
Homework 1
Due: Thursday, September 7, 2006, at 5pm

Problem 1 (*Complex numbers.*)

(a)

$$\begin{aligned} |z| &= |ae^{j\alpha}| = |a||e^{j\alpha}| = |a| \\ \arg(z) &= \alpha \text{ mod } 2\pi \\ \operatorname{Re}\{z\} &= \operatorname{Re}\{ae^{j\alpha}\} = \operatorname{Re}\{a \cos(\alpha) + ja \sin(\alpha)\} = a \cos(\alpha) \\ \operatorname{Im}\{z\} &= a \sin(\alpha) \\ z^* &= a \cos(\alpha) - ja \sin(\alpha) = ae^{-j\alpha} \end{aligned}$$

(b)

Cartesian: $e^{j5\pi} = \cos(5\pi) + j \sin(5\pi) = -1$

Polar: $e^{j5\pi} = e^{j\pi}$

Cartesian: $\frac{1}{3}e^{j\pi/4} = \frac{1}{3}(\cos(\pi/4) + j \sin(\pi/4)) = \frac{\sqrt{2}}{6} + j \frac{\sqrt{2}}{6}$

Cartesian: $(\frac{1}{3}e^{j\pi/4})^* = \frac{\sqrt{2}}{6} - j \frac{\sqrt{2}}{6}$

Polar: $(\frac{1}{3}e^{j\pi/4})^* = \frac{1}{3}e^{j(-\pi/4)}$

Cartesian: $j^{-j} = (e^{j\pi/2})^{-j} = e^{\pi/2}$

Polar: $j^{-j} = e^{\pi/2} = e^{\pi/2} \cdot e^{j0}$

(c) See plots in Figure 1.

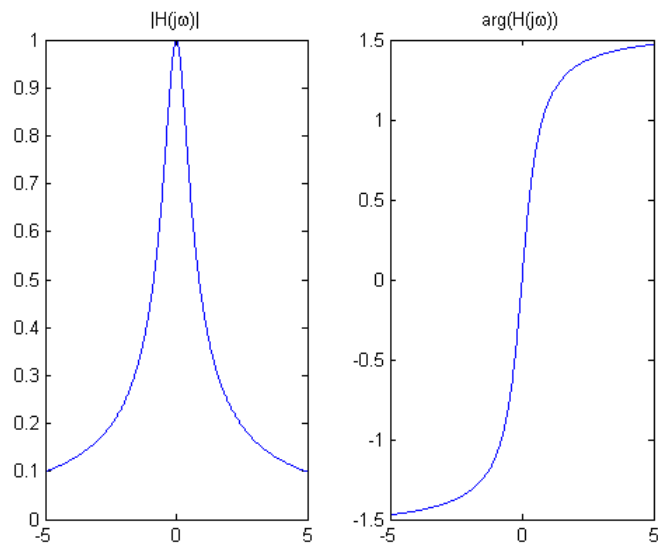


Figure 1: Problem 1(c).

Problem 2 (*Elementary functions and their graphs.*)

(a)

$$\begin{aligned} \operatorname{Re}\{y(t)\} &= \cos(2\pi t) \\ \operatorname{Im}\{y(t)\} &= \sin(2\pi t) \\ |y(t)| &= 1 \\ \arg(y(t)) &= 2\pi(t - [t]) \end{aligned}$$

See plots in Figure 2.

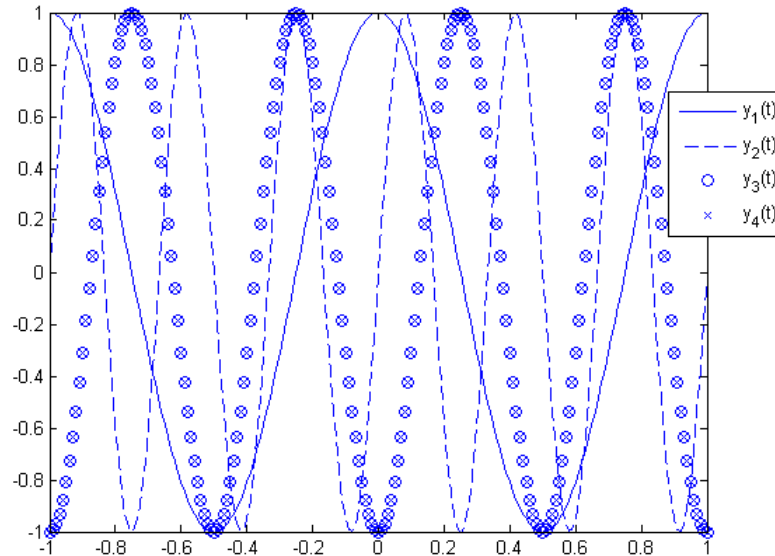


Figure 2: Problem 2(a).

(b)

$\cos(\frac{\pi}{3}t)$ has a period of $\frac{2\pi}{\pi/3} = 6$.

$\cos(\frac{2\pi}{5}t)$ has a period of $\frac{2\pi}{2\pi/5} = 5$.

The fundamental period is 30, the least common multiple of 6 and 5.

(c)

$w(t) = \cos(\frac{\pi}{3}t) + \cos(t)$ is aperiodic.

$\cos(\frac{\pi}{3}t)$ has a period of $\frac{2\pi}{\pi/3} = 6$ and $\cos(t)$ has a period of 2π . Since one is rational and one is irrational, they have no common multiples.

Problem 3 (*Transformations of functions.*)

See Figure 3.

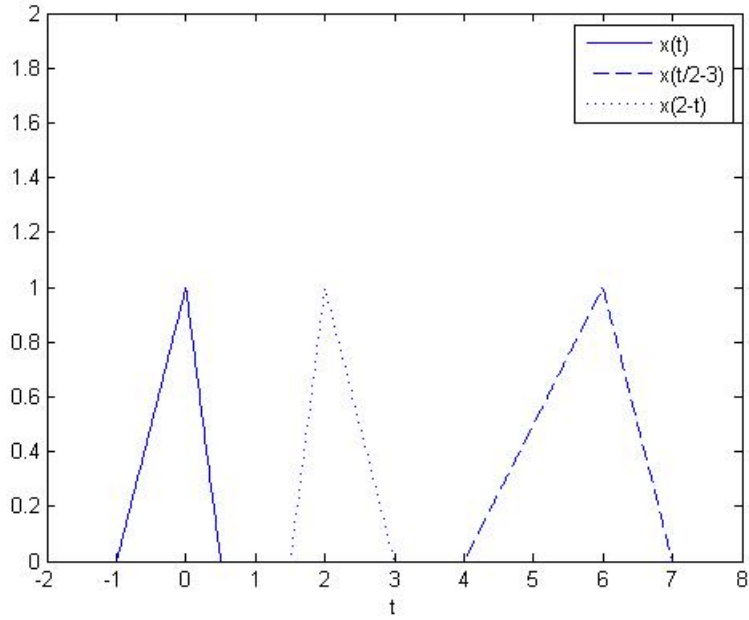


Figure 3: Problem 3.

Problem 4 (*Periodic discrete-time signals.*)

(a)

$y_1[n] = \cos(\frac{\pi}{3}n)$ has a period of 6.

$y_2[n] = \cos(\frac{3\pi}{5}n)$ has a period of 10.

See plots in Figure 4.

(b)

The fundamental period is 30, the least common multiple of 6 and 10.

See plots in Figure 4.

(c)

$z[n] = \cos(2n)$ is aperiodic. There is no integer n such that $2n$ is divisible by 2π .

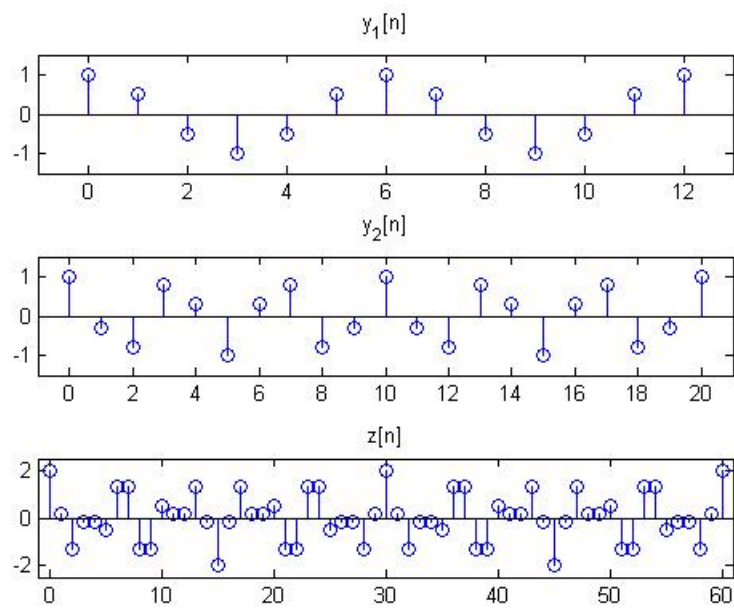


Figure 4: Problem 4.

Problem 5 (*Properties of systems.*)

Let $f(t) \triangleq k_1x_1(t) + k_2x_2(t)$ where k_1, k_2 are constants.

Let $g(t) \triangleq x(t - \tau)$ where τ is a constant.

(a) $\mathcal{H}[x(t)] = x(at + b)$ is linear and not time-invariant.

$$\begin{aligned}\mathcal{H}[f(t)] &= f(at + b) = k_1x_1(at + b) + k_2x_2(at + b) \\ &= k_1\mathcal{H}[x_1(t)] + k_2\mathcal{H}[x_2(t)] \\ \mathcal{H}[g(t)] &= g(at + b) = x(at + b - \tau) \\ &\neq x(a(t - \tau) + b)\end{aligned}$$

The last line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $x(at + b)$, by τ .

(b) $\mathcal{H}[x(t)] = x(at^2 + b)$ is linear and not time-invariant.

$$\begin{aligned}\mathcal{H}[f(t)] &= f(at^2 + b) = k_1x_1(at^2 + b) + k_2x_2(at^2 + b) \\ &= k_1\mathcal{H}[x_1(t)] + k_2\mathcal{H}[x_2(t)] \\ \mathcal{H}[g(t)] &= g(at^2 + b) = x(at^2 + b - \tau) \\ &\neq x(a(t - \tau)^2 + b)\end{aligned}$$

The last line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $x(at^2 + b)$, by τ .

(c) $\mathcal{H}[x(t)] = x(at) + b$ is not linear and not time-invariant.

$$\begin{aligned}\mathcal{H}[f(t)] &= f(at) + b = k_1x_1(at) + k_2x_2(at) + b \\ &\neq k_1\mathcal{H}[x_1(t)] + k_2\mathcal{H}[x_2(t)] = k_1x_1(at) + k_1b + k_2x_2(at) + k_2b \\ \mathcal{H}[g(t)] &= g(at) + b = x(at - \tau) + b \\ &\neq x(a(t - \tau)) + b\end{aligned}$$

The last line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $x(at) + b$, by τ .

Problem 6 (*Properties of continuous time systems.*)

(a) $y(t) = x(t)\frac{e^t + e^{-t}}{2}$

(i) memoryless (OWN definition): The current output value $y(t)$ is determined by the value of the current input $x(t)$.

not memoryless (LV/Handout 1 definition): The output at time t requires knowledge of the value of t .

(ii) unstable: For the trivially bounded input $x(t) = 1 \forall t$, the output $y(t)$ is unbounded as $t \rightarrow \pm\infty$.

(iii) causal: The current output value $y(t)$ does not depend on future inputs $x(\tau), \tau > t$.

(iv) linear:

Let $y_1(t) = x_1(t)\frac{e^t + e^{-t}}{2}$ and $y_2(t) = x_2(t)\frac{e^t + e^{-t}}{2}$.

If the input to the system is $f(t) = k_1x_1(t) + k_2x_2(t)$, where k_1, k_2 are constants,

then the output is $g(t) = f(t)\frac{e^t + e^{-t}}{2} = (k_1x_1(t) + k_2x_2(t))\frac{e^t + e^{-t}}{2} = k_1y_1(t) + k_2y_2(t)$.

(v) not time-invariant:

If the input is $f(t) = x(t - \tau)$, where τ is a constant,

then the output is $g(t) = f(t)\frac{e^t + e^{-t}}{2} = x(t - \tau)\frac{e^t + e^{-t}}{2}$.

Since $y(t - \tau) = x(t - \tau)\frac{e^{t - \tau} + e^{-(t - \tau)}}{2}$, $g(t) \neq y(t - \tau)$.

$$(b) \quad y(t) = \int_{-\infty}^{t/2} x(3\alpha) d\alpha = \frac{1}{3} \int_{-\infty}^{3t/2} x(\beta) d\beta$$

- (i) not memoryless: The current output $y(t)$ uses many past values of the input $x(\tau), \tau < t$.
- (ii) unstable: $y(t) = \infty \quad \forall t$ if $x(t) = u(-t)$.
- (iii) not causal: For $t > 0$, the current output value $y(t)$ depends on future inputs $x(\tau), \tau > t$.
- (iv) linear: Integration is linear.
- (v) not time-invariant:

If the input is $f(t) = x(t - \tau)$, where τ is a constant,

$$\text{then the output is } g(t) = \frac{1}{3} \int_{-\infty}^{3t/2} f(\beta) d\beta = \frac{1}{3} \int_{-\infty}^{3t/2} x(\beta - \tau) d\beta = \frac{1}{3} \int_{-\infty}^{3t/2 - \tau} x(\sigma) d\sigma.$$

$$\text{Since } y(t - \tau) = \frac{1}{3} \int_{-\infty}^{3(t-\tau)/2} x(\beta) d\beta, \quad g(t) \neq y(t - \tau).$$

$$(c) \quad y(t) = \text{Im}\{x(t)\}$$

- (i) memoryless: The output at time t_0 depends only on the input at the same time t_0 .
- (ii) stable: If $|x(t)| < \infty$, then $|\text{Im}\{x(t)\}| < \infty$.
- (iii) causal: At each time, the output depends only on the current input.
- (iv) non-linear:

Consider an input $x_1(t) = jt$ which produces the output $y_1(t) = t$
 The input $x_2(t) = jx_1(t) = -t$ produces the output $y_2(t) = 0$, which is not equal to $jy_1(t)$
- (v) time-invariant:

If the input is $f(t) = x(t - \tau)$, where τ is a constant,
 then the output is $g(t) = \text{Im}\{f(t)\} = \text{Im}\{x(t - \tau)\}$.
 Since $y(t - \tau) = \text{Im}\{x(t - \tau)\}$, the system is time-invariant.

Problem 7 (*Properties of discrete time systems.*)

$$(a) \quad y[n] = \sum_{k=-\infty}^n x[k+1] = \sum_{l=-\infty}^{n+1} x[l]$$

- (i) not memoryless: The current output $y[n]$ depends on many past input values $x[\tau], \tau < n$.
- (ii) not stable: $y[n] = \infty \quad \forall n$ if $x[n] = u[-n]$.
- (iii) not causal: The current output $y[n]$ depends on future input $x[n+1]$.
- (iv) linear:

$$\text{Let } y_1[n] = \sum_{l=-\infty}^{n+1} x_1[l] \text{ and } y_2[n] = \sum_{l=-\infty}^{n+1} x_2[l].$$

If the input is $f[n] = k_1 x_1[n] + k_2 x_2[n]$, where k_1, k_2 are constants,

$$\text{then the output is } g[n] = \sum_{l=-\infty}^{n+1} f[l] = \sum_{l=-\infty}^{n+1} (k_1 x_1[l] + k_2 x_2[l]) = k_1 y_1[n] + k_2 y_2[n].$$

- (v) time-invariant:

If the input is $f[n] = x[n - \tau]$, where τ is a constant,

$$\text{then the output is } g[n] = \sum_{l=-\infty}^{n+1} f[l] = \sum_{l=-\infty}^{n+1} x[l - \tau] = \sum_{m=-\infty}^{n+1-\tau} x[m].$$

Therefore $y[n - \tau] = g[n]$.

$$(b) \quad y[n] = x[n] \cdot \sum_{k=-\infty}^{\infty} \delta[n - 3k]$$

Observe that $y[n] = x[n]$ if n is a multiple of 3, and $y[n] = 0$ otherwise.

- (i) memoryless (OWN definition): The current output $y[n]$ does not depend on past or future inputs $x[n]$.
- not memoryless (LV/Handout 1 definition): The output at time n requires knowledge of the value of n .

- (ii) stable: $y[n]$ is bounded if $x[n]$ is bounded.
 (iii) causal: The current output does not depend on future inputs.
 (iv) linear:

Let $y_1[n] = x_1[n] \sum_{k=-\infty}^{\infty} \delta[n-3k]$ and $y_2[n] = x_2[n] \sum_{k=-\infty}^{\infty} \delta[n-3k]$.

If the input is $f[n] = k_1 x_1[n] + k_2 x_2[n]$, where k_1, k_2 are constants, then the output is $g[n] = f[n] \sum_{k=-\infty}^{\infty} \delta[n-3k] = k_1 y_1[n] + k_2 y_2[n]$.

- (v) not time-invariant:

If the input is $f[n] = x[n-\tau]$, where τ is a constant,

then the output is $g[n] = f[n] \sum_{k=-\infty}^{\infty} \delta[n-3k] = x[n-\tau] \sum_{k=-\infty}^{\infty} \delta[n-3k]$.

Since $y[n-\tau] = x[n-\tau] \sum_{k=-\infty}^{\infty} \delta[n-\tau-3k]$, $g[n] \neq y[n-\tau]$.

Problem 8 (Convolution of step functions.)

- (a) $h(t) = u(t) - u(t-1)$, $x(t) = u(t)$

$$\begin{aligned} h(t) * x(t) &= \int_{-\infty}^{\infty} (u(\tau) - u(\tau-1)) \cdot u(t-\tau) d\tau \\ &= \int_{-\infty}^t u(\tau) - u(\tau-1) d\tau \\ &= \int_0^{\min(t,1)} d\tau \cdot u(t) \\ &= \begin{cases} 0, & t < 0 \\ t, & 0 \leq t \leq 1 \\ 1, & t > 1 \end{cases} \end{aligned}$$

See Figure 5.

- (b) $h(t) = u(t) - u(t-1)$, $x(t) = u(t) - u(t-1)$

We will use the linearity and time invariance properties of the system $h(t)$ to make use of part (a) of this problem. Suppose $u(t)$ is the input to the system as in part (a). Call the output $z(t)$. Now, the input is $x(t) = u(t) - u(t-1)$, the difference of $u(t)$ and a shifted copy of $u(t)$. Using the fact that the system is linear and time invariant, the output should now be $z(t) - z(t-1)$.

$$\begin{aligned} h(t) * x(t) &= [u(t) * (u(t) - u(t-1))] - [u(t-1) * (u(t) - u(t-1))] \\ &= z(t) - z(t-1) \\ &= \begin{cases} t, & 0 \leq t \leq 1 \\ 2-t, & 1 \leq t \leq 2 \\ 0, & t < 0 \text{ or } t > 2 \end{cases} \end{aligned}$$

See Figure 6.

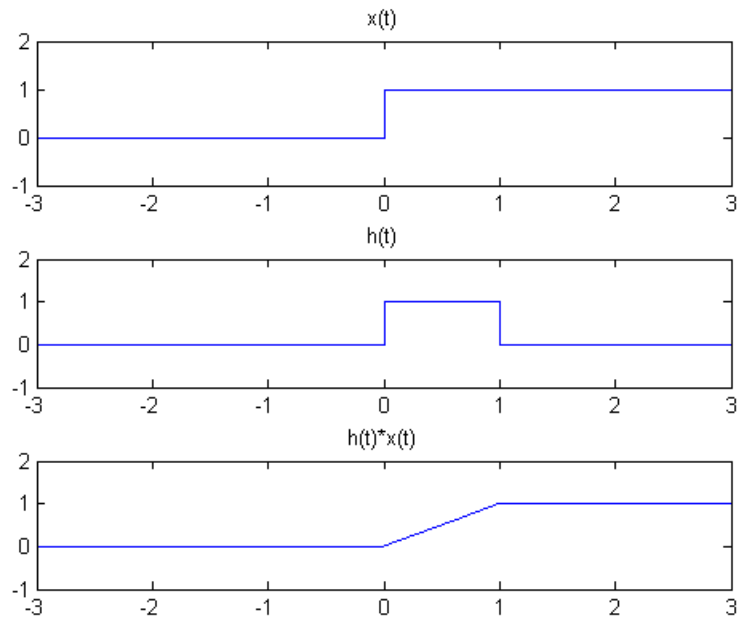


Figure 5: Problem 8(a).

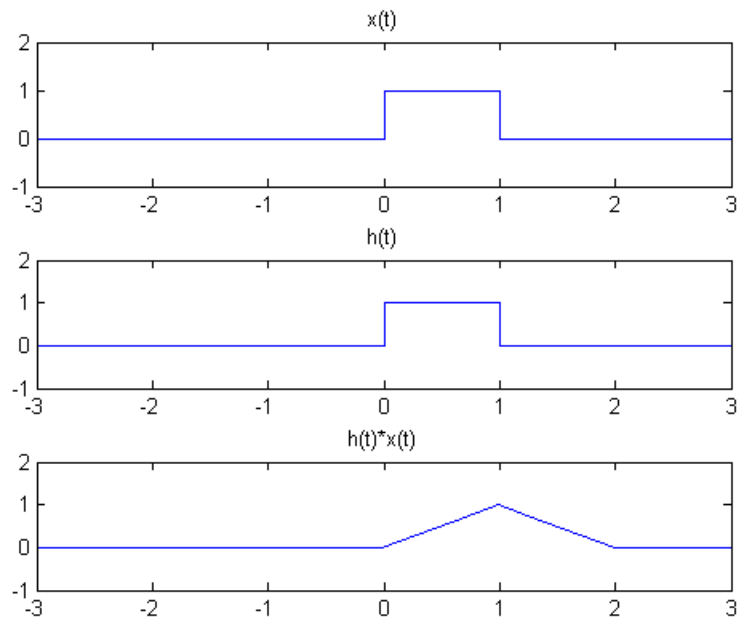


Figure 6: Problem 8(b).

Problem 9 (*LTI systems.*)

We observe that $z(t) = -x(t-1) + 2x(t-2)$. Because the system is LTI, the new output is equal to $-y(t-1) + 2y(t-2)$. The output is plotted in Figure 7.

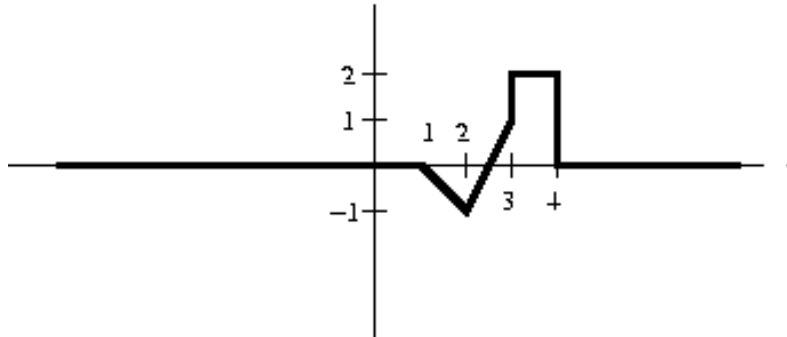


Figure 7: Problem 9.

Problem 10 (*Convolution via matlab.*)

The following Matlab code generates Figure 8.

```
t = -2:.01:2;
xa = (t>=0);
ha = zeros(size(t));
ha(t>=0) = exp(-(0:.01:2));
za = conv(ha,xa);

xb = ha;
hb = zeros(size(t));
hb(t>=0) = exp(-.5.*(0:.01:2));
zb = conv(hb,xb);
```

The Matlab command $\mathbf{z} = \text{conv}(\mathbf{h}, \mathbf{x})$ convolves vectors \mathbf{h} (of length a) and \mathbf{x} (of length b), and produces a vector \mathbf{z} of length $a+b-1$. Matlab effectively computes the convolution sum of the discrete-time signals $g[n]$ and $y[n]$, whose nonzero values are contained in \mathbf{h} and \mathbf{x} , and assumes that the signals are zero otherwise. Thus \mathbf{z} contains the nonzero values of $g[n]*y[n]$. Since our functions $h(t)$ and $x(t)$ are not zero outside of $t \in [-2, 2]$, we need to discard the extraneous (and incorrect) outputs.

Furthermore, because we are approximating $\int h(\tau)x(t-\tau)d\tau$ with $\sum_{\tau} h(\tau)x(t-\tau)\Delta\tau$, we need to multiply the output of `conv`, $\sum_{\tau} h(\tau)x(t-\tau)$, by $\Delta\tau = 0.01$.

```
za = za(201:601)./100;
zb = zb(201:601)./100;
```

Now we are ready to plot the figures.

```
subplot(2,3,1); plot(t,ha); axis([-2 2 -.5 1.5]); title('8a: h(t)');
subplot(2,3,2); stairs(t,xa); axis([-2 2 -.5 1.5]); title('8a: x(t)');
subplot(2,3,3); plot(t,za); axis([-2 2 -.5 1.5]); title('8a: conv(h,x)');

subplot(2,3,4); plot(t,hb); axis([-2 2 -.5 1.5]); title('8b: h(t)');
subplot(2,3,5); plot(t,xb); axis([-2 2 -.5 1.5]); title('8b: x(t)');
subplot(2,3,6); plot(t,zb); axis([-2 2 -.5 1.5]); title('8b: conv(h,x)');
```

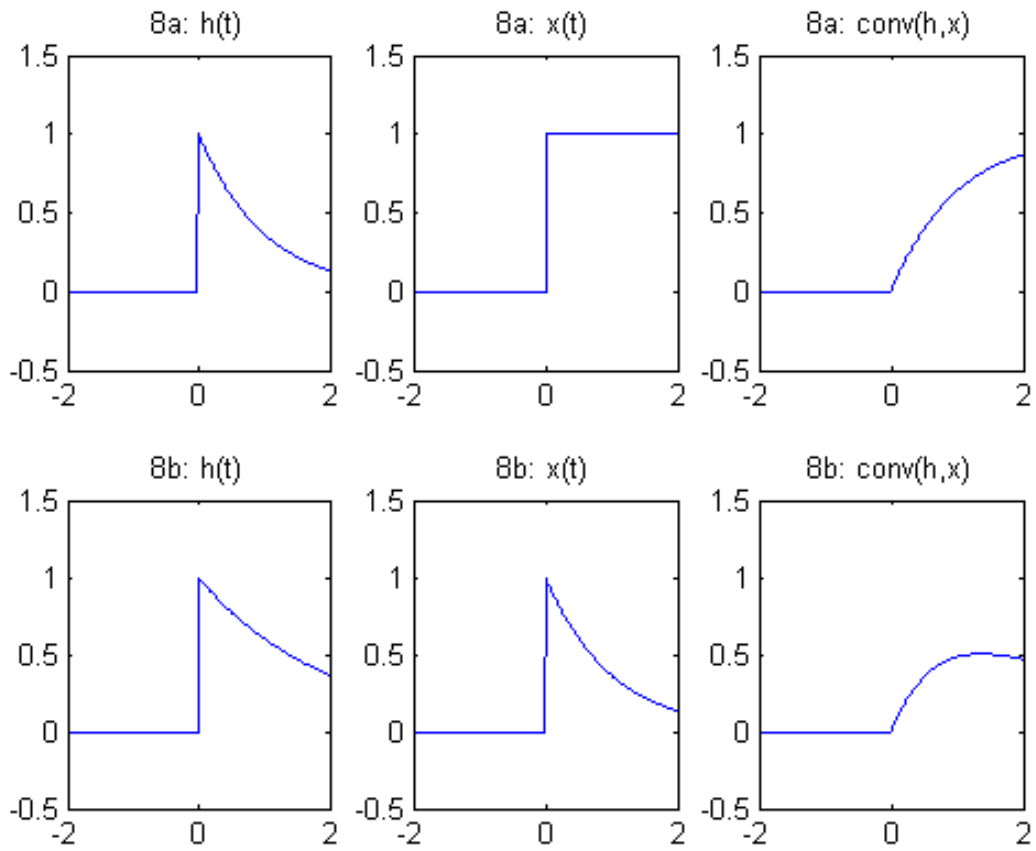


Figure 8: Problem 10.