K^2+4K+1}}{2}$ varies from $1+\frac{\sqrt{3}}{2}$ (evaluated at $K=-2-\sqrt{3}$) to 1 (evaluated at $K=-\infty$).
3. When $-2-\sqrt{3} < K < -2+\sqrt{3}$: the roots for Equation (1) are now $\frac{-K\pm j\sqrt{-K^2-4K-1}}{2}$ because $K^2+4K+1 < 0$. This is a circle of radius $\sqrt{3}$ centered at 1. (This can be seen through showing $\|\frac{-K\pm j\sqrt{-K^2-4K-1}}{2} - 1\|^2 = 3$.)

Combining these cases, the root locus for $K < 0$ is shown below.



Problem 11 *(Pole/Zero plots.)*

(a) Pole-zero plot (a) $\Leftrightarrow$ magnitude response (5).

A pole at $z = a$ and a zero at $z = 1/a$ will cancel each other's effects on the magnitude response, resulting in a constant magnitude response.

(b) Pole-zero plot (b) $\Leftrightarrow$ magnitude response (1).

Magnitude response must have two symmetric peaks around $\omega = \pi/2$ and $\omega = -\pi/2$.

(c) Pole-zero plot (c) $\Leftrightarrow$ magnitude response (3).

Magnitude response must peak near $\omega = 0$. Although we might expect two peaks corresponding to two poles, if the poles are sufficiently close their peaks will merge.

(d) Pole-zero plot (d) $\Leftrightarrow$ magnitude response (4).

Magnitude response must approach but not reach 0 at $\omega = 0$.

(e) Pole-zero plot (e) $\Leftrightarrow$ magnitude response (2).

Magnitude response must hit 0 at $\omega = 0$.