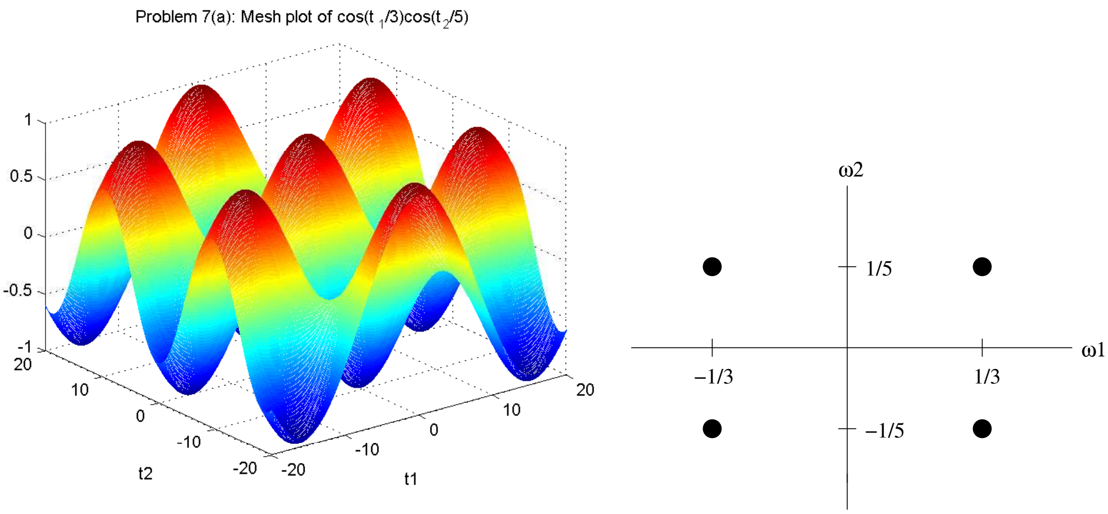


(a) (b)

Figure 1: Problem 12 (a)

Problem 12 *(Image Processing)*

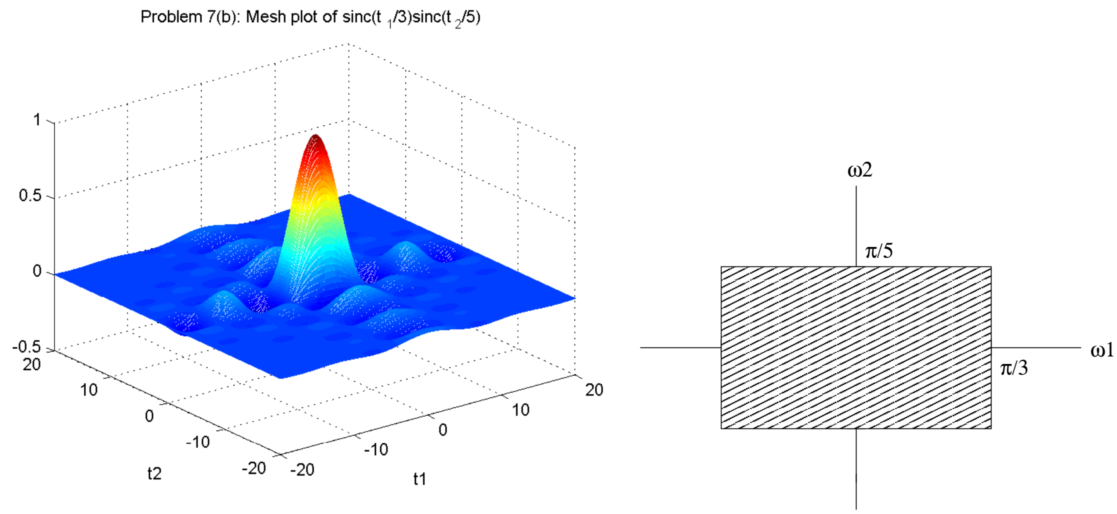
(a)

$$\begin{aligned}
X(\omega_1,\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(t_1/3)\cos(t_2/5)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1dt_2\\
&= \int_{-\infty}^{\infty}\cos(t_2/5)e^{-j\omega_2 t_2}\left(\int_{-\infty}^{\infty}\cos(t_1/3)e^{-j\omega_1 t_1}dt_1\right)dt_2\\
&= \left(\int_{-\infty}^{\infty}\cos(t_1/3)e^{-j\omega_1 t_1}dt_1\right)\left(\int_{-\infty}^{\infty}\cos(t_2/5)e^{-j\omega_2 t_2}dt_2\right)\\
&= \pi\left[\delta(\omega_1-1/3)+\delta(\omega_1+1/3)\right]\cdot\pi\left[\delta(\omega_2-1/5)+\delta(\omega_2+1/5)\right]\\
&= \pi^2\left[\delta(\omega_1-1/3)\delta(\omega_2-1/5)+\delta(\omega_1-1/3)\delta(\omega_2+1/5)+\delta(\omega_1+1/3)\delta(\omega_2-1/5)+\right.\\
&\quad\left.\delta(\omega_1+1/3)\delta(\omega_2+1/5)\right]
\end{aligned}$$

Looking at this equation, we see immediately that $X(\omega_1,\omega_2)$ contains four impulses in the $\omega_1-\omega_2$ plane, and is equal to 0 everywhere else. This is plotted in Figure 1 (b).

(b)

$$\begin{aligned}
X(\omega_1,\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}sinc(t_1/3)sinc(t_2/5)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1dt_2\\
&= \int_{-\infty}^{\infty}sinc(t_2/5)e^{-j\omega_2 t_2}\left(\int_{-\infty}^{\infty}sinc(t_1/3)e^{-j\omega_1 t_1}dt_1\right)dt_2\\
&= \left(\int_{-\infty}^{\infty}sinc(t_1/3)e^{-j\omega_1 t_1}dt_1\right)\left(\int_{-\infty}^{\infty}sinc(t_2/5)e^{-j\omega_2 t_2}dt_2\right)\\
&= \left(\int_{-\infty}^{\infty}3\frac{\sin(t_1\cdot\pi/3)}{\pi t_1}e^{-j\omega_1 t_1}dt_1\right)\left(\int_{-\infty}^{\infty}5\frac{\sin(t_2\cdot\pi/5)}{\pi t_2}e^{-j\omega_2 t_2}dt_2\right)\\
&= \begin{cases}15 & |\omega_1|<\pi/3 \quad and \quad |\omega_2|<\pi/5\\ 0 & else\end{cases}
\end{aligned}$$



(a) (b)

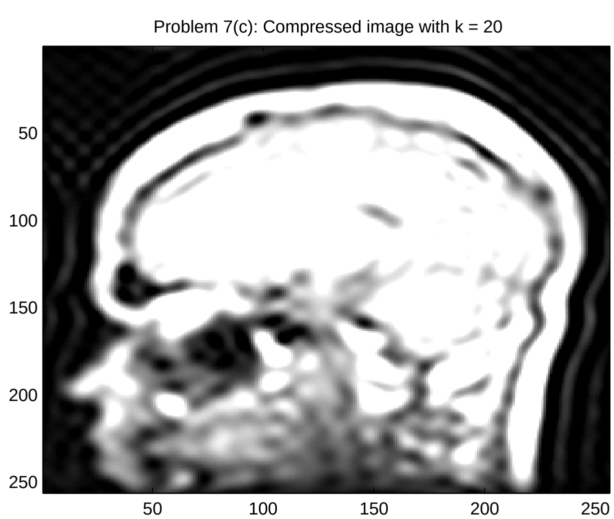
Figure 2: Problem 12 (b)

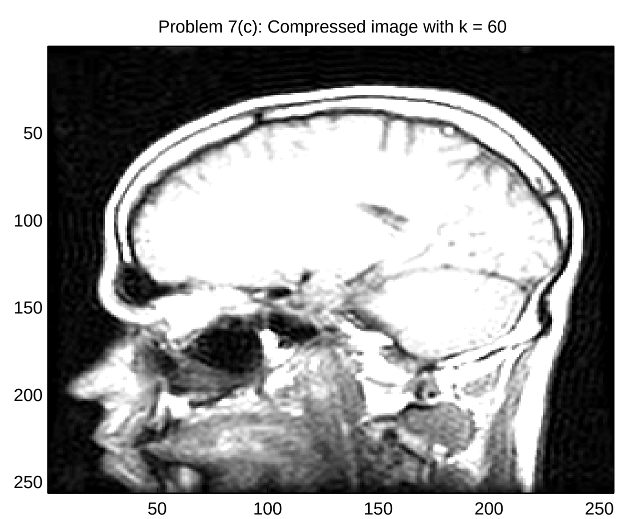
We see that $X(\omega_1, \omega_2)$ is equal to 15 on a rectangular shaped region in the $\omega_1 - \omega_2$ plane, and is equal to 0 everywhere else. This is plotted in the Figure 2 (b). The low frequencies are near the origin, where ω_1 and ω_2 are both close to 0.

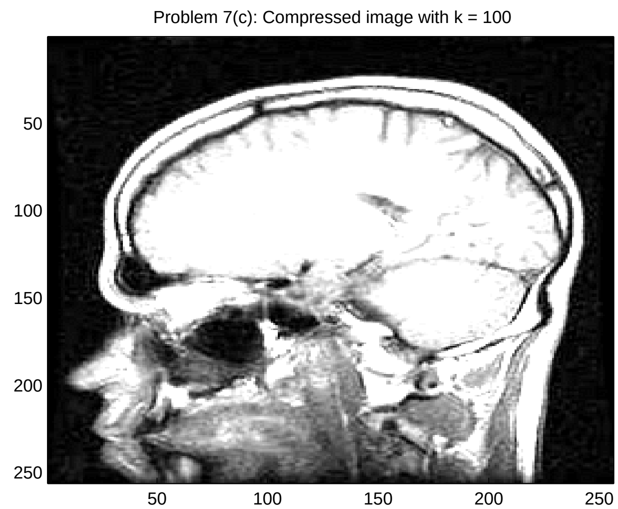
(c)

Three plots of the compressed image are shown below, corresponding to values of k equal to 20, 60, and 100. In the spatial domain, we see that the compression algorithm blurs the sharp edges in the image and smooths out finally textured regions. For very small values of k, the algorithm introduces a 'ringing' artifact around the edge of the skull.

In the Fourier plane, the algorithm sets all of the Fourier coefficients outside of a square region to zero. Thus, the algorithm preserves the Fourier coefficients corresponding to low frequencies, but sets the high frequency coefficients to zero. The compressed image can be stored in a smaller file because the file must only contain the non-zero Fourier coefficients (plus a short header to store the value of k).







(d)

A plot of the noisy image and the denoised image, with $k = 80$ chosen as the optimal cut-off, are shown below. (Note to graders: since this is based on a highly subjective evaluation, any reasonable value of the optimal cut-off is acceptable, as long as the student includes a plot and argues that this cut-off leads to the best looking image.)

When the cut-off frequency is large, the image appears very sharp, but there is a significant amount of noise visible. When the cut-off frequency is small, more noise is removed, but the image becomes blurry and distorted. Since the noise contains both large and small frequencies, we cannot completely remove the noise and preserve the image quality. We must find a balance between denoising and preserving the image.