Homework 6 Solutions

Problem 1 *(DTFT.)*

To sketch $W(e^{j\omega})$, we use the multiplication property,

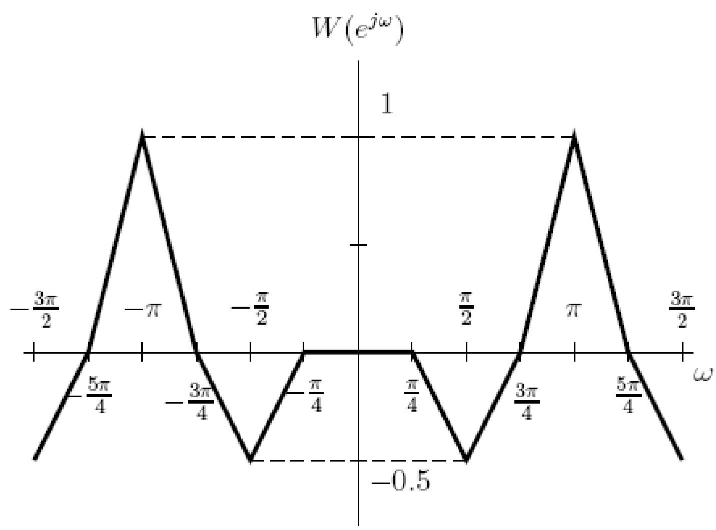
$$w[n] = x[n]p[n] \leftrightarrow W(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) * P(e^{j\omega})$$

$X(e^{j\omega})$ is as shown and

$$P(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \pi - 2\pi l) + \delta(\omega + \pi - 2\pi l)) - \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{2} - 2\pi l) + \delta(\omega + \frac{\pi}{2} - 2\pi l))$$

The area under the impulses at $\omega = \pm\pi \pm 2\pi l$ is equal to 2π not just π because the impulses from the term $\sum_{l=-\infty}^{\infty}(\delta(\omega - \pi - 2\pi l))$ overlap with the impulse from the term $\sum_{l=-\infty}^{\infty}(\delta(\omega + \pi - 2\pi l))$ due to the 2π periodicity of DTFT.

These impulses and the impulses at $\omega = \pm\frac{\pi}{2} \pm 2\pi l$ cause replicas of $X(e^{j\omega})$ that are centered at these frequencies. The result $W(e^{j\omega})$ is shown below:



Problem 2 *(Sampling theorem.)*

- OWN 7.21(a)

  The Nyquist rate for the given signal is $2 \times 5000\pi = 10000\pi$. Therefore, to be able to recover $x(t)$ from $x_p(t)$, the sampling period must be at most $T_{max} = (2\pi)/(10000\pi) = 2 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} < T_{max}$, $x(t)$ can be recovered from $x_p(t)$

- OWN 7.21(b)

  The Nyquist rate for the given signal is $2 \times 15000\pi = 30000\pi$. Therefore, to be able to recover $x(t)$ from $x_p(t)$, the sampling period must be at most $T_{max} = (2\pi)/(30000\pi) = 0.66 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} > T_{max}$, $x(t)$ cannot be recovered from $x_p(t)$

- OWN 7.21(d)

  Since $x(t)$ is real, $X(j\omega)$ is conjugate symmetric. Therefore if $X(j\omega) = 0$ for $\omega > 5000\pi$, $X(j\omega) = 0$ for $\omega < -5000\pi$. The answer to this part is identical to that of part (a).

- OWN 7.21(g)

  If $|X(j\omega)| = 0$ for $\omega > 5000\pi$, then $X(j\omega) = 0$ for $\omega > 5000\pi$. However, the question gives us no information about whether or not there exists some ω_M such that $X(j\omega) = 0$ for $\omega < -\omega_M$. Therefore, we cannot determine whether we can recover $x(t)$ from $x_p(t)$. (Full credit given for answering no, since you can construct an $x(t)$ matching the given properties that cannot be recovered from $x_p(t)$.)

Problem 3 *(Sampling.)*

- OWN 7.23(a) We can express $p(t)$ as $p(t) = p_1(t) - p_1(t - \Delta)$, where

  $$p_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - n(2\Delta))$$

  Now, using Table 4.2, the Fourier transform of $p_1(t)$ is given by

  $$P_1(j\omega) = \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi k}{\Delta}\right)$$

  To find the Fourier transform of $q(t) = p_1(t - \Delta)$, we can find the Fourier series coefficients and use equation 4.22 in OWN. The Fourier series coefficients are given by

  $$a_k = \frac{1}{T}\int_T q(t) e^{-jk(2\pi/T)t} dt = \frac{1}{2\Delta}\int_0^{2\Delta} \delta(t - \Delta) e^{-jk(\pi/\Delta)t} dt = \frac{1}{2\Delta} e^{-jk\pi}$$

  Using equations 4.22 and 4.23, the Fourier transform of $q(t)$ is given by

  $$\sum_{k=-\infty}^{\infty} \frac{\pi}{\Delta} e^{-jk\pi} \delta\left(\omega - \frac{\pi k}{\Delta}\right)$$

  Finally, using the linearity of the Fourier transform, we find that

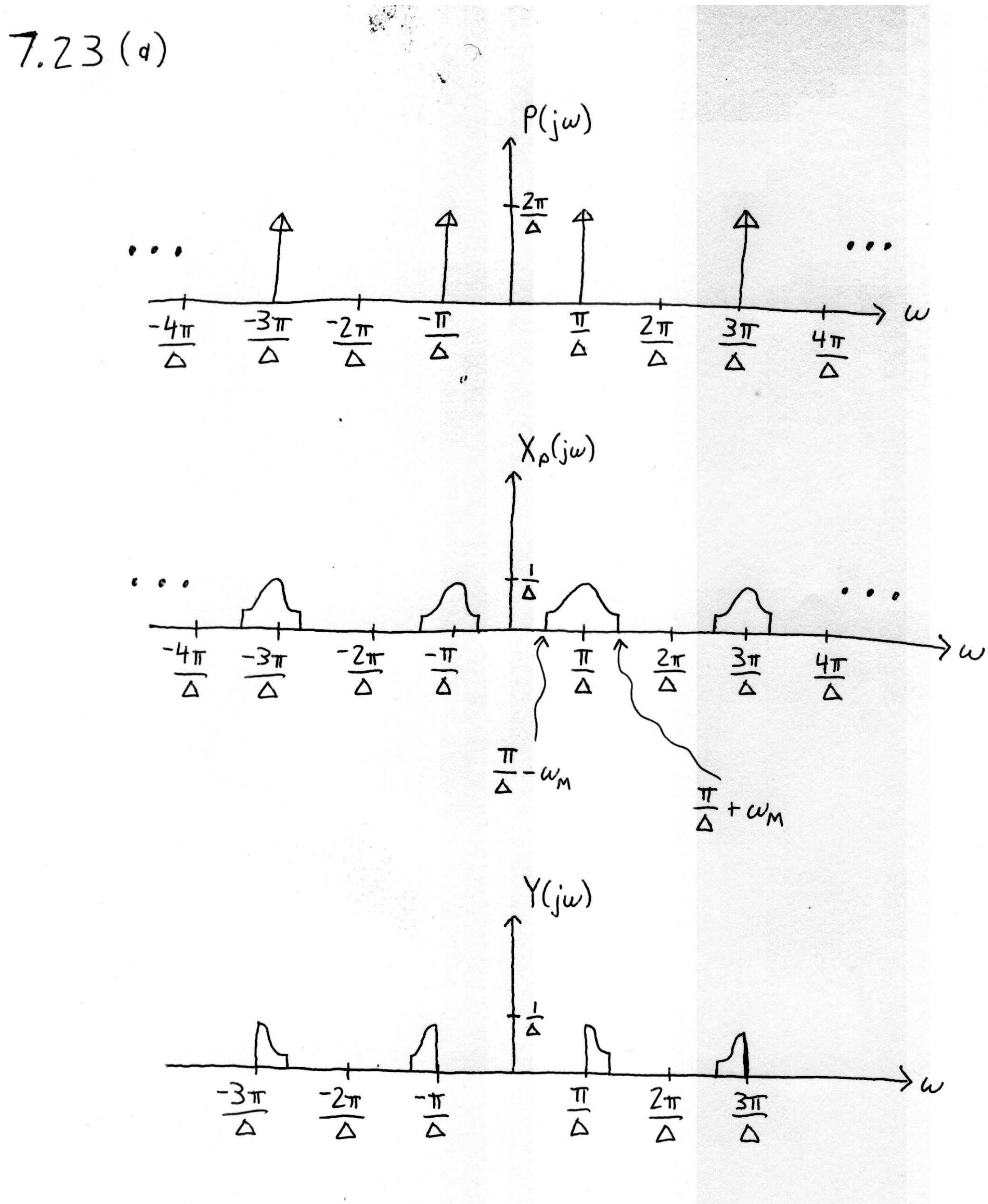
  $$P(j\omega) = \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{k\pi}{\Delta}\right) - \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} e^{-jk\pi} \delta\left(\omega - \frac{k\pi}{\Delta}\right) = \frac{2\pi}{\Delta} \sum_{k\ odd} \delta\left(\omega - \frac{\pi k}{\Delta}\right)$$

  Now, because $x_p(t) = x(t)p(t)$, we know that $X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \star P(j\omega)$

  $$X_p(j\omega) = \frac{1}{\Delta} \sum_{k\ odd} X\left(\omega - \frac{k\pi}{\Delta}\right)$$

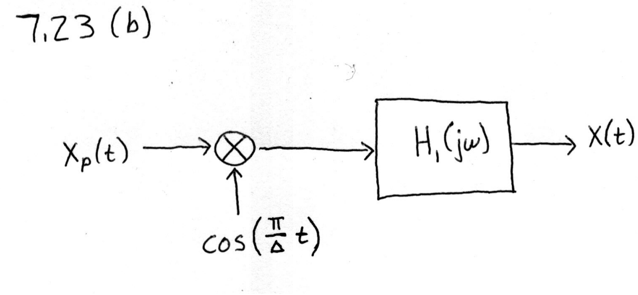
  If $\Delta < \pi/(2\omega_M)$, then $\omega_M < \pi/(2\Delta)$. In this situation, the various copies of $X(j\omega)$ do not overlap in $X_p(j\omega)$. $P(j\omega)$, $X_p(j\omega)$, and $Y(j\omega)$ are shown in the following figure.

7.23 (a)



- OWN 7.23(b)

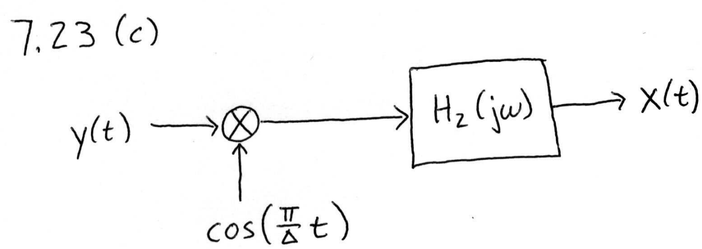
  To recover $x(t)$ from $x_p(t)$, we need to shift one copy of the original spectrum $X(j\omega)$ to the origin, and then filter out all of the other copies of $X(j\omega)$ in $X_p(j\omega)$. One simple way to shift the spectrum is to multiply $x_p(t)$ by $\cos(t\pi/\Delta)$. The following system will recover $x(t)$ from $x_p(t)$



$$H_1(j\omega) = \begin{cases} \Delta & |\omega| < \omega_M \\ 0 & else \end{cases}$$

- OWN 7.23(c)

  To recover $x(t)$ from $y(t)$, we need to shift the two parts of $X(j\omega)$ that are located near π/Δ and $-\pi/\Delta$ back to the origin, and then filter out the components of the spectrum at higher frequencies. The following system will recover $x(t)$ from $y(t)$



$$H_2(j\omega) = \begin{cases} 2\Delta & |\omega| < \omega_M \\ 0 & else \end{cases}$$

- OWN 7.23(d)

  Looking at the plot of $X_p(j\omega)$ in part *(a)*, we see that aliasing is avoided when $\omega_M \leq \pi/\Delta$. Therefore, the maximum value of Δ which allows $x(t)$ to be recovered is $\Delta_{max} = \pi/\omega_M$

Problem 4 *(Discrete-time Processing of Continuous-time Signals.)*

OWN Problem 7.29

The outputs of each block of the overall system for filtering a continuous-time signal using a discrete-time filter are given and plotted below. Note: Following OWN notation, in this problem we use Ω to denote the frequency variable of the discrete-time signal.

$$
\begin{aligned}
x_p(t) &= x_c(t)p(t) \\
X_p(j\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - \frac{2\pi k}{T})) \\
x[n] &= x_c(nT) \\
X(e^{j\Omega}) &= X_p(j\frac{\Omega}{T}) \\
y[n] &= x[n] * h[h] \\
Y(e^{j\Omega}) &= X(e^{j\Omega})H(e^{j\Omega}) \\
y_p(t) &= \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT) \\
Y_p(j\omega) &= Y(e^{j\omega T}) \\
y_c(t) &= y_p(t) * h(t) \\
Y_c(j\omega) &= Y_p(j\omega)H(j\omega)
\end{aligned}
$$

Problem 5 *(Discrete-time Processing of Continuous-time Signals.)*

(a)

Compute $Y_c(j\omega)$ by taking the Fourier transform of both sides of the differential equation for the continuous-time LTI system, then find $y_c(t)$ in OWN Table 4.2.

$$
\begin{aligned}
\frac{dy_c(t)}{dt} + y_c(t) &= x_c(t) = \delta(t) \\
j\omega Y_c(j\omega) + Y_c(j\omega) &= X_c(j\omega) = 1
\end{aligned}
$$

$$
\begin{aligned}
Y_c(j\omega) &= \frac{1}{1 + j\omega} \\
y_c(t) &= e^{-t}u(t)
\end{aligned}
$$

(b)

$$
\begin{aligned}
y[n] &= y_c(nT) = e^{-nT}u[n] \\
Y(e^{j\omega}) &= \frac{1}{1 - e^{-T}e^{-j\omega}} \\
w[n] &= \delta[n] \\
W(e^{j\omega}) &= 1 \\
H(e^{j\omega}) &= \frac{W(e^{j\omega})}{Y(e^{j\omega})} \\
&= 1 - e^{-T}e^{-j\omega} \\
h[n] &= \delta[n] - e^{-T}\delta[n-1]
\end{aligned}
$$

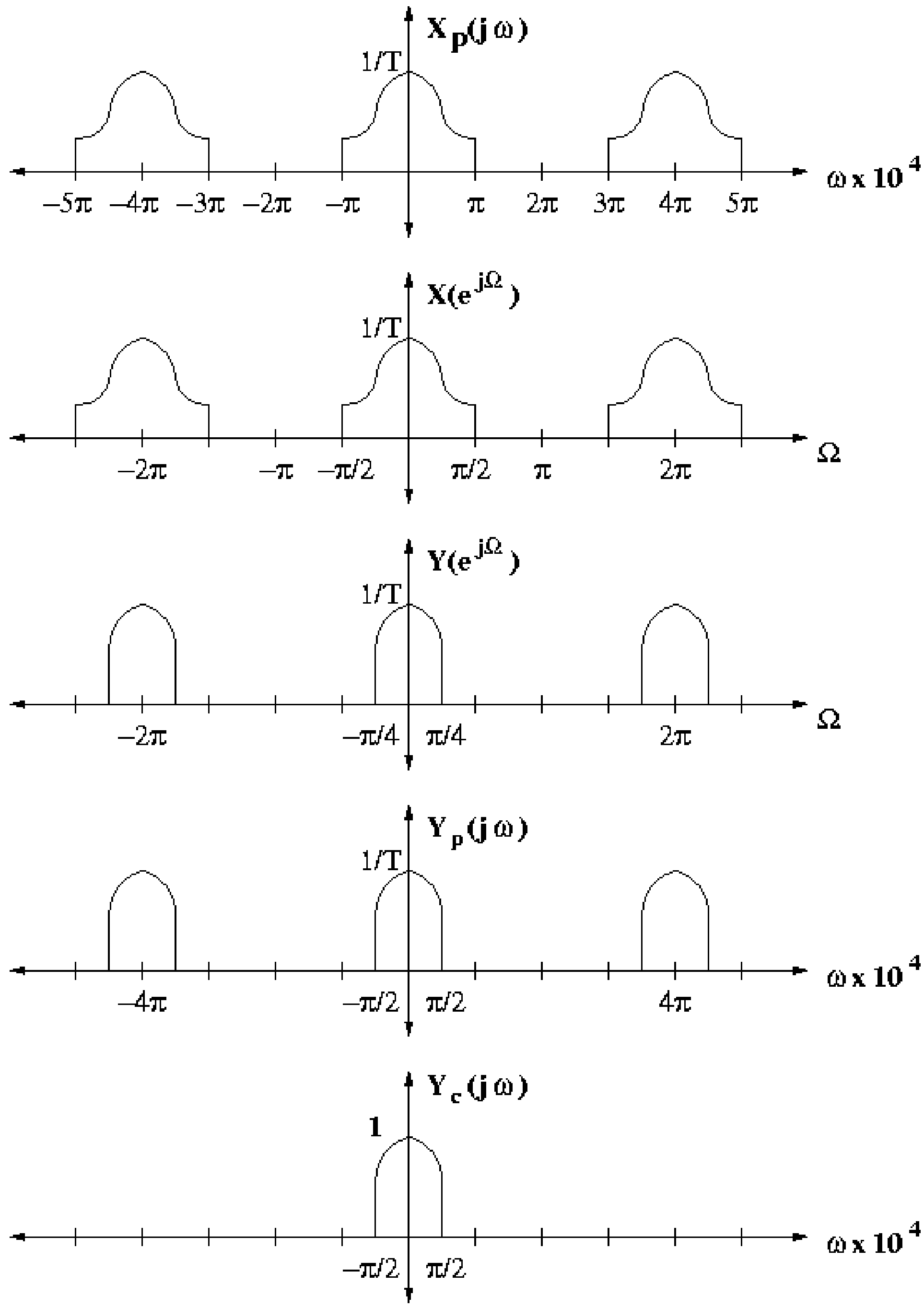


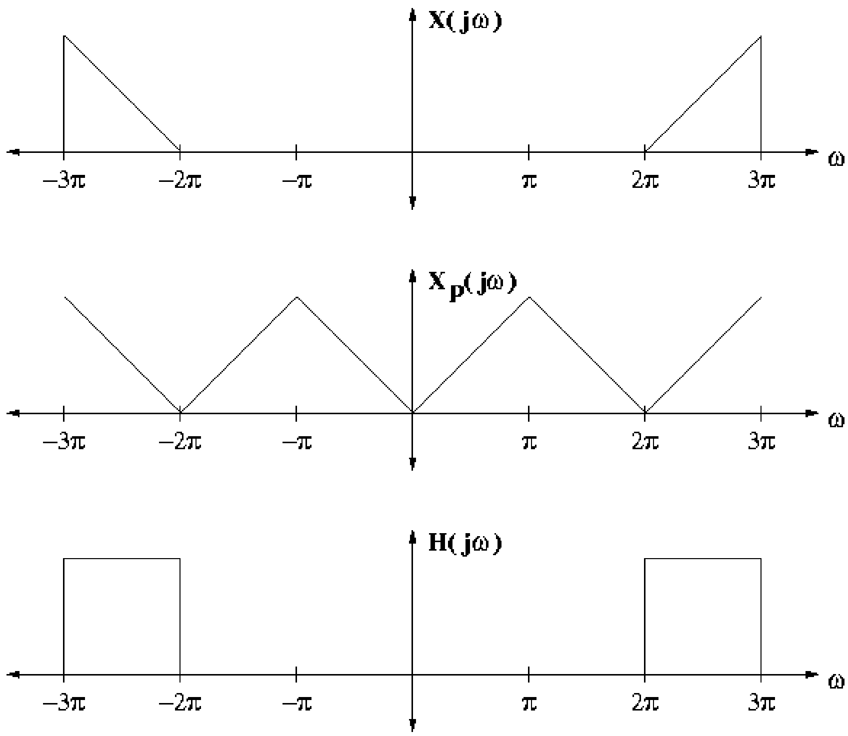
Figure 1: Problem 4.

Problem 6 *(Band-pass Sampling.)*

(a)

$x(t)$ has Nyquist rate 6π. Sampling $x(t)$ at frequency $\omega_s = 2\pi$ produces

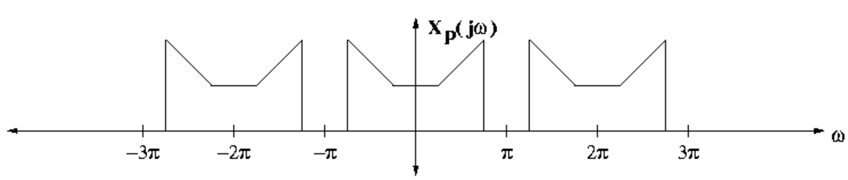
$$X_p(j\omega) = \sum_{k=-\infty}^{\infty} X(j(\omega - 2\pi k)).$$



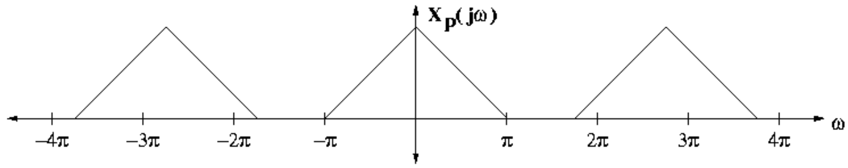
As we can see from the figure above, the shifted replicas of $X(j\omega)$ do not overlap. Since there is no aliasing, we can perfectly reconstruct $x(t)$ by bandpass filtering the sampled signal with $H(j\omega)$.

(b)

$x(t)$ has Nyquist rate $\frac{11\pi}{2}$. If we sample $x(t)$ at frequency $\omega_s = 2\pi$, the shifted replicas of $X(j\omega)$ would overlap and we get aliasing, as seen in the figure below.



The smallest sampling frequency that avoids aliasing is $\omega_s = \frac{11\pi}{4}$. The sampled signal $x_p(t)$ has Fourier transform shown in the figure below.



(c)

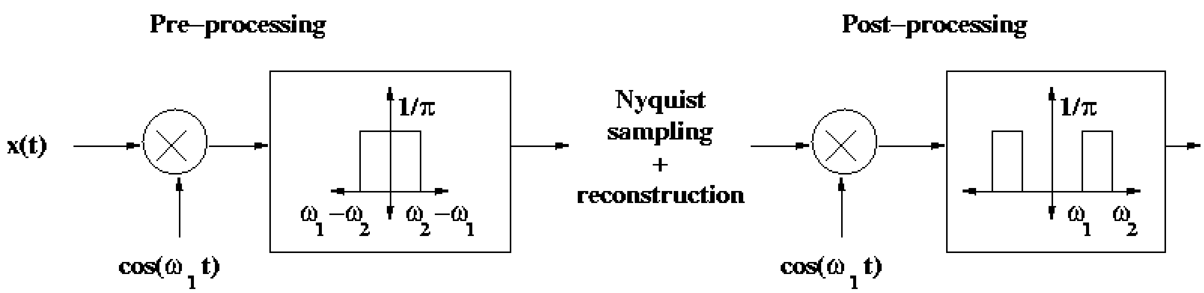
For shifted replicas of $X(j\omega)$ to be nonoverlapping, ω_1 must be an integer multiple of the spectral support,

$$\omega_1 = k(\omega_2 - \omega_1)$$

where k is a nonnegative integer.

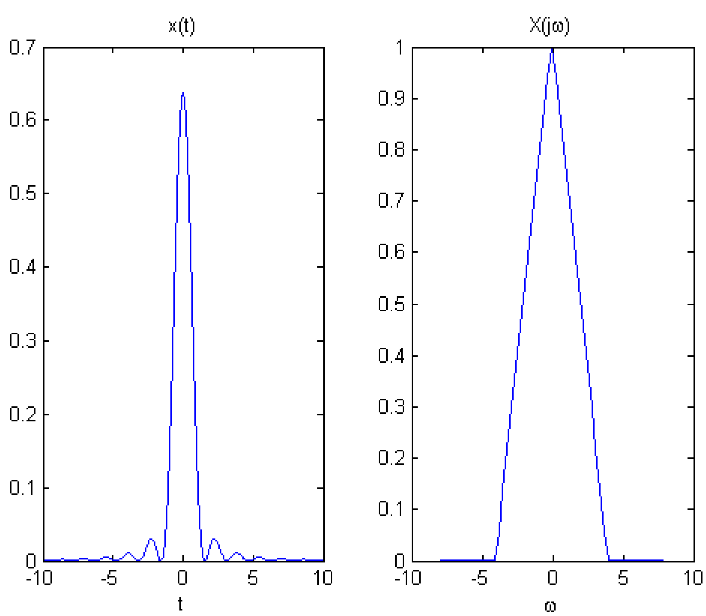
(d)

If we preprocess the signal before sampling by making the bandpass signal into a lowpass signal, then we can sample the lowpass signal at the Nyquist rate, which is twice the spectral support of the bandpass signal. We can then recover the original bandpass signal with reconstruction and postprocessing. See the figure below.

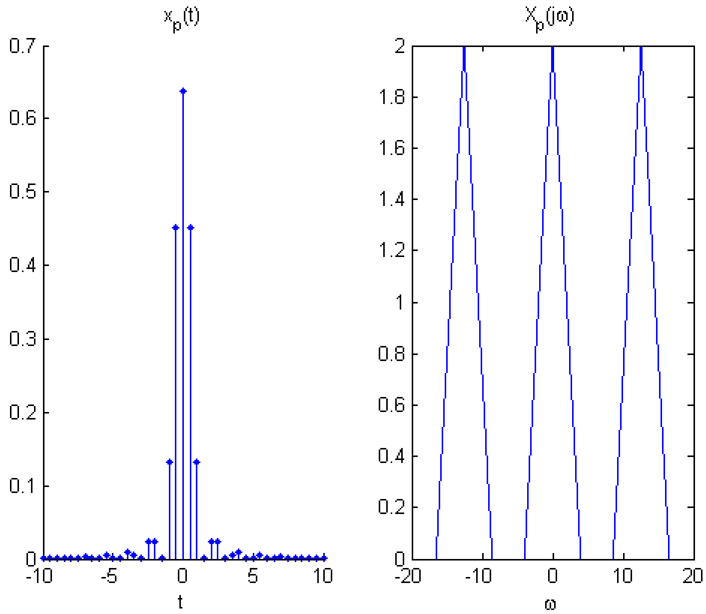


Problem 7 *(Understanding aliasing.)*

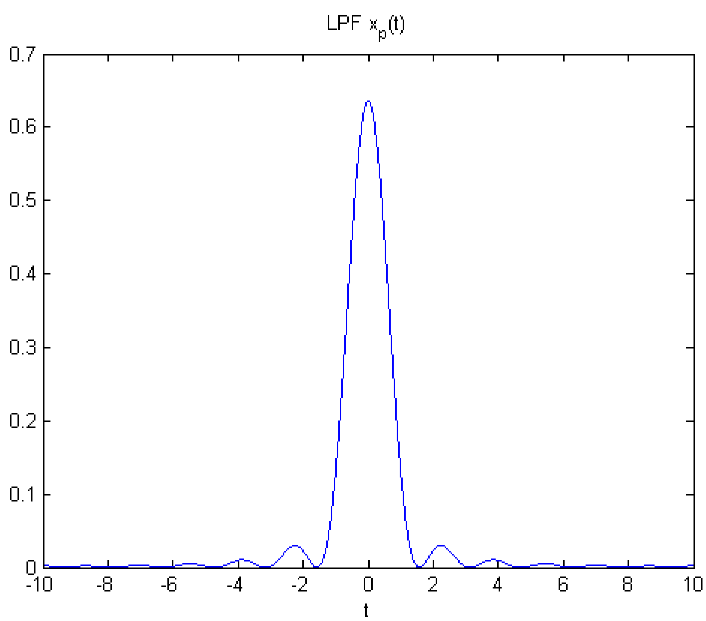
(a)



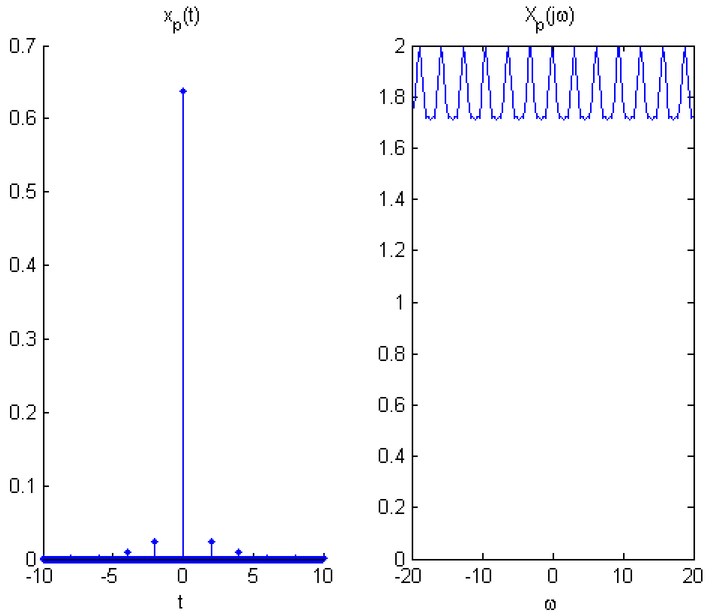
(b)



(c)

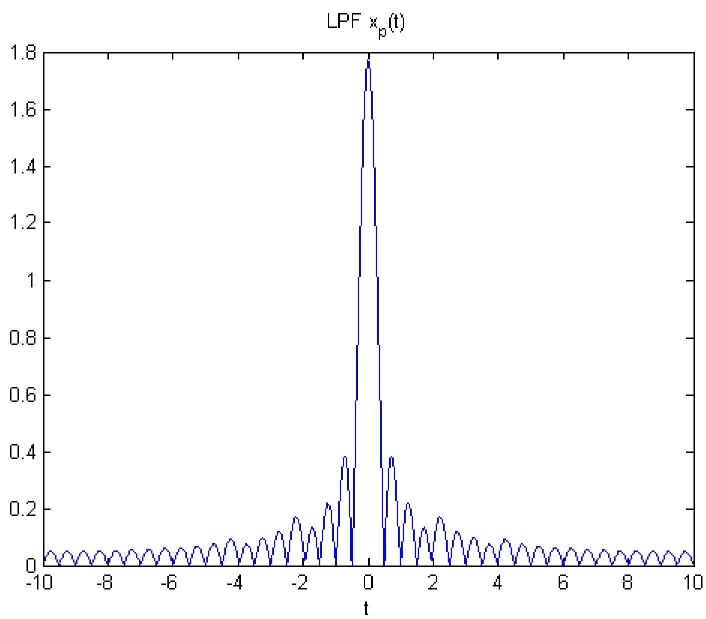


(d)



$x_p(t)$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-10
-5
0
5
10
t
$X_p(j\omega)$
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-20
-10
0
10
20
ω

(e)



LPF $x_p(t)$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-10
-8
-6
-4
-2
0
2
4
6
8
10
t