Homework 2 Solutions

(send your grades to ee120staff@gmail.com; check the course website for more details)

Problem 1 *(Linear Algebra)*

Although the purpose of this exercise was for you solve it by hand in order to get gain more familiarity with matrix operations, you can also do it in Matlab.

(a) $|A|$ is the determinant of matrix A, sometimes referred to as $det(A)$ (only defined for square matrices). In order to compute the determinant, we need to introduce the concept of a Minor matrix. A Minor M_{ij} of a matrix A is a submatrix formed by removing the i^{th} row and j^{th} column of A. Therefore, if A is an $n \times n$ matrix then M_{ij} will be $(n-1) \times (n-1)$. Also, define a cofactor $A_{ij} = (-1)^{i+j}|M_{ij}|$. If we denote a_{ij} as the element in the i^{th} row and j^{th} column of A, then we can compute $|A|$ as follows (we will use the elements and cofactors of the first row, but any row or column will also be sufficient):

$$|A| = \sum_{k=1}^{n} a_{1k} A_{1k}$$

(I didn't tell you how to compute the determinant of the minors, but you can apply the definition recusively until you reach a matrix of size 1, in which case the value of the determinant is the same as the single element).

$$\Rightarrow |A| = \begin{vmatrix} 2 & 1 & 2 \\ -4 & 5 & 0 \\ 4 & -7 & 3 \end{vmatrix} = 58$$

$tr(C)$, called the trace of C, is simply the sum of elements in the main diagonal (only defined for square matrices):

$$tr(C) = \sum_{i=1}^{n} c_{ii} = 3 + 5 - 7 = 1$$

(b) The rank of matrix is the number of linearly independent rows or columns (turns out to be the same) in the matrix. A set of vectors is said to be linearly independent when no vector in the set can be written as a linear combination of other vectors in the set. If you notice, the rows of A fit the definition. However, the third row of B is the sum of the second row and twice the first row. Therefore, B contains only two rows (and columns) that are linearly independent.

$$\Rightarrow rank(A) = 3, rank(B) = 2$$

(c) Before we go into matrix multiplication, let's define the dot-product of two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ (must be of the same length) denoted as $< \overrightarrow{x}, \overrightarrow{y} >$ or $\overrightarrow{x} \cdot \overrightarrow{y}$ as:

$$\overrightarrow{x} \cdot \overrightarrow{y} = \sum_{i=1}^{n} x_n^* y_n$$

Two vectors are said to be orthogonal when their dot-product is 0. The dot-product can also be defined for vectors with infinite dimensions and continuous vectors (functions), but this usually involves infinite sums/integrals (more on that later).

The product K_3 of two matrices K_1K_2 is defined only when K_1 is $m \times k$ and K_2 is $k \times n$ (the number of columns in K_1 has to equal the number of rows in K_2); the output matrix K_3 will be $m \times n$. The element in the i^{th} row and j^{th} column of K_3 is the dot-product of the i^{th} row of K_1 and the j^{th} column of K_2. In general $K_1K_2 \neq K_2K_1$ (it may not be even defined).

$$\Rightarrow AB = \begin{bmatrix} 2 & 1 & 2 \\ -4 & 5 & 0 \\ 4 & -7 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 0 \\ -2 & 2 & 0 & 3 \\ -2 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -6 & 12 & 12 & 9 \\ -10 & 6 & -8 & 15 \\ 8 & 2 & 20 & -12 \end{bmatrix}$$

$B^\top$ is the transpose of matrix B. Rows of $B^\top$ are the columns of B and vice verca ($B^\top_{ij} = B_{ji}$.

$$\Rightarrow B^\top A = \begin{bmatrix} 0 & -2 & -2 \\ 1 & 2 & 4 \\ 2 & 0 & 4 \\ 0 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ -4 & 5 & 0 \\ 4 & -7 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -6 \\ 10 & -17 & 14 \\ 20 & -26 & 16 \\ 0 & -6 & 9 \end{bmatrix}$$

(d) A^{-1}, the inverse of matrix A (defined only for square matrices), is defined such that $A^{-1}A = AA^{-1} = I$, where I is the identity matrix (ones on the main diagonal and zeros everywhere else) of the same size as A. The inverse matrix exists only when the matrix has full rank (all the rows are linearly independent). To compute A^{-1}, we follow three simple steps:

(1) Replace every element of A by its cofactor

(2) Transpose the output from step 1.

(3) Divide the output from step 2 by $|A|$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0.25862068965517 & -0.29310344827586 & -0.17241379310345 \\ 0.20689655172414 & -0.03448275862069 & -0.13793103448276 \\ 0.13793103448276 & 0.31034482758621 & 0.24137931034483 \end{bmatrix}$$

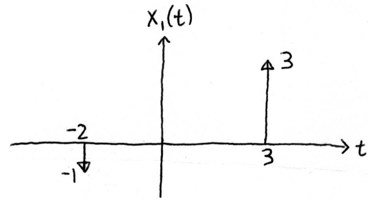
$$C^{-1} = \begin{bmatrix} 0.37053571428571 & 0.09821428571429 & -0.03125000000000 \\ -0.11607142857143 & 0.05357142857143 & -0.06250000000000 \\ -0.34375000000000 & -0.18750000000000 & -0.03125000000000 \end{bmatrix}$$

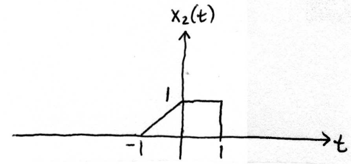
$$(AC)^{-1} = C^{-1}A^{-1} = \begin{bmatrix} 0.11183805418719 & -0.12169027093596 & -0.08497536945813 \\ -0.02755541871921 & 0.01277709359606 & -0.00246305418719 \\ -0.13200431034483 & 0.09752155172414 & 0.07758620689655 \end{bmatrix}$$

Problem 2 *(Graphical Convolution.)*

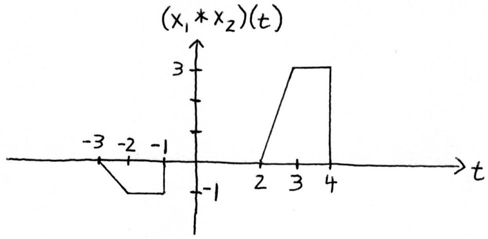
① (d) $x_1(t) = -\delta(t+2) + 3\cdot\delta(t-3)$

$$x_2(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 1 & 0 \le t \le 1 \\ 0 & |t| > 1 \end{cases}$$



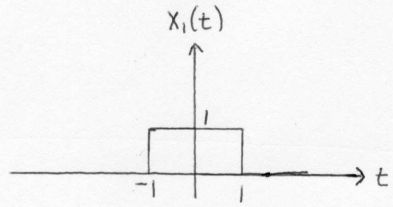


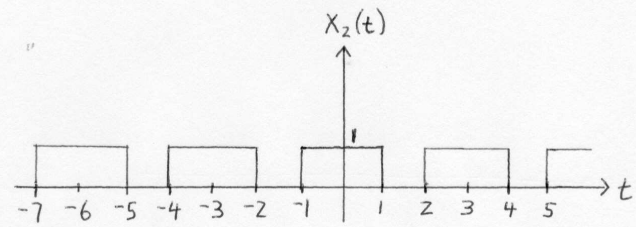
Convolving with delta functions is easy.
Just remember that $x(t) * \delta(t-t_0) = x(t-t_0)$



(b) $x_1(t) = \begin{cases} 1 & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$

$$x_2(t) = \sum_{k=-\infty}^{\infty} x_1(t+3k)$$





We can see that $x_2(t) = \sum_{k=-\infty}^{\infty} x_1(t) * \delta(t+3k)$

$$= x_1(t) * \sum_{k=-\infty}^{\infty} \delta(t+3k)$$

So, $x_1(t) * x_2(t) = x_1(t) * \left(x_1(t) * \sum_{k=-\infty}^{\infty} \delta(t+3k) \right)$

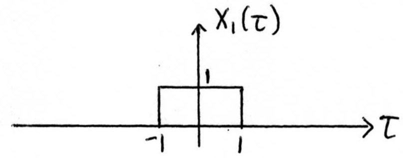
$$= \left(x_1(t) * x_1(t) \right) * \sum_{k=-\infty}^{\infty} \delta(t+3k)$$

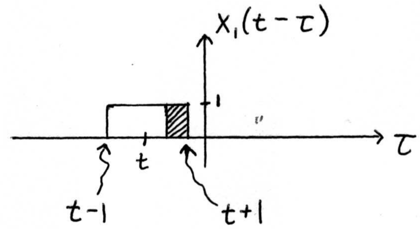
$$= \tilde{x}(t) * \sum_{k=-\infty}^{\infty} \delta(t+3k)$$

$$= \sum_{k=-\infty}^{\infty} \tilde{x}(t+3k)$$

where $\tilde{x}(t) = x_1(t) * x_1(t)$

Now, let's graphically convolve $x_1(t)$ with $x_1(t)$





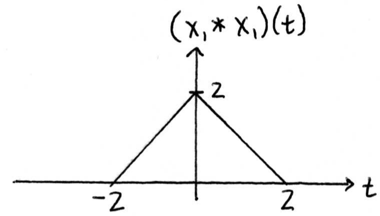
If $\quad t+1 < -1 \quad$ then $\quad (x_1 * x_1)(t) = 0$

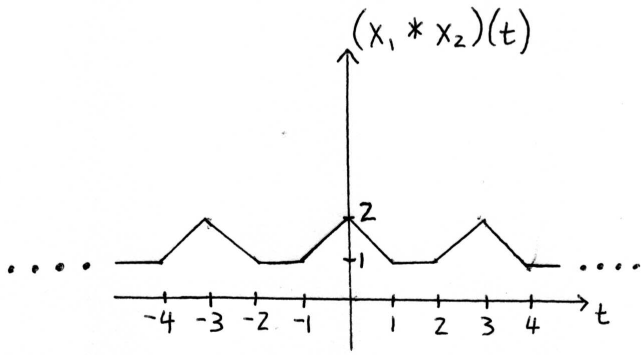
If $\quad -1 \leq t+1 \leq 1 \quad$ then $\quad (x_1 * x_1)(t) = (t+1) - (-1) = t+2$

If $\quad -1 \leq t-1 \leq 1 \quad$ then $\quad (x_1 * x_1)(t) = 1 - (t-1) = 2-t$

If $\quad t-1 > 1 \quad$ then $\quad (x_1 * x_1)(t) = 0$

$$\Rightarrow (x_1 * x_1)(t) = \begin{cases} 2 - |t| & \text{if } |t| \leq 2 \\ 0 & \text{if } |t| > 2 \end{cases}$$





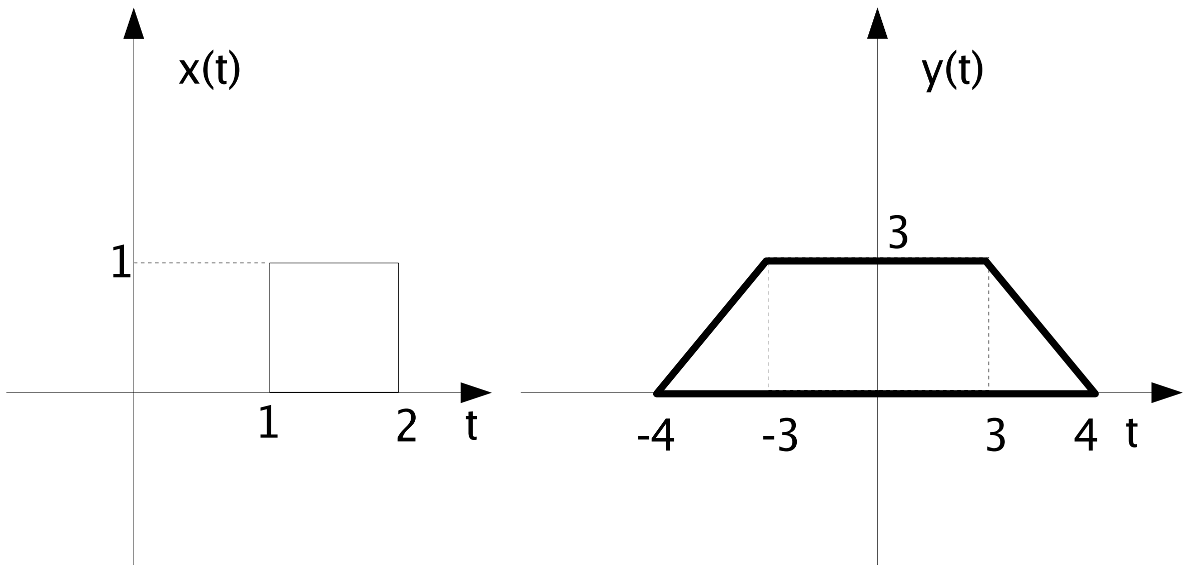


Figure 1: Problem 2(c)

(c) We will be encountering box and triangle functions very often in this class; so we might as well give them special names. So let us define:

$$\Pi(t) = u(t - \frac{1}{2}) - u(t + \frac{1}{2})$$

$$\Lambda(t) = \Pi(t) * \Pi(t)$$

$\Pi(t)$ is basically a box centered around $t = 0$ and has a base and height of 1. $\Lambda(t)$ is a triangle centered around $t = 0$ and has a base of 2 and height of 1. You will encounter this notation in many text books.

In this problem we are given $y(t), x(t)$ and are asked to find $h(t)$ such that $y(t) = x(t) * h(t)$. The first thing to notice is that $x(t) = \Pi(t - \frac{3}{2})$. Second, from our experience in doing graphical convolutions with box functions, we know that $h(t)$ has to be a rectangle of the form $a\Pi(\frac{t-t_0}{b})$. Therefore, we need to determine its length b and height a and the center point on the t-axis t_0. We also know that the length of the interval in which $y(t) > 0$ cannot exceed the sum of the two intervals in which $x(t)$ and $h(t)$ are nonzero respectively. Therefore, $b = 8 - 1 = 7$. The maximum value of $y(t)$ (happens to be 3) is the area of the box $x(t)$ (happens to be 1) scaled by a. Therefore, $a = 3$. Also, notice that $y(t)$ is centered around $t = 0$, while $x(t)$ is delayed by $\tau = \frac{3}{2}$. Therefore, in order to compensate of the delay of $x(t)$, $h(t)$ has to be delayed by $t_0 = -3/2$.

$$\Rightarrow h(t) = 3\Pi(\frac{t + \frac{3}{2}}{7})$$

We can also approach this problem graphically, by going back to the definition of convolution $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$. We first flip one of the signals ($x(t)$ in this case, since we are trying to determine $h)(t)$). Figure 3 shows a flipped version of $x(t)$. How do we position $h(t)$ in order to maximize the value of the convolution at $t = 0$? we need complete overlap between $x(-\tau)$ and $h(\tau)$. Since $y(t)$ is symmetric around the axis, we know that if we drag $x(-\tau)$ in either direction along the axis by the same amount, we should get the same result. Therefore, $h(\tau)$ has to be symmetric around the axis of $x(-\tau)$, which is why we get the above result.

(d) Let $z(t) = \Pi(t) * \Lambda(t)$. Clearly, $z(t)$ has its maximum value at $t = 0$. This maximum vaue is the area of overlap (the sum of the two trapezoids). Therefore, $z_{max} = 0.75$.

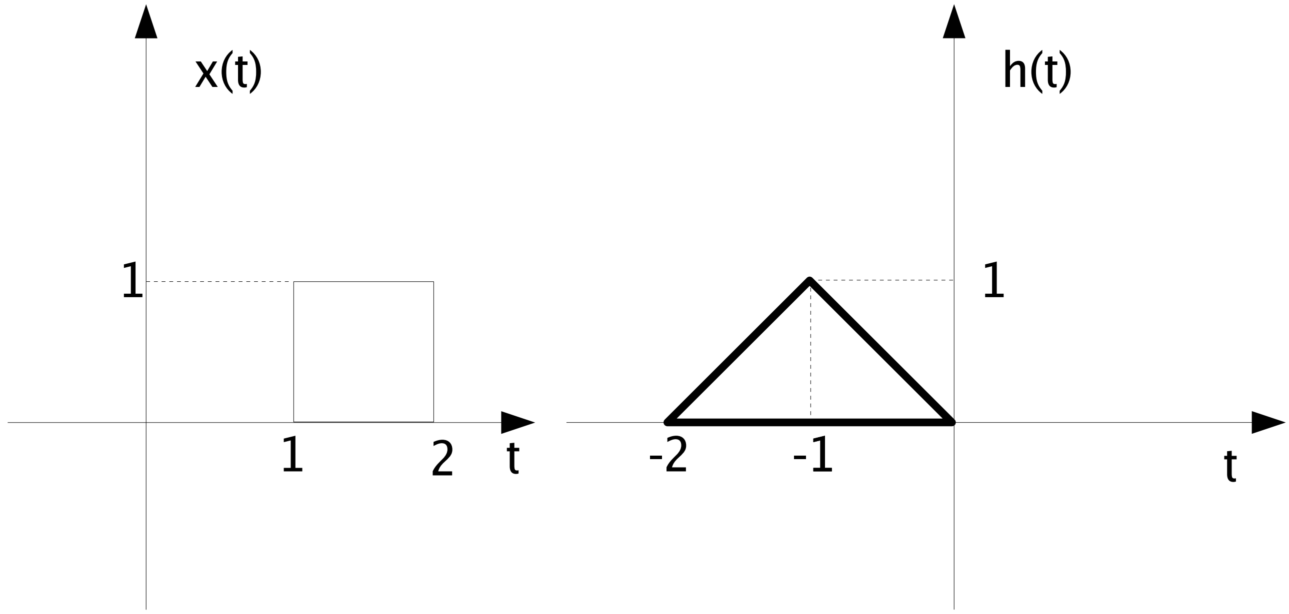


Figure 2: Problem 2(d)

$$y(t) = x(t) * h(t) = \Pi(t - \frac{3}{2}) * \Lambda(t+1) = \Pi(t) * \delta(t - \frac{3}{2}) * \Lambda(t) * \delta(t+1)$$

$$\Rightarrow y(t) = \Pi(t) * \Lambda(t) * \delta(t - \frac{3}{2}) * \delta(t+1) = z(t) * \delta(t - \frac{1}{2})$$

Therefore, as we can see $y(t)$ is just a delayed version of $z(t)$. Therefore, $y_{max} = z_{max} = 0.75, y(\frac{1}{2}) = z(0)$.

Once again we can also look at this problem graphically. Going back to the definition of convolution $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$. In order to perform this convolution graphically, we first flip one of the signals (doesn't matter which one) along the t-axis. Figure 3 shows a flipped version of $x(t)$. The next step is the *drag* step. How far do we have to drag $x(-\tau)$ before its center completely overlaps with the center of $h(\tau)$? when they overlap, what is the area?

Problem 3 *(Convolution.)*

- *2.21(a)*

  $x[n] = \alpha^n u[n]$

  $h[n] = \beta^n u[n]$

  $\alpha \neq \beta$

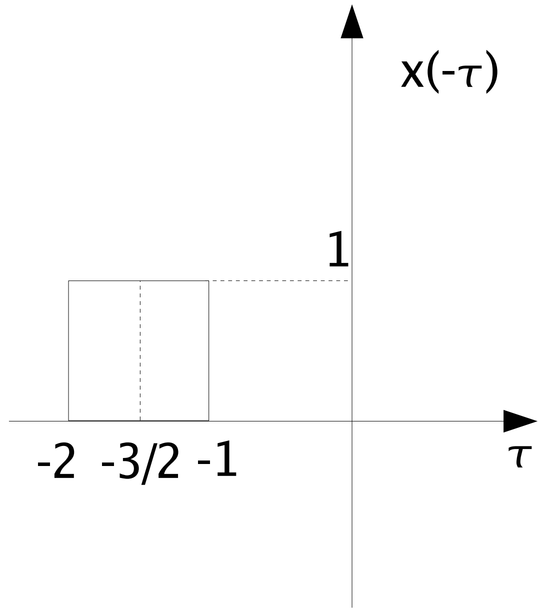
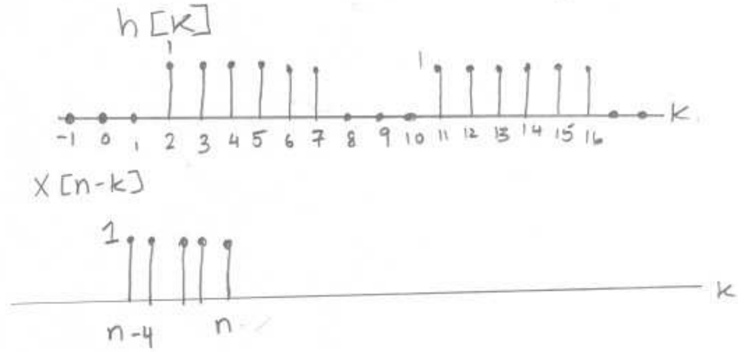


Figure 3: $x(-t)$ the time reversed (flipped) version of $x(t)$ in problems 2cd

$$
\begin{aligned}
y[n] &= (x \star h)[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{\infty} \alpha^k u[k] \beta^{n-k} u[n-k] \\
&= \sum_{k=0}^{\infty} \alpha^k \beta^{n-k} u[n-k] \\
&= \beta^n \cdot \sum_{k=0}^{\infty} (\frac{\alpha}{\beta})^k u[n-k] \\
&= \beta^n u[n] \cdot \sum_{k=0}^{n} (\frac{\alpha}{\beta})^k \\
&= \beta^n u[n] \frac{1-(\frac{\alpha}{\beta})^{n+1}}{1-(\frac{\alpha}{\beta})} \\
&= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} u[n]
\end{aligned}
$$

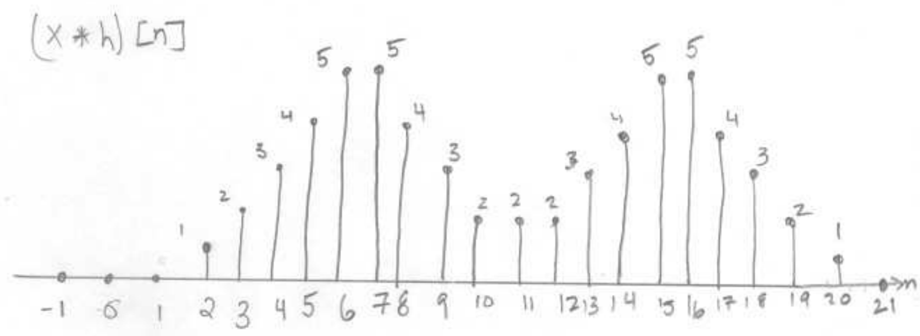
- *2.21(d)*

d) OWN 2.21d $(x * h)[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$



$(x * h)[n]$ = # of overlapping non-zero samples.

$$= \begin{cases} 0, & n \le 1 \\ n-1, & 2 \le n \le 6 \\ 12-n & 7 \le n \le 10 \\ 2 & n = 11 \\ n-10 & 12 \le n \le 15 \\ 21-n & 16 \le n \le 21 \\ 0 & \text{else} \end{cases}$$



- *2.22(a)*
  $x(t) = e^{-\alpha t}u(t)$
  $h(t) = e^{-\beta t}u(t)$

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau)d\tau \\
&= \begin{cases} \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau & t \geq 0 \\ 0 & t < 0 \end{cases}
\end{aligned}$$

If $\alpha \neq \beta$, then:

$$\begin{aligned}
y(t) &= e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau \;\; u(t) \\
&= e^{-\beta t}\frac{e^{-(\alpha-\beta)t}-1}{\beta-\alpha} \quad u(t)
\end{aligned}$$

If $\alpha = \beta$, then:

$$\begin{aligned}
y(t) &= e^{-\beta t}\int_0^t 1 d\tau \;\; u(t) \\
&= te^{-\beta t}u(t)
\end{aligned}$$

See plot below.

- *2.22(e)*
  First, we observe that $h(t)$ can be written as:

$$h(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ 0 & else \end{cases}$$

Because $x(t)$ is periodic, $y(t)$ will also be periodic. We will compute $y(t)$ for one period. For $-\frac{1}{2} < t < \frac{1}{2}$, we have

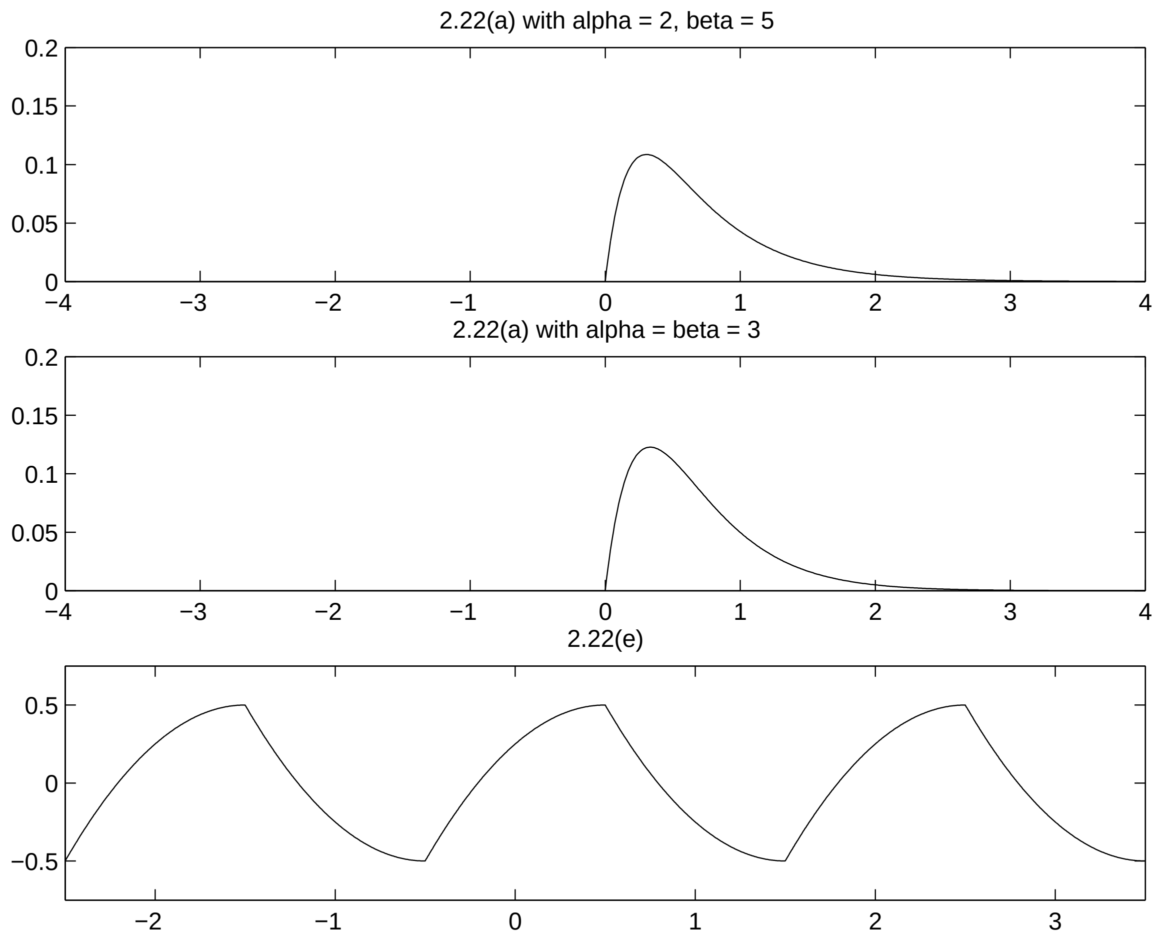
$$\begin{aligned}
y(t) &= x(t) \star h(t) \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_0^{t+1/2}(1-\tau)d\tau + \int_{t+1/2}^{1}(-1)(1-\tau)d\tau \\
&= [\tau - \frac{1}{2}\tau^2]_{\tau=0}^{t+1/2} + [\frac{1}{2}\tau^2 - \tau]_{\tau=t+1/2}^{1} \\
&= ((t+\frac{1}{2}) - \frac{1}{2}(t+\frac{1}{2})^2) + (\frac{1}{2}-1) - (\frac{1}{2}(t+\frac{1}{2})^2 - (t+\frac{1}{2})) \\
&= \frac{1}{4} + t - t^2
\end{aligned}$$

For $\frac{1}{2} < t < \frac{3}{2}$, we have

$$\begin{aligned} y(t) &= \int_0^{t-1/2} (-1)(1-\tau)d\tau + \int_{t-1/2}^{1} (1-\tau)d\tau \\ &= [\frac{1}{2}\tau^2 - \tau]_{\tau=0}^{t-1/2} + [\tau - \frac{1}{2}\tau^2]_{\tau=t-1/2}^{1} \\ &= (\frac{1}{2}(t-\frac{1}{2})^2) + (1-\frac{1}{2}) - ((t-\frac{1}{2}) - \frac{1}{2}(t-\frac{1}{2})^2) \\ &= t^2 - 3t + \frac{7}{4} \end{aligned}$$

$y(t)$ will have a period of 2.

The plots of the solutions to problems 2.22(a) and (e) are shown in the following figure.



Problem 4 *(Impulse response and system properties.)*

- *2.28(b)*

  Not causal, because $h[-1] = 1.25 \neq 0$

  Stable, because $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-2}^{\infty} (0.8)^n = \frac{1.5625}{1-0.8} = 7.81215 < \infty$

- *2.28(g)*

  Causal, because $h[n] = 0$ for all $n < 0$

  Stable, because $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=1}^{\infty} n(\frac{1}{3})^n = 0.75 < \infty$

- *2.29(d)*

  Not causal, because $h(-2) = e^{-4} \neq 0$

  Stable, because $\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{-1} e^{2t}dt = \frac{1}{2}e^{-2t} < \infty$

- *2.29(g)*

  Causal, because $h(t) = 0$ for all $t < 0$

  Unstable, because $\int_{-\infty}^{\infty} |h(t)|dt = \infty$

  (Observe that $-e^{(t-100)/100}$ approaches $-\infty$ as t goes to ∞)

Problem 5 *(Noise suppression system for airplanes.)*

- *(a)*

  Assume that the two delay elements both store the value 0 at the beginning of time.

Linearity First, assume that an input signal $x_1[n]$ produces an output signal

$$y_1[n] = \frac{2}{3}x_1[n] + \frac{1}{3}x_1[n-1] + \frac{1}{3}x_1[n-2]$$

Similarly, assume that an input signa; $x_2[n]$ produces an output signal

$$y_2[n] = \frac{2}{3}x_2[n] + \frac{1}{3}x_2[n-1] + \frac{1}{3}x_2[n-2]$$

If we use the signal $x[n] = \alpha x_1[n]$ as the input to the system, the output will be

$$y[n] = \frac{2}{3}\alpha x_1[n] + \frac{1}{3}\alpha x_1[n-1] + \frac{1}{3}\alpha x_1[n-2] = \alpha y_1[n]$$

If we use the signal $x[n] = x_1[n] + x_2[n]$ as the input to the system, the output will be

$$y_1[n] = \frac{2}{3}(x_1[n] + x_2[n]) + \frac{1}{3}(x_1[n-1] + x_2[n-1]) + \frac{1}{3}(x_1[n-2] + x_2[n-2]) = y_1[n] + y_2[n]$$

Therefore, the system is linear.

Time-invariance Again assume that an input signal $x_1[n]$ produces an output signal

$$y_1[n] = \frac{2}{3}x_1[n] + \frac{1}{3}x_1[n-1] + \frac{1}{3}x_1[n-2]$$

If we use the signal $x[n] = x_1[n-n_0]$ as the input to the system, the output will be

$$y[n] = \frac{2}{3}x_1[n-n_0] + \frac{1}{3}x_1[n-n_0-1] + \frac{1}{3}x_1[n-n_0-2] = y_1[n-n_0]$$

Therefore, the system is time-invariant.

Impulse response We can see from the figure that the impulse response of the system is

$$h[n] = \frac{2}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$$

- *(b)*

Causality Yes, the system is causal, because the current output $y[n]$ depends only on current and past inputs.

Memorylessness No, the system is not memoryless, because $y[n]$ depends on $x[n-1]$ and $x[n-2]$.

Stability Yes, the system is stable, because

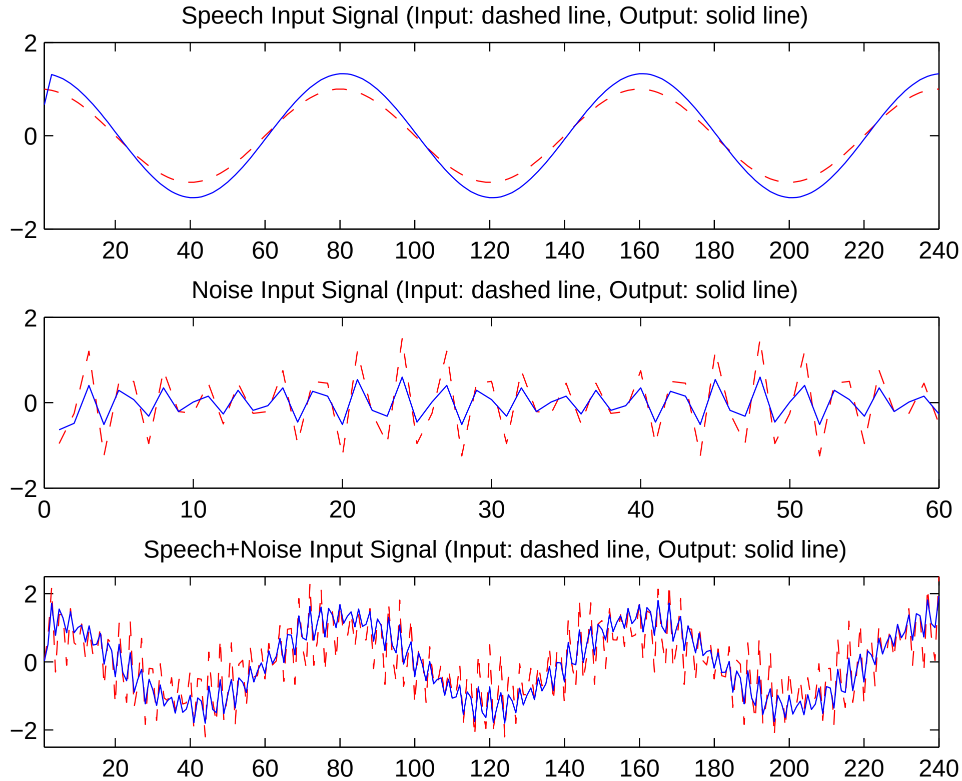
$$\sum_{n=-\infty}^{\infty} |h[n]| = \frac{4}{3} < \infty$$

Stability is important because we want all bounded input signals to produce bounded outputs. Unbounded outputs can cause buffer overflows which usually return incorrect results.

- *(c)*

The following figures show the input and output signals for the three cases. We see that the system significantly attenuates the noise. When the input to the system is the combined signal, the output is much less noisy than the input. The following Matlab script was used to generate the figures.

```
h = [2/3 1/3 1/3];
index = 1:1:240;
noise_input = cos(3*pi/4*index) + 0.5*cos(2*pi/3*index);
speech_input = cos(pi/40*index);
total_input = cos(3*pi/4*index) + 0.5*cos(2*pi/3*index)+cos(pi/40*index);
noise_output = conv(noise_input,h);
speech_output = conv(speech_input,h);
total_output = conv(total_input,h);
subplot(3,1,1),plot(index,speech_input,'--r',index,speech_output(1:240),'-b')
axis([1 240 -2 2])
title('Speech Input Signal (Input:  dashed line, Output:  solid line)')
subplot(3,1,2),plot(index(1:60),noise_input(1:60),'--r',index(1:60),noise_output(1:60),'-b')
title('Noise Input Signal (Input:  dashed line, Output:  solid line)')
subplot(3,1,3),plot(index,total_input,'--r',index,total_output(1:240),'-b')
axis([1 240 -2.5 2.5])
title('Speech+Noise Input Signal (Input:  dashed line, Output:  solid line)')
```



- *(d)*

  We can use Matlab to compute the SNR. At the input, the SNR is 0.8000. At the output, the SNR is 7.7331. We see that the filter has increased the SNR dramatically.

```
signal_input_power = (sum(speech_input.^2))/240;
noise_input_power = (sum(noise_input.^2))/240;
snr_input = signal_input_power/noise_input_power;
signal_output_power = (sum((speech_output(1:240)).^2))/240;
noise_output_power = (sum((noise_output(1:240)).^2))/240;
snr_output = signal_output_power/noise_output_power;
```

Problem 6 *(Matlab, vector bases)*

- *(a)*

  $\|\vec{v}_i\| = \sqrt{\vec{v}_i^T \vec{v}_i} = \sqrt{\vec{v}_i \cdot \vec{v}_i}$

  $\theta_{ij} = \arccos((\vec{v}_i \cdot \vec{v}_j)/(\|\vec{v}_i\| \|\vec{v}_j\|))$

  Using the Matlab script shown below, we compute that:

$$\vec{v}_i \cdot \vec{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

  We find that the norm of each of the four vectors is equal to 1.0000, and that the angle between each pair of unique vectors is $\frac{\pi}{2}$. These four vectors form an <u>orthonormal</u> basis set.

```
load('basis.mat');
dot(v1,v1)
dot(v2,v2)
dot(v3,v3)
dot(v4,v4)
dot(v1,v2)
dot(v1,v3)
dot(v1,v4)
dot(v2,v3)
dot(v2,v4)
dot(v3,v4)
acos(dot(v1,v2))
acos(dot(v1,v3))
acos(dot(v1,v4))
acos(dot(v2,v3))
acos(dot(v2,v4))
acos(dot(v3,v4))
```

- *(b)*

$$\begin{aligned}
\overrightarrow{y}^T\overrightarrow{v}_1 &= \alpha_1\overrightarrow{v_1}^T\overrightarrow{v}_1 + \alpha_2\overrightarrow{v_2}^T\overrightarrow{v}_1 + \alpha_3\overrightarrow{v_3}^T\overrightarrow{v}_1 + \alpha_4\overrightarrow{v_4}^T\overrightarrow{v}_1 \\
&= \alpha_1 * 1 + 0 + 0 + 0 \\
&= \alpha_1 \\
\overrightarrow{y}^T\overrightarrow{v}_2 &= \alpha_2 \\
\overrightarrow{y}^T\overrightarrow{v}_3 &= \alpha_3 \\
\overrightarrow{y}^T\overrightarrow{v}_4 &= \alpha_4
\end{aligned}$$

- *(c)*

```
x1_alpha1 = dot(x1,v1);
x1_alpha2 = dot(x1,v2);
x1_alpha3 = dot(x1,v3);
x1_alpha4 = dot(x1,v4);
x2_alpha1 = dot(x2,v1);
x2_alpha2 = dot(x2,v2);
x2_alpha3 = dot(x2,v3);
x2_alpha4 = dot(x2,v4);
x3_alpha1 = dot(x3,v1);
x3_alpha2 = dot(x3,v2);
x3_alpha3 = dot(x3,v3);
x3_alpha4 = dot(x3,v4);
```

$$\begin{aligned}
\overrightarrow{x}_1 &= 5.2630\overrightarrow{v}_1 + 0.0367\overrightarrow{v}_2 + 1.3194\overrightarrow{v}_3 - 4.6446\overrightarrow{v}_4 \\
\overrightarrow{x}_2 &= 1.5750\overrightarrow{v}_1 - 1.9564\overrightarrow{v}_2 + 3.9976\overrightarrow{v}_3 + 4.2687\overrightarrow{v}_4 \\
\overrightarrow{x}_3 &= 0.2789\overrightarrow{v}_1 + 1.2448\overrightarrow{v}_2 - 1.7953\overrightarrow{v}_3 - 3.7327\overrightarrow{v}_4
\end{aligned}$$

Problem 7 *(Complex Numbers)*

(a) Remember that every complex number c can be represented in one of two forms: $c = a + jb$ (Cartesian/Rectangular) or $c = re^{j\theta}$ (Polar), where $a, b, r, \theta \in \mathbb{R}$ and $r \geqslant 0$. $a = \Re e\{c\}$ and $b = \Im m\{c\}$ are called the real and imaginary parts of c respectively, while $r = |c|$ and $\theta = \angle c$ are called the magnitude and the angle (phase) of c respectively. The rectangular representation is usually more convenient for addition/subtraction, while the polar representation is more useful in representing products and powers and ratios. We have the following relationships between a, b, r, θ:

$$r = \sqrt{a^2 + b^2}, \theta = \arctan(\frac{b}{a}), a = r\cos\theta, b = r\sin\theta$$

Now using these relationships, we can find the cartesian and polar representations of $c_1 = \frac{1+j}{1-j}$. For the rectangular form, we multiply the numerator and denominator of c_1 by the complex conjugate of the denominator (to get the complex conjugate of a number we simply flip the sign of the imaginary part and the angle).

$$c_1 = j = e^{j\frac{\pi}{2}}$$

(b) The polar representation is more convenient for finding roots. The N^{th} root of a complex number $c = re^{j\theta}$ can be found by simply raising c to the $\frac{1}{N}$ power: $c^{\frac{1}{N}} = r^{\frac{1}{N}} e^{j\frac{\theta}{N}}$. Notice however, $c = re^{j\theta} = re^{j(\theta+2\pi)} = re^{j(\theta+2n\pi)}$ for all $n \in \mathbb{Z}$. Therefore, $r^{\frac{1}{N}} e^{j\frac{(\theta+2n\pi)}{N}}$ is an N^{th} root for all $n \in \mathbb{Z}$. Does that mean that there is an infinite number of N^{th} roots? Not quite. If you look closely at the expression, you will notice that it repeats every N integers. Therefore, there are only N unique roots (also remember that every polynomial of order N has exactly N complex roots). So we only need to consider $0 \leqslant n < N$.

Now back to our example, $c_2 = -16 = 16e^{j\pi}$. The four fourth roots of c_2 are $2e^{j\frac{\pi}{4}}, 2e^{j\frac{3\pi}{4}}, 2e^{j\frac{5\pi}{4}}, 2e^{j\frac{7\pi}{4}}$.

Problem 8 *(Differential Equations)*

(i)

Let $i(t)$ be the current flowing out of the positive terminal of the signal source $x(t)$. Let $v_R(t), v_L(t), v_C(t)$ be the voltages across the resistor, inductor, and capacitor respectively (in the direction of the current flow). We have the following relations (using KVL, Ohm's law ...):

$$v_R(t) + v_L(t) + v_c(t) = x(t)$$

$$v_R(t) = i(t)R, v_L(t) = L\frac{di(t)}{dt}, i(t) = C\frac{dv_C(t)}{dt}, v_C(t) = y(t)$$

$$\Rightarrow v_R(t) = RC\frac{dy(t)}{dt}, v_L(t) = LC\frac{d^2y(t)}{dt^2}$$

$$y(t) + RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} = x(t)$$

(ii)

In order to find the homogeneous solution $y_h(t)$, we set the input to $x(t) = 0$ and plug $y_h(t) = Ae^{rt}$ into the differential equation to get the following characteristic polynomial:

$$Lr^2 + Rr + \frac{1}{C} = 0$$

solving for the roots of the polynomial we get:

$$r = -\frac{R}{2L} \pm \frac{\sqrt{R^2 - 4\frac{L}{C}}}{2L}$$

Plugging in $R = 2\Omega, L = 1H, C = 0.2F$, we find that the two roots are $r_1 = -1 + j2, r_2 = -1 - j2$.

$$\Rightarrow y_h(t) = K_1 e^{r_1 t} + K_2 e^{r_2 t} = e^{-t}(K_1 e^{j2t} + K_2 e^{-j2t})$$

(iii)

Since all the voltages and currents are real, we only need the real part of $y_h(t)$:

$$\begin{aligned}\Re e\{y_h(t)\} &= e^{-t}(\Re e\{K_1 e^{j2t}\} + \Re e\{K_2 e^{-j2t}\}) \\ &= e^{-t}(\Re e\{K_1\}\cos 2t - \Im m\{K_1\}\sin 2t + \Re e\{K_2\}\cos 2t + \Im m\{K_2\}\sin 2t) \\ &= e^{-t}(\alpha_1 \cos 2t + \alpha_2 \sin 2t)\end{aligned}$$

where α_1, α_2 are arbitrary real constants that are determined from the initial conditions. Also, remember that $\alpha_1 \cos 2t + \alpha_2 \sin 2t$ can also be written as $\alpha \cos(2t + \phi)$, where both α, ϕ are real constants. Therefore,

$$\Re e\{y_h(t)\} = \alpha e^{-t}\cos(2t + \phi)$$

Problem 9 *(Difference Equations)*

(a)

Let us use year 2001 as our referrence year (i.e. $n = 0$).

$$y[n] = 1.06y[n-1] + x[n]$$

$$x[n] = 1500u[n]$$

$$y[n] = 0 \ \ \forall n < 0$$

$x[n]$ is the input to the system (a scaled step function). $y[n]$ is the ouput (account balance). In order to find the impulse response $h[n]$, we solve the difference equation when the input is $\delta[n]$.

$$h[n] = 1.06h[n-1] + \delta[n]$$

$$\Rightarrow h[0] = 1.06h[-1] + \delta[0] = 1$$

$$h[1] = 1.06h[0] + \delta[1] = 1.06$$

$$h[2] = 1.06h[1] + \delta[2] = 1.06^2$$

$$h[3] = 1.06h[2] = 1.06^3$$

$$\Rightarrow h[n] = 1.06^n \forall n \geq 0$$

$$\Rightarrow h[n] = 1.06^n u[n]$$

As we can see from the impulse response, the system is causal, but NOT stable or memoryless. Although not required, we can compute the output:

$$y[n] = x[n] * h[n] = 1500 \sum_{k=-\infty}^{n} h[k] = 1500u[n]\sum_{k=0}^{n} 1.06^k$$

$$y[n] = 1500u[n]\frac{1 - 1.06^{n+1}}{1 - 1.06} = 25000(1.06^{n+1} - 1)u[n]$$

(b)

$$y[n] = (\frac{1}{3} - j\frac{1}{4})y[n-1] + j2x[n]$$

$$\Rightarrow y_R[n] + jy_I[n] = (\frac{1}{3} - j\frac{1}{4})(y_R[n-1] + jy_I[n-1]) + j2(x_R[n] + jx_I[n])$$

$$\Rightarrow y_R[n] + jy_I[n] = \frac{1}{3}y_R[n-1] + j\frac{1}{3}y_I[n-1] - j\frac{1}{4}y_R[n-1] + \frac{1}{4}y_I[n-1] + j2x_R[n] - 2x_I[n]$$

$$\Rightarrow y_R[n] + jy_I[n] = (\frac{1}{3}y_R[n-1] + \frac{1}{4}y_I[n-1] - 2x_I[n]) + j(\frac{1}{3}y_I[n-1] - \frac{1}{4}y_R[n-1] + 2x_R[n])$$

$$\Rightarrow y_R[n] = \frac{1}{3}y_R[n-1] + \frac{1}{4}y_I[n-1] - 2x_I[n]$$

$$y_I[n] = \frac{1}{3}y_I[n-1] - \frac{1}{4}y_R[n-1] + 2x_R[n]$$

The block diagram is shown in Figure 4.

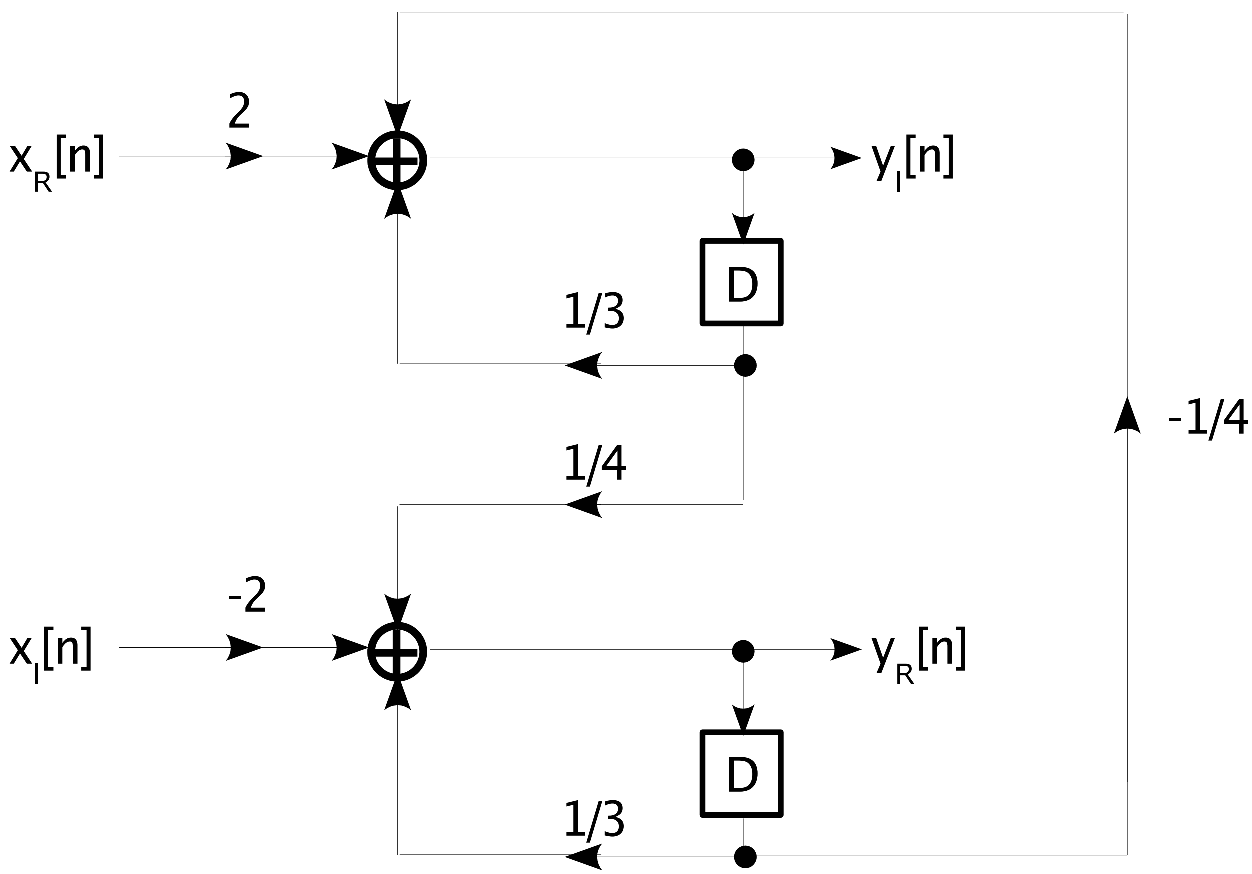


Figure 4: Block diagram