Homework 3 Solutions

(Send your grades to ee120staff@gmail.com. Check the course website for details)

Problem 1 *(Noise suppression system for airplanes, continued.)*

(a) From Homework 2, the impulse response of the noise suppression filter is $g[n] = \frac{2}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$. Thus the frequency response is:

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} = \frac{2}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}.$$

(b) See Figure 1.

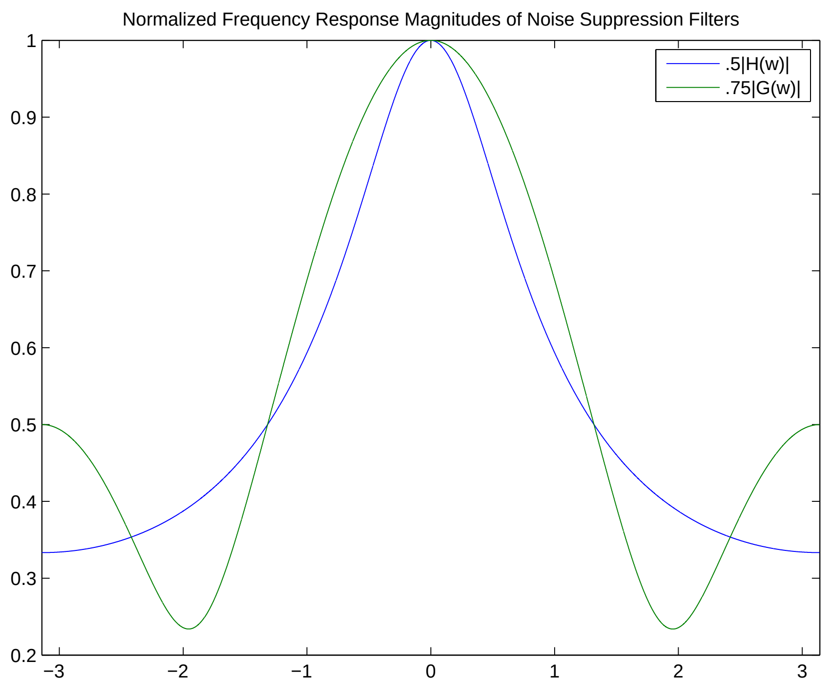

Figure 1: Problem 1b.

Problem 2 *(Frequency responses.)*

The output of an LTI system when the input is a linear combination of complex exponentials has a simple form:

$$e^{j\omega t} * h(t) = H(j\omega)e^{j\omega}, \quad H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

(a) $H(j\omega) = \frac{1}{j\omega}, x(t) = 2e^{j2t} - \cos(-\pi t) = 2e^{j2t} - \frac{e^{j\pi t}}{2} - \frac{e^{-j\pi t}}{2}$

$$\Rightarrow y(t) = x(t) * y(t) = 2H(j2)e^{j2t} - H(j\pi)\frac{e^{j\pi t}}{2} - H(-j\pi)\frac{e^{-j\pi t}}{2}$$
$$= -je^{j2t} - \frac{1}{j2\pi}e^{j\pi t} + \frac{1}{j2\pi}e^{-j\pi t} = -je^{j2t} - \frac{1}{\pi}sin(\pi t)$$

(b) In order to take advantage of the Eigenfunction property, we need to write $x(t)$ as a linear combination of complex exponentials (Fourier Series expansion). $x(t)$ is periodic with fundamental period $T = 10^{-4}s$.

$$\Rightarrow \omega_0 = \frac{2\pi}{T} = 2\pi * 10000 \approx 6.28 \times 10^4$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{\omega_0}{\pi}\int_0^{T_d} e^{-jk\omega_0 t}dt$$
$$= -\frac{e^{-jk\omega_0 t}}{jk\pi}|_{t=0}^{t=T_d}$$
$$a_k = \begin{cases} \frac{\omega_0 T_d}{\pi} = \frac{1}{2} & \text{if } k = 0 \\ \frac{1-e^{-jk\omega_0 T_d}}{jk\pi} = \frac{1-e^{-jk\frac{\pi}{2}}}{jk\pi} & \text{otherwise} \end{cases}$$

Notice that $a_k = a_{-k}^*$, which is what we expect because $x(t)$ is a real signal. Also, $H(j\omega)$ rejects all frequencies $\omega \geqslant 1.5 \times 10^5 rad/s$. Therefore, all harmonics $|k| > 2$ will be gone.

$$H(j\omega) = \begin{cases} 8(1 - \frac{|\omega|}{150000}) & \text{if } |\omega| \leq 150000 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = a_{-2}H(-j2\omega_0)e^{-j2\omega_0 t} + a_{-1}H(-j\omega_0)e^{-j\omega_0 t} + a_0 H(j0) + a_1 H(j\omega_0)e^{j\omega_0 t} + a_2 H(j2\omega_0)e^{j2\omega_0 t}$$
$$H(j0) = 8,\ H(-j\omega_0) = H(j\omega_0) \approx 4.649,\ H(-j2\omega_0) = H(j2\omega_0) \approx 1.298$$
$$a_0 = \frac{1}{2},\ a_1 = \frac{1}{\pi}(1-j) = \sqrt{2}e^{-j\frac{\pi}{4}},\ a_{-1} = \frac{1}{\pi}(1+j) = \sqrt{2}e^{j\frac{\pi}{4}},\ a_2 = \frac{-j}{\pi},\ a_{-2} = \frac{j}{\pi}$$
$$\Rightarrow y(t) = a_0 H(j0) + H(j\omega_0)(a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t}) + H(j2\omega_0)(a_2 e^{2j\omega_0 t} + a_2^* e^{-j2\omega_0 t})$$
$$\Rightarrow y(t) \approx 4 + \frac{13.15}{\pi}\cos(\omega_0 - \frac{\pi}{4}) - \frac{2.59}{\pi}\sin(2\omega_0 t)$$

(c) $h[n] = (\frac{1}{3})^n u[n], x[n] = 3e^{j\frac{3\pi}{4}(n-2)} - \sin(\frac{5\pi}{4}n)$. First we need to find the frequency response $H(e^{j\omega})$:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jk\omega} = \sum_{k=0}^{\infty}(\frac{1}{3})^k e^{-jk\omega} = \sum_{k=0}^{\infty}(\frac{1}{3}e^{-j\omega})^k = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$
$$x[n] = 3e^{-j\frac{3\pi}{2}}e^{j\frac{3\pi}{4}n} + \frac{j}{2}e^{j\frac{5\pi}{4}n} - \frac{j}{2}e^{-j\frac{5\pi}{4}n} = j(3e^{j\frac{3\pi}{4}n} + \frac{1}{2}e^{j\frac{5\pi}{4}n} - \frac{1}{2}e^{-j\frac{5\pi}{4}n})$$
$$\Rightarrow y[n] = j((\frac{3}{1-\frac{1}{3}e^{-j\frac{3\pi}{4}}})e^{j\frac{3\pi}{4}n} + (\frac{\frac{1}{2}}{1-\frac{1}{3}e^{-j\frac{5\pi}{4}}})e^{j\frac{5\pi}{4}n} - (\frac{\frac{1}{2}}{1-\frac{1}{3}e^{j\frac{5\pi}{4}}})e^{-j\frac{5\pi}{4}n})$$

$$= j((\frac{3}{1+\frac{1}{3}e^{j\frac{\pi}{4}}})e^{j\frac{3\pi}{4}n} + (\frac{\frac{1}{2}}{1+\frac{1}{3}e^{-j\frac{\pi}{4}}})e^{j\frac{5\pi}{4}n} - (\frac{\frac{1}{2}}{1+\frac{1}{3}e^{j\frac{\pi}{4}}})e^{-j\frac{5\pi}{4}n})$$

Problem 3 *(Continuous-time Fourier series.)*

(a) $x(t)$ is periodic with period $T = 3$ and fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$, and over one period is defined as

$$x(t) = \begin{cases} 2, & 0 < t \le 1 \\ 1, & 1 < t \le 2 \\ 0, & 2 < t \le 3 \end{cases}.$$

The Fourier series coefficients of $x(t)$ are

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{3}\int_0^3 x(t)dt = 1,$$

and for $k \neq 0$,

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{3}\int_0^1 2e^{-jk\frac{2\pi}{3}t}dt + \frac{1}{3}\int_1^2 e^{-jk\frac{2\pi}{3}t}dt \\
&= \frac{1}{-jk\pi}\left(e^{-jk\frac{2\pi}{3}} - 1\right) + \frac{1}{-jk2\pi}\left(e^{-jk\frac{4\pi}{3}} - e^{-jk\frac{2\pi}{3}}\right) \\
&= \frac{1}{-jk2\pi}\left(\left(e^{-jk\frac{2\pi}{3}} - 1\right) + \left(e^{-jk\frac{4\pi}{3}} - 1\right)\right) \\
&= \frac{1}{-jk2\pi}\left(e^{-jk\frac{\pi}{3}}\left(e^{-jk\frac{\pi}{3}} - e^{jk\frac{\pi}{3}}\right) + e^{-jk\frac{2\pi}{3}}\left(e^{-jk\frac{2\pi}{3}} - e^{jk\frac{2\pi}{3}}\right)\right) \\
&= \frac{e^{-jk\pi/3}\sin(k\pi/3) + e^{-jk2\pi/3}\sin(k2\pi/3)}{k\pi}.
\end{aligned}$$

Now $y(t) = x(3t)$ is periodic with $T = 1$ and $\omega_0 = 2\pi$. By the time scaling property of the CTFS, $y(t)$ has FS coefficients $b_k = a_k$. Note however that $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{3}t}$ and $y(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t}$ have different fundamental frequencies.

(b) $x(t)$ is periodic with $T = 4$ and $\omega_0 = \pi/2$. Example 3.5 on page 193 of OWN shows that a periodic square wave defined over one period as

$$y(t) = \begin{cases} \frac{1}{2}, & |t| < \frac{1}{4} \\ 0, & \frac{1}{4} < |t| < 2 \end{cases}$$

has FS coefficients $b_k = \frac{\sin(k\pi/8)}{2k\pi}$. Since $a_k = (-1)^k \frac{\sin(k\pi/8)}{2k\pi} = b_k e^{j\pi k}$, by the time shifting property of the CTFS, $x(t) = y(t+2)$. Thus $x(t)$ is a period square wave defined over one period as

$$x(t) = \begin{cases} \frac{1}{2}, & 7/4 < t < 9/4 \\ 0, & 0 < t < 7/4 \text{ and } 9/4 < t < 4 \end{cases}.$$

(c) Let $x(t)$ be a periodic signal with fundamental period T and FS coefficients a_k. By the time shifting property of the CTFS, the FS coefficients of $x(t-t_0)$ are $b_k = a_k e^{-jk\frac{2\pi}{T}t_0}$. Similarly, the FS coefficients of $x(t+t_0)$ are $c_k = a_k e^{jk\frac{2\pi}{T}t_0}$. Therefore, the FS coefficients of $x(t-t_0) + x(t+t_0)$ are

$$d_k = b_k + c_k = \left(e^{-jk\frac{2\pi}{T}t_0} + e^{jk\frac{2\pi}{T}t_0}\right)a_k = 2\cos(k2\pi t_0/T)a_k.$$

Problem 4 *(CTFS Properties.)*

OWN 3.42

$x(t)$ is a real-valued signal with fundamental period T and Fourier Series Coefficients a_k. we need to show the following:

(a) $a_k = a^*_{-k}$ and a_0 is real.

From the definition, $a_0 = \frac{1}{T}\int_T x(t)dt$. Since $x(t)$ is real, the integral can only be real.

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \quad \text{taking the complex conjugate of both sides}$$

$$\Rightarrow a_k^* = \{\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\}^* = \frac{1}{T}\int_T x(t)^* e^{jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_T x(t)e^{jk\omega_0 t}dt = a_{-k}$$

This implies that $\Re e\{a_k\} = \Re e\{a_{-k}\}$ and $\Im m\{a_k\} = -\Im m\{a_{-k}\}$. The real part is even and the imaginary part is odd.

(b) $x(t)$ is even (i.e $x(t) = x(-t)$).

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega t}$$

$$x(t) = x(-t) \Leftrightarrow \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega t}$$

$$\Rightarrow a_k = a_{-k}$$

Therefore, $a_k = a_{-k} = a_k^*$. This is true only if $\Im m\{a_k\} = 0$.

(c) $x(t)$ is odd (i.e $x(t) = -x(-t)$).

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega t}$$

$$x(t) = -x(-t) \Leftrightarrow \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} = \sum_{k=-\infty}^{\infty} -a_k e^{-jk\omega t}$$

$$\Rightarrow a_k = -a_{-k}$$

Therefore, $a_k = -a_{-k} = -a_k^*$. This is true only if $\Re e\{a_k\} = 0$. Since a_0 cannot be imaginary, it must be 0.

(d) We know that we can write the even part of $x(t)$ as $\frac{x(t)+x(-t)}{2}$.

$$\Rightarrow \frac{x(t)+x(-t)}{2} = \frac{1}{2}(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} + \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega t})$$

$$= \frac{1}{2}(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} + \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega t}) = \frac{1}{2}\sum_{k=-\infty}^{\infty} (a_k + a_{-k}) e^{jk\omega t} = \sum_{k=-\infty}^{\infty} \frac{1}{2}(a_k + a_k^*) e^{jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} \Re e\{a_k\} e^{jk\omega t}$$

(e) We know that we can write the odd part of $x(t)$ as $\frac{x(t)-x(-t)}{2}$.

$$\Rightarrow \frac{x(t)-x(-t)}{2} = \frac{1}{2}(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} - \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega t})$$

$$= \frac{1}{2}(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} - \sum_{k=-\infty}^{\infty} a_{-k} e^{jk\omega t}) = \frac{1}{2}\sum_{k=-\infty}^{\infty} (a_k - a_{-k}) e^{jk\omega t} = \sum_{k=-\infty}^{\infty} \frac{1}{2}(a_k - a_k^*) e^{jk\omega t}$$

$$= \sum_{k=-\infty}^{\infty} j\Im m\{a_k\} e^{jk\omega t}$$

Problem 5 *(CTFS Properties.)*

OWN 3.44

(a) From (1) and (2), $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}, \quad a_{-k} = a_k^*$

(b) From (3), $x(t) = a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t}$.

(b) From (4):

$$x(t) = a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t}$$
$$x(t-3) = -a_1 e^{j\omega_0 t} - a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t}$$
$$x(t-3) = -x(t) \Leftrightarrow a_2 = a_2^* = 0$$
$$\Rightarrow x(t) = a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t}$$

(c) $|x(t)|^2 = x(t)x^*(t) = (a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t})(a_1^* e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}) = 2|a_1|^2 + a_1^2 e^{j2\omega_0 t} + {a_1^*}^2 e^{-j2\omega_0 t}$. When we integrate over a period, the last two terms will disappear.

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{6}\int_{-3}^{3} 2|a_1|^2 dt = 2|a_1|^2 = \frac{1}{2}$$
$$\Rightarrow |a_1| = \frac{1}{2}$$

Therefore, from (5) and (6), $a_1 = a_1^* = \frac{1}{2}$.

$$\Rightarrow x(t) = \frac{1}{2}(e^{j\frac{\pi}{3}t} + e^{-j\frac{\pi}{3}t}) = \cos(\frac{\pi}{3}t)$$
$$\Rightarrow A = 1, \;\; B = \frac{\pi}{3}, \;\; C = 0$$

Problem 6 *(CTFS Properties.)*

(a) $y_1(t) = x(t - \frac{T}{2})$ has Fourier series coefficients b_k. From the time-shifting property, we know that $b_k = a_k e^{-jk\omega_0 \frac{T}{2}} = a_k e^{-jk\pi} = a_k(-1)^k$.

$y_2(t) = Od\{y(t)\} = \frac{y(t)-y(-t)}{2}$ has Fourier series coefficients c_k. From the properties of Fourier series, we know that $c_k = j\Im m\{b_k\} = j(-1)^k \Im m\{a_k\}$. However, this property only holds when the signal is *real*. The question did not specify $x(t)$ to be real. If we assume that $x(t)$ is complex, we can still use the *Time Reversal* property.

$$y_2(t) = Od\{y(t)\} = \frac{y(t) - y(-t)}{2} \Leftrightarrow c_k = \frac{b_k - b_{-k}}{2} = \frac{a_k(-1)^k - a_{-k}(-1)^{-k}}{2} = \frac{1}{2}(-1)^k(a_k - a_{-k})$$

Notice that when $x(t)$ is real, $a_k^* = a_{-k}$, which leads to $a_k - a_{-k} = a_k - a_k^* = 2j\Im m\{a_k\}$.

(b) $y_3(t) = x(\frac{T}{4} - t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}(\frac{T}{4}-t)}$

$$\Rightarrow y_3(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{2}} e^{-jk\frac{2\pi t}{T}}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}(j)^{-k} e^{jk\frac{2\pi t}{T}}$$

Therefore, $y_3(t)$ is periodic with fundamental period T and Fourier series coefficients $d_k = j^{-k}a_{-k}$.

Problem 7 *(DTFS/Frequency responses.)*

OWN 3.16

(a) $x_1[n] = (-1)^n = e^{j\pi n}$. The output $y_1[n] = (x_1 * h)[n] = 0$, since $H(e^{j\pi}) = 0$.

(b) $x_2[n] = 1 + \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$. The DC component e^{0n} disappears while the remaining part $\sin(\frac{3\pi}{8}n + \frac{\pi}{4})$ passes without any distortion. Therefore, $y_2[n] = (x_2 * h)[n] = \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$.

(c) $x_3[n] = \sum_{k=-\infty}^{\infty} (\frac{1}{2})^{n-4k} u[n-4k]$

$$x_3[n-4] = \sum_{k=-\infty}^{\infty} (\frac{1}{2})^{n-4-4k} u[n-4-4k]$$

$$= \sum_{k=-\infty}^{\infty} (\frac{1}{2})^{n-4(k+1)} u[n-4(k+1)] \quad (\text{replace } k \text{ by } m = k+1)$$

$$= \sum_{m=-\infty}^{\infty} (\frac{1}{2})^{n-4m} u[n-4m] = x_3[n]$$

Therefore, $x_3[n]$ is periodic with period $N = 4$.

$$x_3[n] = \sum_{k=0}^{3} a_k e^{jkn\frac{\pi}{2}} = a_0 + a_1 e^{j\frac{n\pi}{2}} + a_2 e^{jn\pi} + a_3 e^{jn\frac{3\pi}{2}}$$

However, notice that $H(e^{j0}) = 0, \quad H(e^{j\frac{\pi}{2}}) = 0, \quad H(e^{j\pi}) = 0, \quad H(e^{j\frac{3\pi}{2}}) = 0$. Therefore, $y_3[n] = (x_3 * h)[n] = 0$ (we don't need to compute the Fourier series coefficients).

Problem 8 *(Discrete-time Fourier series.)*

(a)

(a) $x[n]$ is periodic with $N = 7$ and $\omega_0 = 2\pi/7$. The Fourier series coefficients of $x[n]$ are specified over one period $(0 \leq k \leq 6)$ as $a_0 = \frac{5}{7}$ and

$$\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega n} \\
&= \frac{1}{7} \sum_{n=0}^{4} e^{-jk\omega_0 n} \\
&= \frac{1}{7} \frac{1 - e^{-jk\omega_0 5}}{1 - e^{-jk\omega_0}} \\
&= \frac{1}{7} \frac{e^{-jk\omega_0 \frac{5}{2}}}{e^{-jk\omega_0 \frac{1}{2}}} \frac{\left(e^{jk\omega_0 \frac{5}{2}} - e^{-jk\omega_0 \frac{5}{2}}\right)}{\left(e^{jk\omega_0 \frac{1}{2}} - e^{-jk\omega_0 \frac{1}{2}}\right)} \\
&= \frac{1}{7} e^{-j\frac{4\pi}{7}k} \frac{\sin(\frac{5\pi}{7}k)}{\sin(\frac{\pi}{7}k)}.
\end{aligned}$$

(b) $x[n]$ is periodic with $N = 6$ and $\omega_0 = \pi/3$. The DTFS coefficients of $x[n]$ are specified over one period $(0 \leq k \leq 5)$ as $a_0 = \frac{4}{6}$ and

$$\begin{aligned}
a_k &= \frac{1}{6} \sum_{n=0}^{3} e^{-jk\omega_0 n} \\
&= \frac{1}{6} \frac{1 - e^{-jk\omega_0 4}}{1 - e^{-jk\omega_0}} \\
&= \frac{1}{6} \frac{e^{-jk\omega_0 \frac{4}{2}}}{e^{-jk\omega_0 \frac{1}{2}}} \frac{\left(e^{jk\omega_0 \frac{4}{2}} - e^{-jk\omega_0 \frac{4}{2}}\right)}{\left(e^{jk\omega_0 \frac{1}{2}} - e^{-jk\omega_0 \frac{1}{2}}\right)} \\
&= \frac{1}{6} e^{-j\frac{\pi}{2}k} \frac{\sin(\frac{2\pi}{3}k)}{\sin(\frac{\pi}{6}k)}
\end{aligned}$$

(b)

(a) $x[n]$ is periodic with $N = 8$ and $\omega_0 = \frac{\pi}{4}$, and has DTFS coefficients

$$\begin{aligned}
a_k &= \cos\left(\frac{k\pi}{4}\right) + \sin\left(\frac{k3\pi}{4}\right) \\
&= \frac{1}{2}\left(e^{jk\frac{\pi}{4}} + e^{-jk\frac{\pi}{4}}\right) + \frac{1}{2j}\left(e^{jk\frac{3\pi}{4}} - e^{-jk\frac{3\pi}{4}}\right).
\end{aligned}$$

Now, looking at the synthesis equation for the DTFS, $a_k = \frac{1}{8}\sum_{n=\langle 8 \rangle} x[n] e^{-jk\frac{\pi}{4}n}$, we see that $x[1] = x[-1] = 4$, $x[3] = 4j$, and $x[-3] = -4j$. Thus we can express one period $(0 \leq n \leq 7)$ of $x[n]$ as

$$x[n] = 4\delta[n-1] + 4j\delta[n-3] - 4j\delta[n-5] + 4\delta[n-7].$$

(b) $x[n]$ is periodic with $N=8$, $\omega_0=\frac{\pi}{4}$, and DTFS coefficients $a_k=\sin\left(\frac{k\pi}{3}\right)=\frac{1}{2j}\left(e^{jk\frac{\pi}{3}}-e^{-jk\frac{\pi}{3}}\right)$ for $0\le k\le 6$, and $a_7=0$.

$$\begin{aligned}
x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}\\
&= \frac{1}{2j}\sum_{k=0}^{6}\left(e^{jk\frac{\pi}{3}}e^{jk\frac{\pi}{4}n}-e^{-jk\frac{\pi}{3}}e^{jk\frac{\pi}{4}n}\right)\\
&= \frac{1}{2j}\sum_{k=0}^{6}e^{jk\left(\frac{\pi}{4}n+\frac{\pi}{3}\right)}-\frac{1}{2j}\sum_{k=0}^{6}e^{jk\left(\frac{\pi}{4}n-\frac{\pi}{3}\right)}\\
&= \frac{1}{2j}\frac{\left(1-e^{j7\alpha}\right)}{\left(1-e^{j\alpha}\right)}-\frac{1}{2j}\frac{\left(1-e^{j7\beta}\right)}{\left(1-e^{j\beta}\right)}\\
&= \frac{1}{2j}\frac{e^{j7\alpha/2}}{e^{j\alpha/2}}\frac{\left(e^{-j7\alpha/2}-e^{j7\alpha/2}\right)}{\left(e^{-j\alpha/2}-e^{j\alpha/2}\right)}-\frac{1}{2j}\frac{e^{j7\beta/2}}{e^{j\beta/2}}\frac{\left(e^{-j7\beta/2}-e^{j7\beta/2}\right)}{\left(e^{-j\beta/2}-e^{j\beta/2}\right)}\\
&= \frac{1}{2j}\left[-e^{j\frac{3\pi}{4}n}\frac{\sin(\frac{7}{2}(\frac{\pi}{4}n+\frac{\pi}{3}))}{\sin(\frac{1}{2}(\frac{\pi}{4}n+\frac{\pi}{3}))}+e^{j\frac{3\pi}{4}n}\frac{\sin(\frac{7}{2}(\frac{\pi}{4}n-\frac{\pi}{3}))}{\sin(\frac{1}{2}(\frac{\pi}{4}n-\frac{\pi}{3}))}\right]
\end{aligned}$$

where we denoted $\alpha=\frac{\pi}{4}n+\frac{\pi}{3}$ and $\beta=\frac{\pi}{4}n-\frac{\pi}{3}$.

Problem 9 *(Parseval's Relation.)*

(a) First let's consider the periodic signal $x(t)=\sum_{n=-\infty}^{\infty}f(t-4n)$ and derive its Fourier series coefficients a_k. We will then derive the Fourier series coefficients of $y(t)$ using the convolution property.

$x(t)$ is periodic with fundamental period $T=4$. Therefore, $\omega_0=\frac{2\pi}{T}=\frac{\pi}{2}$.

$$\Rightarrow x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}$$

$$\Rightarrow a_k=\frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt=\frac{-1}{4}\int_{\frac{-1}{2}}^{\frac{1}{2}}e^{-jk\omega_0 t}dt$$

$$=\frac{1}{4jk\omega_0}e^{-jk\omega_0 t}|_{-1/2}^{1/2}=\frac{1}{4jk\omega_0}(e^{-jk\frac{\omega_0}{2}}-e^{jk\frac{\omega_0}{2}})$$

$$=\frac{-1}{2k\omega_0}(\frac{1}{2j}(e^{jk\frac{\omega_0}{2}}-e^{-jk\frac{\omega_0}{2}}))=\frac{-\sin(k\frac{\omega_0}{2})}{2k\omega_0}=\frac{-\sin(k\frac{\pi}{4})}{k\pi}$$

$$\Rightarrow a_k=\begin{cases}\frac{-1}{4} & \text{if } k=0\\ \frac{-\sin(k\frac{\pi}{4})}{k\pi} & \text{otherwise}\end{cases}$$

Let b_k be the Fourier series coefficients of $y(t)$. Since $y(t)$ is periodic with fundamental period T, then we know from the convolution property that $b_k=Ta_k^2$.

$$\Rightarrow y(t)=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t},\quad b_k=\begin{cases}\frac{1}{4} & \text{if } k=0\\ \frac{4\sin^2(k\frac{\pi}{4})}{(k\pi)^2} & \text{otherwise}\end{cases}$$

The power of the signal $y(t)$ is defined as $\frac{1}{T}\int_T|y(t)|^2dt$. From Parseval's Relation, we know that $\frac{1}{T}\int_T|y(t)|^2dt=\sum_{k=-\infty}^{\infty}|b_k|^2$. Therefore, in order to approximate $y(t)$ as a *finite* linear sum of complex exponentials, we need to retain the coefficients that contain most of the power. We also know that the Fourier series coefficients b_k are real, positive and even and strictly decreasing as $|k|$ increases.

$$\Rightarrow \hat{y}(t) = \sum_{k=-M_1}^{M_2} b_k e^{jk\omega_0 t}$$

$$P_y = \frac{1}{T}\int_T |y(t)|^2 dt = \frac{1}{4}\int_{-1}^{1} |g(t)|^2 dt$$

$$= \frac{1}{2}\int_{-1}^{0} (t+1)^2 dt = \frac{1}{2}\frac{(t+1)^3}{3}|_{t=-1}^{t=0} = \frac{1}{6}$$

Therefore, we need to choose M_1 and M_2 such that $\sum_{k=-M}^{M} |b_k|^2 \geq \frac{.9}{6} = 0.15$.

$$b_0^2 = \frac{1}{16} = 0.0625, \;\; b_1^2 = b_{-1}^2 = (\frac{2}{\pi^2})^2 \approx 0.041, \;\; b_2^2 = b_{-2}^2 = (\frac{4}{(2\pi)^2})^2 \approx 0.1$$

Notice that $b_0^2 + b_1^2 + b_{-1}^2 + b_2^2 \geq 0.15$. Also, this sum is minimum (i.e. if we remove any of the terms, the inequaltiy no longer holds).

$$\hat{y}(t) = b_0 + b_1 e^{j\omega_0 t} + b_1 e^{-j\omega_0 t} + b_2 e^{j2\omega_0 t} = b_0 + 2b_1\cos(\omega_0 t) + b_2 e^{j2\omega_0 t}$$

(b) $z(t) = y(t)\cos(20\pi t)$. Since $\cos(20\pi t) = \frac{1}{2}(e^{j40\omega_0 t} + e^{-j40\omega_0 t})$ is also periodic with $T = 4$, then $z(t)$ also has a fundamental period $T = 4$. Let the c_k be the Fourier series coefficients of $z(t)$. Also, let $z(t) = z_1(t) + z_2(t)$, where $z_1(t) = \frac{1}{2}e^{j40\omega_0 t}y(t)$ and $z_2(t) = \frac{1}{2}e^{-j40\omega_0 t}y(t)$. Let r_k and s_k be the Fourier series coefficients of $z_1(t)$ and $z_2(t)$ respectively.

$$r_k = \frac{1}{2}b_{k-40}, \quad s_k = \frac{1}{2}b_{k+40}$$

$$c_k = r_k + s_k$$

Since the FS coefficients of $y(t)$ diminish very quickly, at least one term in this sum will be insignificant.

$$P_z = \frac{1}{T}\int_T |z(t)|^2 dt = \frac{1}{T}\int_T |y(t)|^2 \cos^2(20\pi t) dt$$

$$= \frac{1}{T}\int_T |y(t)|^2 (\frac{1}{2} + \frac{1}{2}\cos(40\pi t))dt = \frac{P_y}{2} + \frac{1}{2T}\int_T |y(t)|^2\cos(40\pi t)dt$$

$$\frac{1}{2T}\int_T |y(t)|^2\cos(40\pi t)dt \approx 0$$

$$P_z = \frac{P_y}{2}$$

The approximation $\frac{1}{2T}\int_T |y(t)|^2\cos(40\pi t)dt \approx 0$ is valid since $\cos(40\pi t)$ varies at a much higher rate than $y(t)$. Because we are scaling the shifted verions of b_k by a factor of $\frac{1}{2}$ in order to get r_k and s_k, we will need twice the terms we used in part (a). In other words $\widehat{z}(t) = \widehat{y}(t)\cos(40\omega_0 t)$.

Problem 10 *(Fourier Series and Gibbs phenomenon - Matlab.)*

(a)

$$\begin{aligned} c_k &= \int_0^1 p(t)e^{-j2\pi kt}dt \\ &= \int_0^{1/2} e^{-j2\pi kt}dt - \int_{1/2}^1 e^{-j2\pi kt}dt \\ &= \frac{1}{-j2\pi k}\left(e^{-j\pi k} - 1 - e^{-j2\pi k} + e^{-j\pi k}\right) \\ &= \frac{1 - e^{-j\pi k}}{j\pi k} \end{aligned}$$

$$c_0 = \int_0^1 p(t)dt = 0$$

(b) The following Matlab code generates Figure 2.

```
function [] = gibbs();
[t10, p10, y10] = FS(10);
[t100, p100, y100] = FS(100);
[t1000, p1000, y1000] = FS(1000);
figure;
plot(t1000,y1000,'g-');
hold on;
stairs(t1000,p1000,'k--');
plot(t100,y100,'k-');
plot(t10,y10,'b-');
title('Fourier series convergence and Gibbs phenomenon');
xlabel('t');
ylabel('p_N(t)');

function [t, p, y] = FS(N)
k = (-N:N);
t = linspace(-.5,.5,20*N+1);
p = (t>=0);
p = 2.*p - 1;
c = (1 - exp(-j*pi.*k))./(j*pi.*k);
c(N+1) = 0; % c_k at k=0
y = zeros(size(t));
for i=1:length(c)
 y = y + c(i)*exp(j*2*pi*k(i).*t);
end
y = real(y);
```

The partial sum approximations at $t = 0$ are $p_N(0) = 0$, which does not agree with the value of the function $p(0) = 1$.

(c) The maximum overshoot stays constant as the number of terms in the partial sum approximation increases, $max|p(t) - p_N(t)| \approx 1.18$.

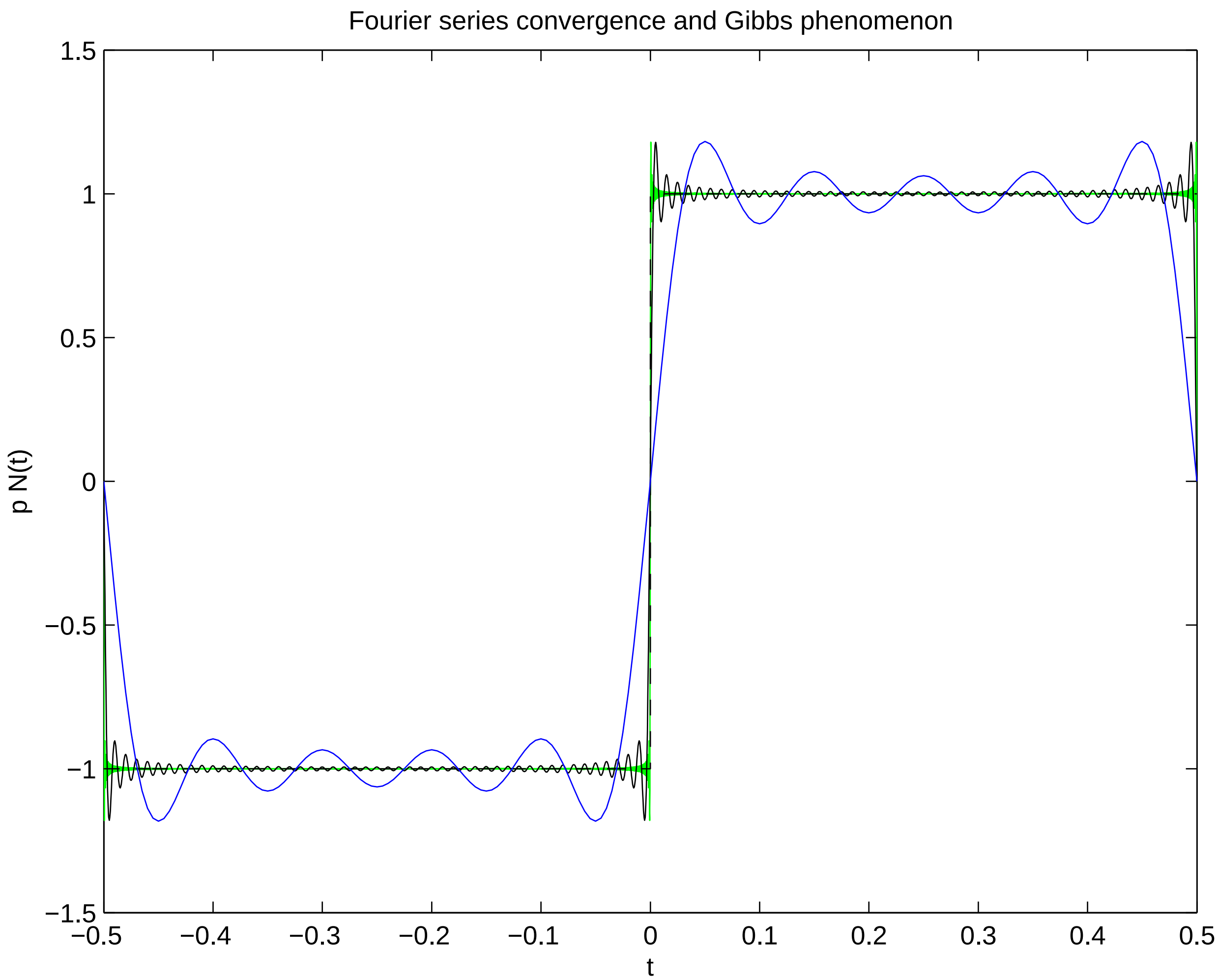

Figure 2: Problem 10b.

(d) As the number of terms in the partial sum approximation increases, the time-locations of the maximum overshoot gets closer and closer to the points of discontinuity at $t = 0, \pm 0.5$.

Problem 11 *(Orthogonality.)*

(i) In order to find an orthormal basis, we follow the Gram-Schmidt algorithm. Since we have four vectors, we will have at most four basis vectors. Lets call them $\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3$, and $\widehat{\beta}_4$.

$$\widehat{\beta}_1 = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \frac{\overrightarrow{v_1}}{\sqrt{46}} = [0.1474 \quad 0.5898 \quad 0.2949 \quad 0 \quad 0.7372]^\top$$

$$\widehat{\beta}_2 = \frac{\overrightarrow{v_2} - (\overrightarrow{v_2}^\top \widehat{\beta}_1)\widehat{\beta}_1}{\|\overrightarrow{v_2} - (\overrightarrow{v_2}^\top \widehat{\beta}_1)\widehat{\beta}_1\|} = [0.0130 \quad -0.6962 \quad -0.2733 \quad 0 \quad 0.6637]^\top$$

$$\widehat{\beta}_3 = \frac{\overrightarrow{v_3} - (\overrightarrow{v_3}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_3}^\top \widehat{\beta}_1)\widehat{\beta}_1}{\|\overrightarrow{v_3} - (\overrightarrow{v_3}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_3}^\top \widehat{\beta}_1)\widehat{\beta}_1\|} = [0.0090 \quad 0.4009 \quad -0.9151 \quad 0 \quad 0.0435]^\top$$

$$\widehat{\beta}_4 = \frac{\overrightarrow{v_4} - (\overrightarrow{v_4}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{v_4}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_4}^\top \widehat{\beta}_1)\widehat{\beta}_1}{\|\overrightarrow{v_4} - (\overrightarrow{v_4}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{v_4}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_4}^\top \widehat{\beta}_1)\widehat{\beta}_1\|} = [0 \quad 0 \quad 0 \quad 0 \quad 0]^\top$$

Therefore, we have only three basis vectors for $\mathbf{S}$ (not surprising since $\overrightarrow{v_4} = \overrightarrow{v_1} + 2\overrightarrow{v_2}$).

(ii)

$$\overrightarrow{v_1} = w_{11}\widehat{\beta}_1 + w_{12}\widehat{\beta}_2 + w_{13}\widehat{\beta}_3 = (\overrightarrow{v_1}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_1}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_1}^\top \widehat{\beta}_3)\widehat{\beta}_3 = \sqrt{46}\widehat{\beta}_1$$

$$\overrightarrow{v_2} = w_{21}\widehat{\beta}_1 + w_{22}\widehat{\beta}_2 + w_{23}\widehat{\beta}_3 = (\overrightarrow{v_2}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_2}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_2}^\top \widehat{\beta}_3)\widehat{\beta}_3 = 6.1926\widehat{\beta}_1 + 6.6822\widehat{\beta}_2$$

$$\overrightarrow{v_3} = w_{31}\widehat{\beta}_1 + w_{32}\widehat{\beta}_2 + w_{33}\widehat{\beta}_3 = (\overrightarrow{v_3}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_3}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_3}^\top \widehat{\beta}_3)\widehat{\beta}_3 = 12.0902\widehat{\beta}_1 + 10.0461\widehat{\beta}_2 + 9.6386\widehat{\beta}_3$$

$$\overrightarrow{v_4} = w_{41}\widehat{\beta}_1 + w_{42}\widehat{\beta}_2 + w_{43}\widehat{\beta}_3 = (\overrightarrow{v_4}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_4}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_4}^\top \widehat{\beta}_3)\widehat{\beta}_3 = 19.1675\widehat{\beta}_1 + 13.3645\widehat{\beta}_2$$

Note: the answer to this problem is *not* unique. However, all answers must satisfy the following:

$$\widehat{\beta}_i^\top \widehat{\beta}_j = \delta[i-j] \quad i,j = 1,2,3$$
$$\overrightarrow{v_i} - w_{i1}\widehat{\beta}_1 + w_{i2}\widehat{\beta}_2 + w_{i3}\widehat{\beta}_3 = 0 \quad i = 1,2,3,4$$

Also, if you are using matlab, you probably won't be getting the answers to be exactly what you expect due to finite precision.

Problem 12 *(Projections.)*

(i) All vectors in $\mathbf{S}^\perp$ must be orthogonal to every vector in $\mathbf{S}$:

$$\overrightarrow{x} \in \mathbf{S}^\perp \Leftrightarrow \overrightarrow{x}^\top \widehat{\beta}_i = 0 \quad \forall_{i=1,2,3}$$

Since $\mathbf{S}$ has rank 3, then the rank of $\mathbf{S}^\perp$ is $5-3=2$. Therefore, we need to find two basis vectors $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$. If we take any random vector $\overrightarrow{y}$ and project it onto $\mathbf{S}$ to get $\overrightarrow{\hat{y}}$, then we know that the error vector $\overrightarrow{y} - \overrightarrow{\hat{y}}$ will be orthogonal to $\mathbf{S}$ (Orthogonality Principal). Since $\mathbf{S}$ is a "very thin slice" of $\mathbb{R}^5$, any vector we choose at random will most likely NOT be in $\mathbf{S}$. Since we are going to do a projection anyway, let's choose $\overrightarrow{b} = [1 \quad -1 \quad 4 \quad 7 \quad -7]^\top$.

$$\overrightarrow{\hat{b}} = \overrightarrow{b} - (\overrightarrow{b}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{b}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{b}^\top \widehat{\beta}_1)\widehat{\beta}_1 = [-0.7568 \quad -0.8536 \quad 4.0569 \quad 0 \quad -6.7885]^\top$$

$$\overrightarrow{b_e} \perp \mathbf{S} = \overrightarrow{b} - \overrightarrow{\hat{b}} = [1.7568 \quad -0.1464 \quad -0.0569 \quad 7.0000 \quad -0.2115]^\top$$
$$\Rightarrow \widehat{\alpha}_1 = \frac{\overrightarrow{b_e}}{\|\overrightarrow{b_e}\|} = [0.2433 \quad -0.0203 \quad -0.0079 \quad 0.9693 \quad -0.0293]^\top$$

In order to get the other basis vector $\widehat{\alpha}_2$, we pick another random vector $\overrightarrow{c} = [1 \;\; 2 \;\; 3 \;\; 5 \;\; 40]^\top$. This time however, we need to project $\overrightarrow{c}$ onto the space spanned by $\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3$ and $\widehat{\alpha}_1$.

$$\widehat{\alpha}_2 = \frac{\overrightarrow{c} - (\overrightarrow{c}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{c}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{c}^\top \widehat{\beta}_1)\widehat{\beta}_1 - (\overrightarrow{c}^\top \widehat{\alpha}_1)\widehat{\alpha}_1}{\| \overrightarrow{c} - (\overrightarrow{c}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{c}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{c}^\top \widehat{\beta}_1)\widehat{\beta}_1 - (\overrightarrow{c}^\top \widehat{\alpha}_1)\widehat{\alpha}_1 \|} = [-0.9586 \;\; 0.0799 \;\; 0.0311 \;\; 0.2460 \;\; 0.1154]^\top$$

(ii) (see part *(i)*)

(iii) (see part *(i)*)

(iv) (see part *(i)*) The projection of $\overrightarrow{b}$ onto $\mathbf{S}^\perp$ is simply the error vector:

$$\overrightarrow{b_e} = [1.7568 \;\; -.1464 \;\; -.0569 \;\; 7.0000 \;\; -0.2115]^\top$$