# Homework 4 Solution
## Due: Thursday, September 28, 2006, at 5pm

(*Submit your grades to ee120staff@gmail.com*)

**Problem 1** *(Properties of the CTFS.)*

Note: the homework uses the notation $Z[k]$ to refer to the Fourier series coefficients $c_k$ corresponding to $z(t)$.

- *3.46(a)*

$$\begin{aligned} c_k &= \frac{1}{T}\int_T z(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_T \left(\sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t}\right) e^{-jk\omega_0 t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_n b_m \left(\frac{1}{T}\int_T e^{j(n+m-k)\omega_0 t}dt\right) \end{aligned}$$

For fixed values of $k$ and $n$, if $m = k-n$, then

$$\frac{1}{T}\int_T e^{j(n+m-k)\omega_0 t}dt = \frac{1}{T}\int_T dt = 1$$

If $m \neq k-n$, then set $\ell = n+m-k$ and note that

$$\frac{1}{T}\int_T e^{j(n+m-k)\omega_0 t}dt = \frac{1}{T}\int_T e^{j\ell(2\pi/T)t}dt = \frac{1}{j\ell 2\pi}\left[e^{j\ell(2\pi/T)t}\right]_{t=0}^{T} = \frac{1}{j\ell 2\pi}(1-1) = 0$$

Thus, we see that

$$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

- *3.46(b)* The signal in Figure P3.46(a) has a fundamental period of $T=3$, and $\omega_0 = \frac{2\pi}{3}$.
  We can observe that $x_1(t) = x(t)y(t)$, where $x(t) = \cos(20\pi t)$ and $y(t)$ is a periodic square wave.

$$x(t) = \cos(20\pi t) = \frac{1}{2}e^{j20\pi t} + \frac{1}{2}e^{-j20\pi t} = \frac{1}{2}e^{j30\omega_0 t} + \frac{1}{2}e^{-j30\omega_0 t}$$

This means that $a_{30} = a_{-30} = \frac{1}{2}$ and $a_k = 0$ for all other $k$. We can write this as $a_k = \frac{1}{2}\delta(k-30) + \frac{1}{2}\delta(k+30)$.

By looking at Example 3.5 on page 193 of OWN, we see that the Fourier series coefficients $b_k$ corresponding to $y(t)$ are given by $b_0 = \frac{2}{3}$ and $b_k = \frac{\sin(k\omega_0)}{k\pi}$ when $k \neq 0$. Applying the convolution formula from part *3.46(a)*, we find that

$$c_k = \frac{\sin((k-30)2\pi/3)}{2(k-30)\pi} + \frac{\sin((k+30)2\pi/3)}{2(k+30)\pi} \qquad k \neq 30, -30$$

$$c_{30} = c_{-30} = \frac{1}{3}$$

- *3.46(c)* Suppose that $y(t) = x^*(t)$. Then $z(t) = x(t)y(t) = |x(t)|^2$. From problem *3.42(a)* (Problem 4 of Homework 3), we know that $b_k = a^*_{-k}$. Using *3.46(a)*, we find that

  $$c_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n} = \sum_{n=-\infty}^{\infty} a_n a^*_{n-k}$$

  From the Fourier representation of $z(t)$, we have

  $$c_k = \frac{1}{T}\int_0^T |x(t)|^2 e^{-j\omega_0 kt} dt = \sum_{n=-\infty}^{\infty} a_n a^*_{n-k}$$

  Evaluating this equation at $k = 0$, we get

  $$\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |a_n|^2$$

**Problem 2** *(Properties of the DTFS.)*

- *(a)* OWN 3.48(a). The Fourier series coefficients of $x[n-n_0]$ can be written as

  $$\begin{aligned}
\hat{a}_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n-n_0]e^{-j2\pi nk/N} \\
&= \frac{1}{N}e^{-j2\pi n_0 k/N} \sum_{n=\langle N\rangle} x[n-n_0]e^{-j2\pi(n-n_0)k/N} \\
&= e^{-j2\pi n_0 k/N}\frac{1}{N}\sum_{m=\langle N\rangle} x[m]e^{-j2\pi(m)k/N} \\
&= e^{-j2\pi n_0 k/N} a_k
\end{aligned}$$

- *b* OWN3.48(e). The Fourier series coefficients of $x^*[-n]$ are given by

  $$\begin{aligned}
\hat{a}_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x^*[-n]e^{-j2\pi nk/N} \\
&= \left(\sum_{n=\langle N\rangle} x[-n]e^{j2\pi nk/N}\right)^* \\
&= \left(\sum_{m=\langle N\rangle} x[m]e^{-j2\pi mk/N}\right)^* \\
&= a_k^*
\end{aligned}$$

**Problem 3** *(Properties of the DTFS.)*

- *(a)* OWN 3.48(f). We first observe that when N is even,

$$(-1)^n x[n] = e^{j\pi n} x[n] = e^{j(N/2)(2\pi/N)n} x[n])$$

Now, we can apply the frequency shifting property of the DTFS from Table 3.2, and we see that the Fourier series coefficients of $(-1)^n x[n]$ are given by

$$\hat{a}_k = a_{k-N/2}$$

- *(b)* OWN 3.48(h). We can write $y[n]$ as

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}(-1)^n x[n]$$

To solve this problem, we must consider two separate cases, when $N$ is even and when $N$ is odd.

Case 1: If $N$ is even, then $(-1)^n x[n]$ also has period $N$. Using the result from 3.48(e), we know that the Fourier series coefficients of $(-1)^n x[n]$ are given by

$$b_k = a_{k-N/2}$$

Finally, using the linearity of the DTFS, the Fourier series coefficients of $y[n]$ are given by

$$\hat{a}_k = \frac{1}{2}(a_k + a_{k-N/2})$$

Case 2: If $N$ is odd, on the other hand, then $(-1)^n x[n]$ will have period $2N$. The Fourier series coefficients of $(-1)^n x[n]$ can be found as

$$\begin{aligned}
b_k &= \frac{1}{2N} \sum_{n=\langle 2N \rangle} e^{j\pi n} x[n] e^{-jk2\pi/(2N)n} \\
&= \frac{1}{2}\frac{1}{N} \sum_{n=1}^{2N} x[n] e^{-j2\pi n((k-N)/2)/N} \\
&= \frac{1}{2}\frac{1}{N} \sum_{n=1}^{N} x[n] \left( e^{-j2\pi n((k-N)/2)/N} + e^{-j2\pi(n+N)((k-N)/2)/N} \right) \\
&= \frac{1}{2}\frac{1}{N} \sum_{n=1}^{N} x[n] \left( e^{-j2\pi n((k-N)/2)/N} + e^{-j2\pi n((k-N)/2)/N} e^{-j2\pi((k-N)/2)} \right)
\end{aligned}$$

In the last step, we have used the fact that the period of $x[n]$ is $N$.

If $k$ is odd, then $k-N$ is even and $(k-N)/2$ is an integer. This means that

$$e^{-j2\pi((k-N)/2)} = 1$$

and that $b_k$ is given by

$$b_k = \frac{1}{2}\frac{1}{N} \sum_{n=1}^{N} x[n] \left( 2e^{-j2\pi n((k-N)/2)/N} \right) = a_{(k-N)/2}$$

If $k$ is even, then $k-N$ is odd, which means that

$$e^{-j2\pi((k-N)/2)} = -1$$

and we see that $b_k = 0$

Finally, using the linearity of the DTFS, the Fourier series coefficients of $y[n]$ are given by

$$\hat{a}_k = \begin{cases} \frac{1}{2}(a_k + a_{(k-N)/2}) & k \ \ odd \\ \frac{1}{2}a_k & k \ \ even \end{cases}$$

**Problem 4** *(FT.)*

- *(a)* OWN 4.22(e)

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\left(\int_{-3}^{-2} -e^{j\omega t}d\omega + \int_{-2}^{-1}(\omega+1)e^{j\omega t}d\omega + \int_{1}^{2}(\omega-1)e^{j\omega t}d\omega + \int_{2}^{3} e^{j\omega t}d\omega\right) \end{aligned}$$

$$\begin{aligned} \int_{-3}^{-2} -e^{j\omega t}d\omega &= \left[\frac{-1}{jt}e^{j\omega t}\right]_{\omega=-3}^{-2} \\ &= -\frac{1}{jt}e^{-j2t} + \frac{1}{jt}e^{-j3t} \end{aligned}$$

$$\begin{aligned} \int_{-2}^{-1}(\omega+1)e^{j\omega t}d\omega &= \left[(\omega+1)\frac{1}{jt}e^{j\omega t}\right]_{\omega=-2}^{-1} - \int_{-2}^{-1}\frac{1}{jt}e^{j\omega t}d\omega \\ &= 0 + \frac{1}{jt}e^{-j2t} + \left[\frac{1}{t^2}e^{j\omega t}\right]_{\omega=-2}^{-1} \\ &= \frac{1}{jt}e^{-j2t} + \frac{1}{t^2}e^{-jt} - \frac{1}{t^2}e^{-j2t} \end{aligned}$$

$$\begin{aligned} \int_{1}^{2}(\omega-1)e^{j\omega t}d\omega &= \left[(\omega-1)\frac{1}{jt}e^{j\omega t}\right]_{\omega=1}^{2} - \int_{1}^{2}\frac{1}{jt}e^{j\omega t}d\omega \\ &= \frac{1}{jt}e^{j2t} - 0 + \left[\frac{1}{t^2}e^{j\omega t}\right]_{\omega=1}^{2} \\ &= \frac{1}{jt}e^{j2t} + \frac{1}{t^2}e^{j2t} - \frac{1}{t^2}e^{jt} \end{aligned}$$

$$\begin{aligned} \int_{2}^{3} e^{j\omega t}d\omega &= \left[\frac{1}{jt}e^{j\omega t}\right]_{\omega=2}^{3} \\ &= \frac{1}{jt}e^{j3t} - \frac{1}{jt}e^{j2t} \end{aligned}$$

$$
\begin{aligned}
x(t) &= \frac{1}{j2\pi t}e^{-j3t} + \frac{1}{j2\pi t}e^{j3t} + \frac{1}{2\pi t^2}e^{-jt} - \frac{1}{2\pi t^2}e^{jt} - \frac{1}{2\pi t^2}e^{-j2t} + \frac{1}{2\pi t^2}e^{j2t} \\
&= \frac{\cos(3t)}{j\pi t} + \frac{\sin(t)}{j\pi t^2} - \frac{\sin(2t)}{j\pi t^2}
\end{aligned}
$$

- *(b)*

OWN 4.23(a) For the given signal $x_0(t)$, the Fourier transform is given by

$$
\begin{aligned}
X_0(j\omega) &= \int_0^1 e^{-t}e^{-j\omega t}dt \\
&= \int_0^1 e^{-(1+j\omega)t}dt \\
&= \left[\frac{-1}{1+j\omega}e^{-(1+j\omega)t}\right]_{t=0}^1 \\
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega}
\end{aligned}
$$

We know that $x_1 t = x_0 t + x_0(-t)$. Using the linearity and time reversal properties of the Fourier transform, we have

$$
\begin{aligned}
X_1(j\omega) &= X_0(j\omega) + X_0(-j\omega) \\
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega} + \frac{1-e^{-(1-j\omega)}}{1-j\omega} \\
&= \frac{(1-e^{-(1+j\omega)})(1-j\omega) + (1-e^{-(1-j\omega)})(1+j\omega)}{1+\omega^2} \\
&= \frac{1-e^{-1}e^{-j\omega} - j\omega + j\omega e^{-1}e^{-j\omega} + 1 - e^{-1}e^{j\omega} + j\omega - j\omega e^{-1}e^{j\omega}}{1+\omega^2} \\
&= \frac{2-2e^{-1}\cos(\omega) + 2\omega e^{-1}\sin(\omega)}{1+\omega^2}
\end{aligned}
$$

OWN 4.23(b) We know that $x_2(t) = x_0(t) - x_0(-t)$. Using the linearity and time reversal properties of the Fourier transform, we have

$$
\begin{aligned}
X_2(j\omega) &= X_0(j\omega) - X_0(-j\omega) \\
&= \frac{1-e^{-(1+j\omega)}}{1+j\omega} - \frac{1-e^{-(1-j\omega)}}{1-j\omega} \\
&= \frac{(1-e^{-(1+j\omega)})(1-j\omega) - (1-e^{-(1-j\omega)})(1+j\omega)}{1+\omega^2} \\
&= \frac{1-e^{-1}e^{-j\omega} - j\omega + j\omega e^{-1}e^{-j\omega} - 1 + e^{-1}e^{j\omega} - j\omega + j\omega e^{-1}e^{j\omega}}{1+\omega^2} \\
&= \frac{2je^{-1}\sin\omega - 2j\omega + 2j\omega e^{-1}\cos(\omega)}{1+\omega^2} \\
&= j\left[\frac{-2\omega + 2e^{-1}\sin(\omega) + 2\omega e^{-1}\cos(\omega)}{1+\omega^2}\right]
\end{aligned}
$$

**Problem 5** *(FT.)*

- *(a)* OWN 4.29, only signals $x_a(t)$ and $x_c(t)$

  We can express the Fourier transform of $x_a(t)$ as

$$\begin{aligned} X_a(j\omega) &= |X_a(j\omega)|e^{j\angle X_a(j\omega)} \\ &= |X(j\omega)|e^{j(\angle X(j\omega)-a\omega)} \\ &= |X(j\omega)|e^{j\angle X(j\omega)-ja\omega)} \\ &= X(j\omega)e^{-ja\omega} \end{aligned}$$

  Using the time shifting property in Table 4.1, we see that $x_a(t) = x(t-a)$.

  Similarly, we can express the Fourier transform of $x_c(t)$ as

$$\begin{aligned} X_c(j\omega) &= |X_c(j\omega)|e^{j\angle X_c(j\omega)} \\ &= |X(j\omega)|e^{-j\angle X(j\omega)} \\ &= X^*(j\omega) \end{aligned}$$

  Using the conjugation and time reversal properties in Table 4.1, we see that $x_c(t) = x^*(-t)$.

- *(d)* OWN 4.41

OWN 4.41(a)

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\left[X(j\omega)\star Y(j\omega)\right]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}X(j\theta)Y(j(\omega-\theta))d\theta\right]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j(\omega-\theta))e^{j\omega t}d\omega\right]d\theta \end{aligned}$$

OWN 4.41(b) Using the frequency shifting property in Table 4.1, we see that the inverse Fourier transform of $Y(j(\omega-\theta))$ is $e^{j\theta t}y(t)$. This means that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(j(\omega-\theta))e^{j\omega t}d\omega = e^{j\theta t}y(t)$$

OWN 4.41(c) Combining the results from parts (a) and (b), we have

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)e^{j\theta t}y(t)d\theta \\ &= y(t)\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\theta)e^{j\theta t}dt \\ &= y(t)x(t) \end{aligned}$$

**Problem 6** *(FT.)*

- *(a)* Using the third to last entry in table 4.2, the frequency response is

$$
\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\
&= 2\frac{\frac{1}{1+j\omega} - \frac{1}{4+j\omega}}{\frac{1}{1+j\omega} + \frac{1}{3+j\omega}} \\
&= 2\frac{(4+j\omega)(3+j\omega) - (1+j\omega)(3+j\omega)}{(4+j\omega)(3+j\omega) + (4+j\omega)(1+j\omega)} \\
&= 2\frac{(3+j\omega)(4+j\omega-1-j\omega)}{(4+j\omega)(3+j\omega+1+j\omega)} \\
&= \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)}
\end{aligned}
$$

- *(b)* First, we take a partial fraction expansion of $H(j\omega)$

$$H(j\omega) = \frac{3(3+j\omega)}{(4+j\omega)(2+j\omega)} = \frac{A}{4+j\omega} + \frac{B}{2+j\omega}$$

$$3(3+j\omega) = A(2+j\omega) + B(4+j\omega)$$

  Setting $\omega = -2/j$, we find that $B = 3/2$. Setting $\omega = -4/j$, we find that $A = 3/2$. Using the same entry of Table 4.2, we find that the inverse Fourier transform of $H(j\omega)$ is

$$h(t) = \frac{3}{2}\left(e^{-4t} + e^{-2t}\right)u(t)$$

- *(c)* From part (a), we have

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{9+3j\omega}{8+6j\omega+(j\omega)^2}$$

  Cross multiplying and taking the inverse Fourier transform, we find that

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 3\frac{dx(t)}{dt} + 9x(t)$$

**Problem 7** *(Frequency response of linear time-invariant systems.)*

- *(a)*

  We are given the equation

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) \quad = \quad \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t)$$

  and we will substitute in $x(t) = e^{j\omega t}$ and $y(t) = H(j\omega)e^{j\omega t}$. First, note that

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}$$

which generalizes to

$$\frac{d^k}{dt^k}e^{j\omega t} = (j\omega)^k e^{j\omega t}$$

We find that

$$\begin{aligned}
\sum_{k=0}^{N} a_k H(j\omega)(j\omega)^k e^{j\omega t} &= \sum_{m=0}^{M} b_m (j\omega)^m e^{j\omega t} \\
\sum_{k=0}^{N} a_k H(j\omega)(j\omega)^k &= \sum_{m=0}^{M} b_m (j\omega)^m \\
H(j\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{k=0}^{N} a_k (j\omega)^k}
\end{aligned}$$

- *(b)*

  We are given the equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

and we will substitute in $x[n] = e^{j\omega n}$ and $y[n] = H(e^{j\omega})e^{j\omega n}$.

$$\begin{aligned}
\sum_{k=0}^{N} a_k H(e^{j\omega}) e^{j\omega(n-k)} &= \sum_{m=0}^{M} b_m e^{j\omega(n-m)} \\
e^{j\omega n}\sum_{k=0}^{N} a_k H(e^{j\omega}) e^{-j\omega k} &= e^{j\omega n}\sum_{m=0}^{M} b_m e^{-j\omega m} \\
H(e^{j\omega}) &= \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}
\end{aligned}$$

**Problem 8** *(Frequency response of linear time-invariant systems (continued).)*

- *(a)*

  Using the result from part (a), we see that for this equation $N = 1$, $a_1 = 2$, $a_0 = 6$, $M = 0$, and $b_0 = 1$. The frequency response is given by

$$H(j\omega) = \frac{1}{6 + 2j\omega} = \frac{0.5}{3 + j\omega}$$

The impulse response can be found by using the basic transform pairs in table 4.2

$$h(t) = \frac{1}{2}e^{-3t}u(t)$$

When the input is

$$x(t) = \sin(t/4) = \frac{1}{2j}\left[e^{jt/4} - e^{-jt/4}\right]$$

then the output is given by

$$\begin{aligned} y(t) &= \frac{1}{2j}\left[H(j(1/4))e^{j(1/4)t} - H(j(-1/4))e^{j(-1/4)t}\right] \\ &= \frac{1}{2j}\left[\frac{1}{6+j0.5}e^{j(1/4)t} - \frac{1}{6-j0.5}e^{j(-1/4)t}\right] \end{aligned}$$

Now, we write $\frac{1}{6+j0.5} = re^{j\theta}$ , where $r = \frac{1}{\sqrt{36.25}}$ and $\theta = -\arctan(1/12)$

$$\begin{aligned} y(t) &= \frac{1}{2j}\left[re^{j\theta}e^{j(1/4)t} - re^{-j\theta}e^{j(-1/4)t}\right] \\ &= \frac{1}{2j}\left[re^{j(t/4+\theta)} - re^{-j(t/4+\theta)}\right] \\ &= r\sin(t/4+\theta) \end{aligned}$$

- *(b)*

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+2e^{-j\omega}-3e^{-2j\omega}+e^{-3j\omega}}{2+3e^{-j\omega}-5e^{-2j\omega}+7e^{-3j\omega}-9e^{-4j\omega}}$$

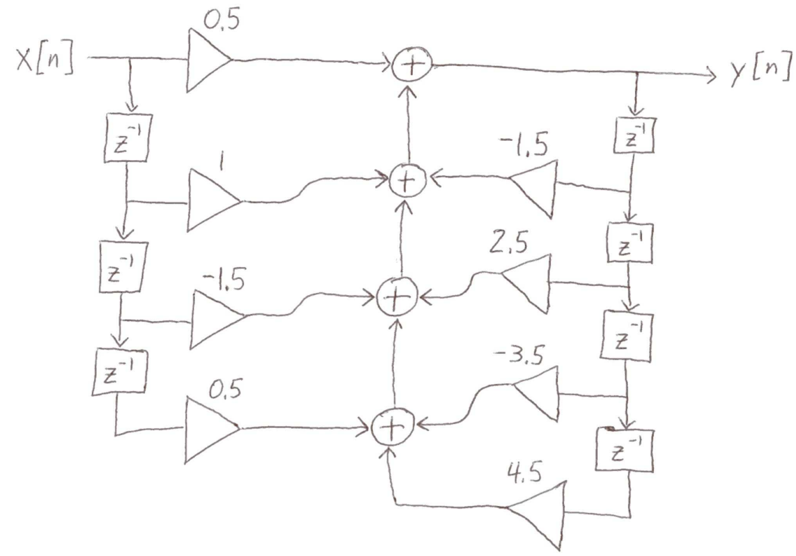
$$Y(e^{j\omega})\left(2+3e^{-j\omega}-5e^{-2j\omega}+7e^{-3j\omega}-9e^{-4j\omega}\right) = X(e^{j\omega})\left(1+2e^{-j\omega}-3e^{-2j\omega}+e^{-3j\omega}\right)$$

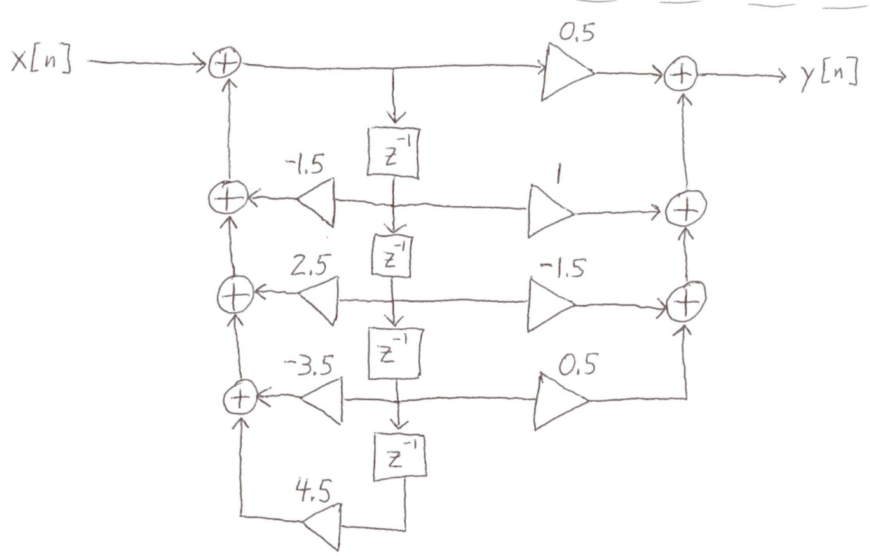
$$2y[n]+3y[n-1]-5y[n-2]+7y[n-3]-9y[n-4] = x[n]+2x[n-1]-3x[n-2]+x[n-3]$$

$$y[n] = 0.5x[n]+x[n-1]-1.5x[n-2]+0.5x[n-3]-1.5y[n-1]+2.5y[n-2]-3.5y[n-3]+4.5y[n-4]$$

This system can be implemented using 7 delay elements. See the block diagram at the top of the following figure.

*Optional*: The system can actually be implemented with only 4 delay elements using the Direct Form II structure. See the bottom block diagram in the following figure.


0.5
x[n]
y[n]
$z^{-1}$
1
-1.5
2.5
-1.5
-3.5
0.5
4.5


x[n]
0.5
y[n]
$z^{-1}$
-1.5
1
2.5
-1.5
-3.5
0.5
4.5