EECS 120 Signals & Systems
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Homework 7 Solutions

Problem 1 *(Downsampling.)*

OWN Problem 7.35

- (a)

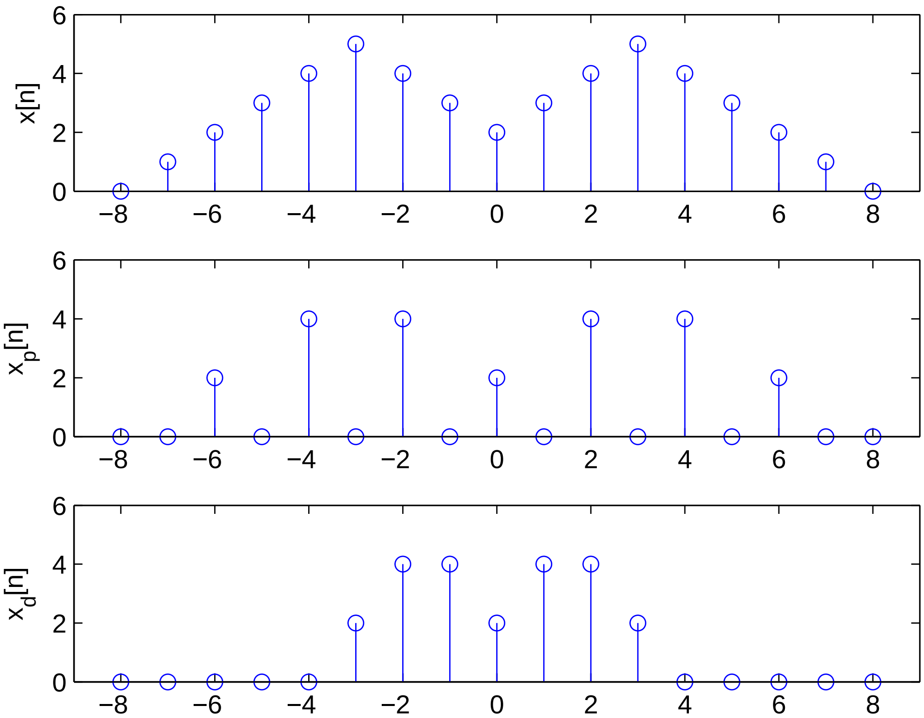


Figure 1: Problem 1 (a)

- (b)

  Discrete-time impulse-train sampling of $x[n]$ generates the signal $x_p[n]$.

$$
\begin{aligned}
x_p[n] &= \begin{cases} x[n], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 1, \pm 3, \ldots \end{cases} \\
&= \sum_{k=-\infty}^{\infty} x[2k]\delta[n-2k] \\
&= x[n]p[n] \\
P(e^{j\omega}) &= \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k) \\
X_p(e^{j\omega}) &= \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta \\
&= \frac{1}{2} \sum_{k=0}^{1} X(e^{j(\omega - \pi k)})
\end{aligned}
$$

Decimation of $x[n]$ generates the signal $x_d[n]$.

$$
\begin{aligned}
x_d[n] &= x[2n] = x_p[2n] \\
X_d(e^{j\omega}) &= X_p(e^{j\frac{\omega}{2}})
\end{aligned}
$$

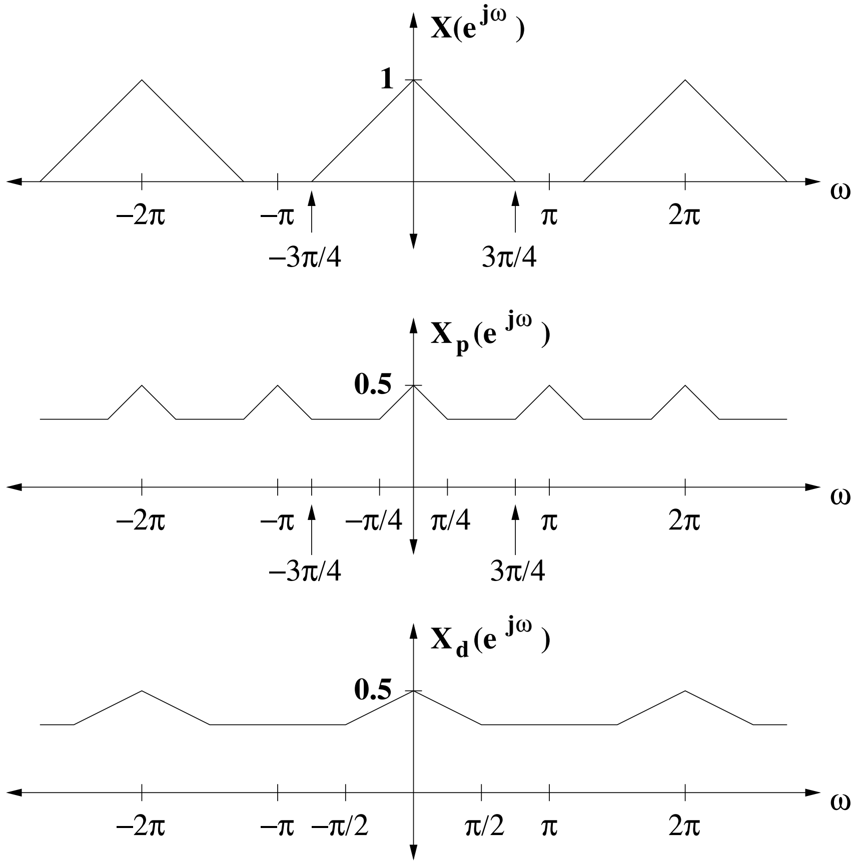


Figure 2: Problem 1 (b)

Problem 2 *(Upsampling.)*

OWN Problem 7.49

- (a)

  Let $x_{d_1}[n]$ and $x_{d_2}[n]$ be two inputs to the system, with corresponding outputs $x_{p_1}[n]$ and $x_{p_2}[n]$. Now, consider an input of the form $x_{d_3}[n] = \alpha_1 x_{d_1}[n] + \alpha_2 x_{d_2}[n]$. With this input, the output of the system will be

$$x_{p_3}[n] = \begin{cases} \alpha_1 x_{d_1}[n/N] + \alpha_2 x_{d_2}[n/N] & n = 0, \pm N, \pm 2N, \ldots \\ 0 & else \end{cases}$$

  Thus, $x_{p_3}[n] = \alpha_1 x_{p_1}[n] + \alpha_2 x_{p_2}[n]$. This means that the system is **linear**.

- (b)

  The system is **not time invariant**. For example, an input $\delta[n]$ gives an output $\delta[n]$, while an input $\delta[n-1]$ gives an output $\delta[n-N]$.

- (c)

$$
\begin{aligned}
X_p(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_p[n]e^{j\omega n} \\
&= \sum_{k=-\infty}^{\infty} x_p[kN]e^{j\omega kN} \\
&= \sum_{k=-\infty}^{\infty} x_d[k]e^{j\omega kN} \\
&= X_d(e^{j\omega N})
\end{aligned}
$$

The plot of $X_p(e^{j\omega})$ is shown in the figure in part (d).

- (d)

  The plot of $X(e^{j\omega})$ is shown in the following figure.

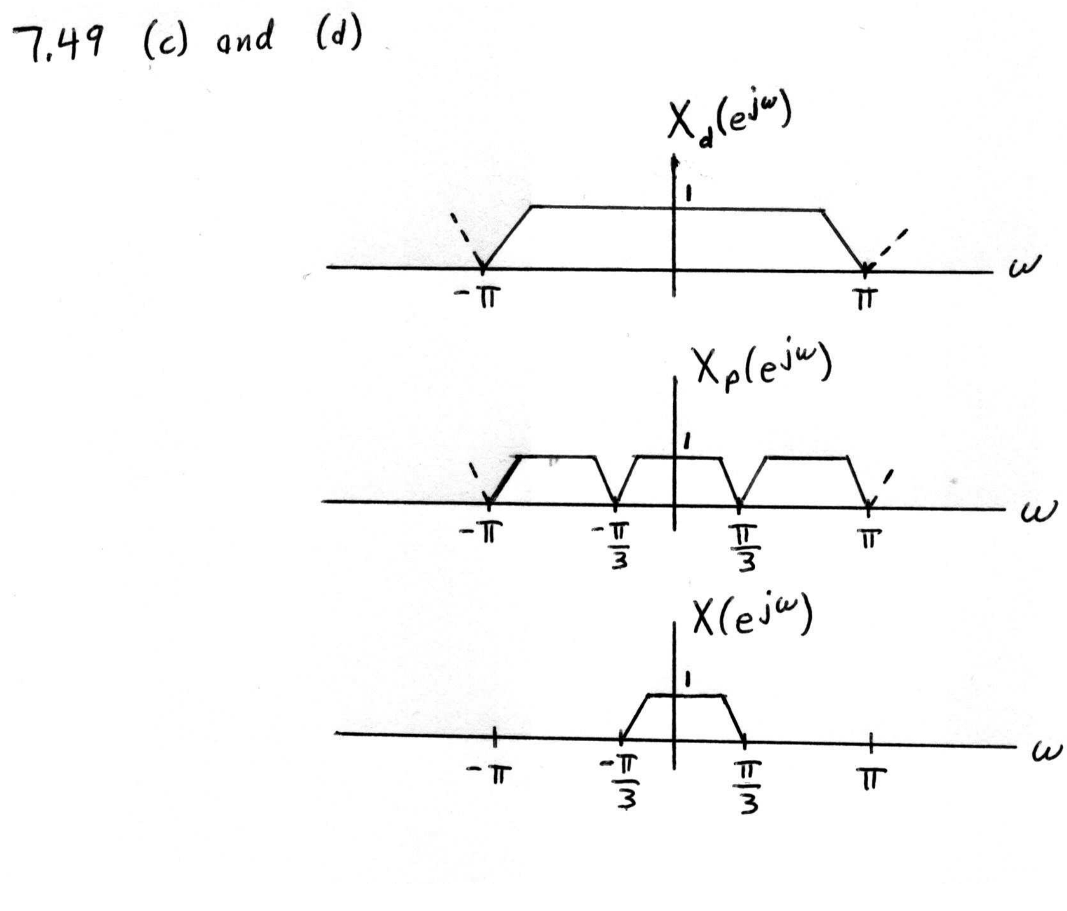


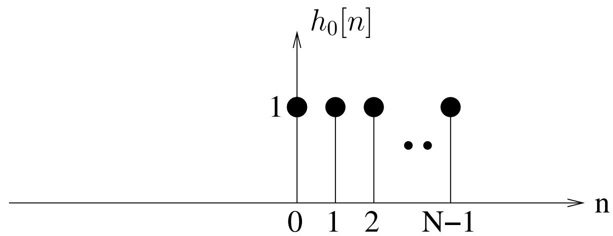
Figure 3: Problem 2 (c) and (d)

Problem 3 *(Zero-Order and First-Order Holds.)*

OWN Problem 7.50

- (a)

$$h_0[n] = \begin{cases} 1 & n = 0, 1, ..., N-1 \\ 0 & else \end{cases}$$



- (b)

The condition $\omega_S = \frac{2\pi}{N} > 2\omega_M$ is equivalent to $\omega_M < \frac{\pi}{N}$. Because

$$X_p(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_S)})$$

the signals $X(e^{j\omega})$ and $X_p(e^{j\omega})$ can be sketched as follows.

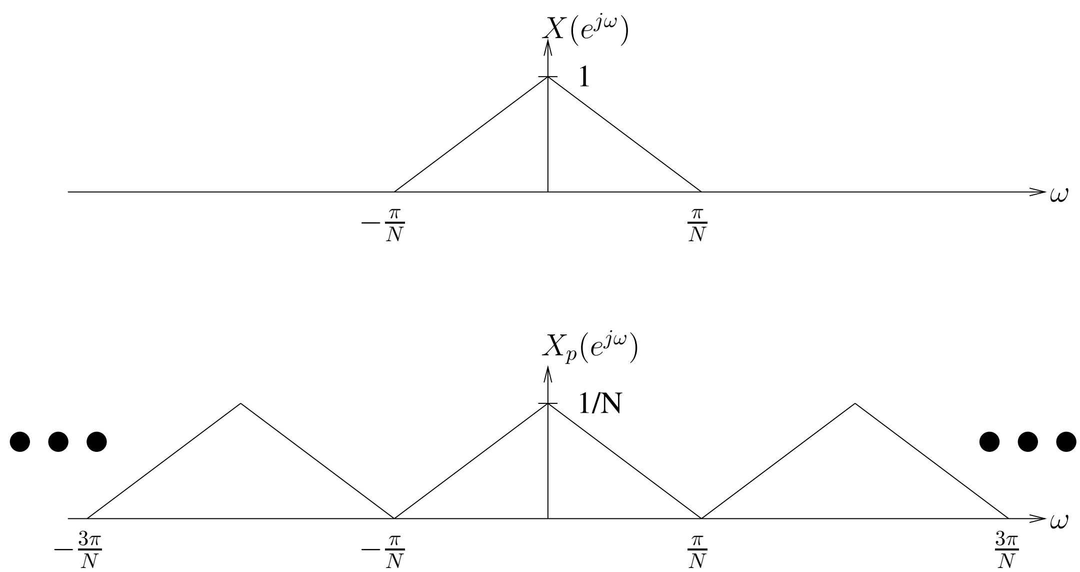


Figure 4: Problem 3 (b)

In order to perfectly recover $x[n]$, we require that

$$H_L(e^{j\omega}) = H_0(e^{j\omega})H(e^{j\omega}) = \begin{cases} N & |\omega| < \pi/N \\ 0 & else \end{cases}$$

The frequency response of the ZOH is given by

$$H_0(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = e^{(-j\omega(N-1)/2)} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Therefore,

$$H(e^{j\omega}) = \frac{H_L(e^{j\omega})}{H_0(e^{j\omega})} = \begin{cases} Ne^{(j\omega(N-1)/2)} \cdot \frac{\sin(\omega/2)}{\sin(\omega N/2)} & |\omega| < \pi/N \\ 0 & else \end{cases}$$

A plot of $|H(e^{j\omega})|$ for $N=2$ is given below.

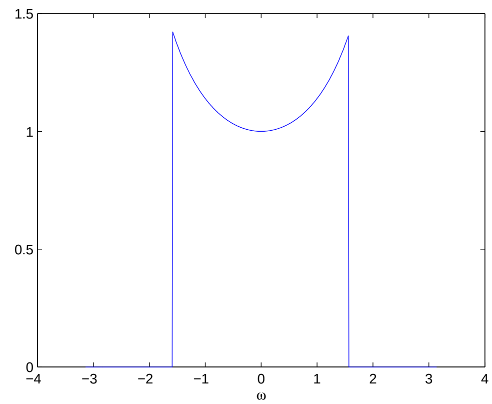
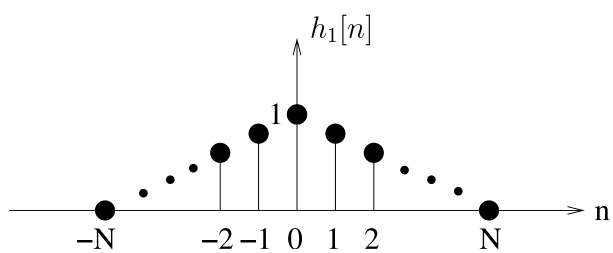


Figure 5: Problem 3 (b)

- (c)

$$h_1[n] = \begin{cases} 1 - \left|\frac{n}{N}\right| & n = -N, ..., N \\ 0 & else \end{cases}$$



- (d)

  First, we observe that $h_1[n] = \frac{1}{N} h_0[n] * h_0[-n]$. Therefore,

$$H_1(e^{j\omega}) = \frac{1}{N} H_0(e^{j\omega}) H_0(e^{-j\omega}) = \frac{1}{N} \frac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$$

In order to perfectly recover $x[n]$, we require $H_L(e^{j\omega}) = H_1(e^{j\omega})H(e^{j\omega})$ to be the same as in part (b). Following the same reasoning as in part (b), we see that

$$H(e^{j\omega}) = \begin{cases} N^2 \frac{\sin^2(\omega/2)}{\sin^2(\omega N/2)} & |\omega| < \pi/N \\ 0 & else \end{cases}$$

A plot of $|H(e^{j\omega})|$ for $N = 2$ is given below.

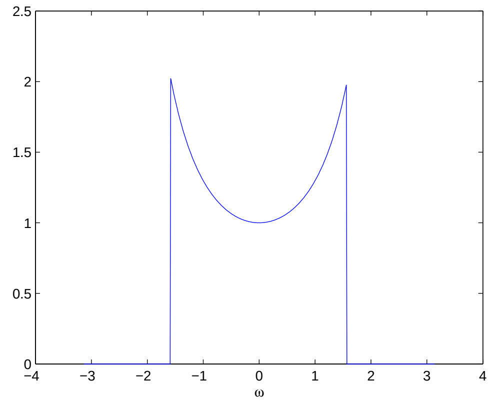


Figure 6: Problem 3 (d)

Problem 4 *(Oversampled D/A.)*

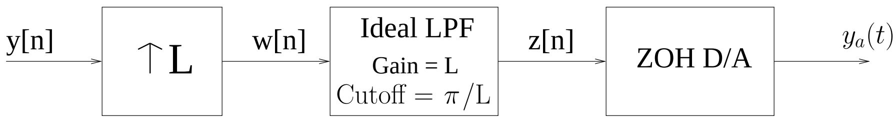


Figure 7: Problem 4. Block diagram of oversampled D/A

- (a)

  We know that that $W(e^{j\Omega}) = Y(e^{jL\Omega})$. Applying a LPF to $w[n]$ gives a signal $z[n]$ with spectrum

  $$Z(e^{j\Omega}) = \begin{cases} L & |\Omega| < \frac{3\pi}{4L} \\ 0 & \frac{3\pi}{4L} < |\Omega| < \pi \end{cases}$$

  The spectra $W(e^{j\Omega})$ and $Z(e^{j\Omega})$ are plotted in the following figure.

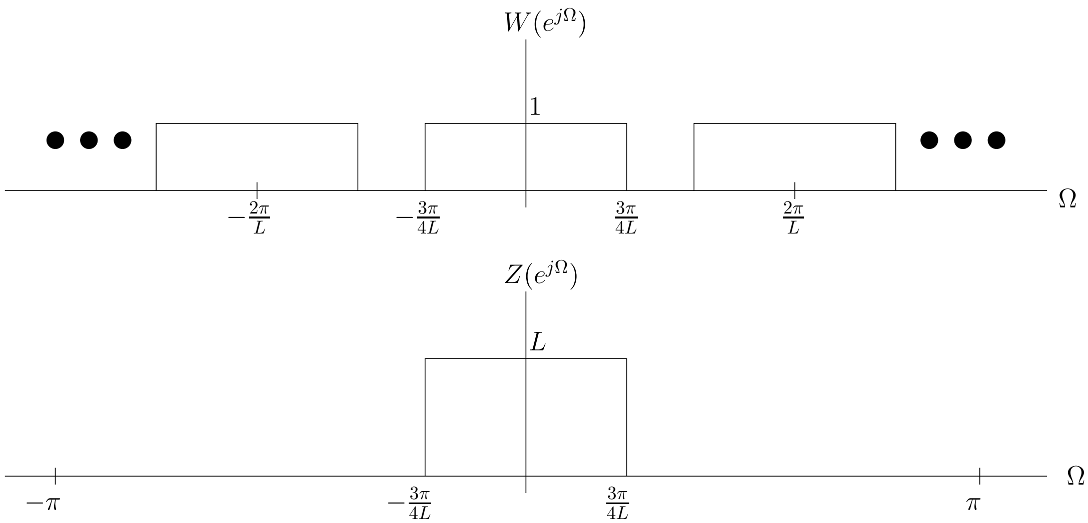


Figure 8: Problem 4 (a)

The specturm of the output of the ZOH is given by $Z(e^{j\omega T})$ multiplied by the frequency response of $p(t)$

$$\begin{aligned} Y_a(j\omega) &= Z(e^{j\omega T}) \cdot e^{-j\omega T/2} \cdot T \cdot \mathrm{sinc}(\omega T/2) \\ &= Z(e^{j\omega/L}) \cdot e^{-j\omega/(2L)} \cdot \frac{1}{L} \cdot \mathrm{sinc}(\omega/(2L)) \end{aligned}$$

where we are using the definition $\mathrm{sinc}(x) = \sin(x)/x$. The magnitude of the output is given by

$$|Y_a(j\omega)| = |Z(e^{j\omega/L})| \cdot \frac{1}{L} \cdot |\mathrm{sinc}(\omega/(2L))|$$

Plots of $|Y_a(j\omega)|$ in the interval $-5\pi L \leq \omega \leq 5\pi L$ are shown in the following figure.

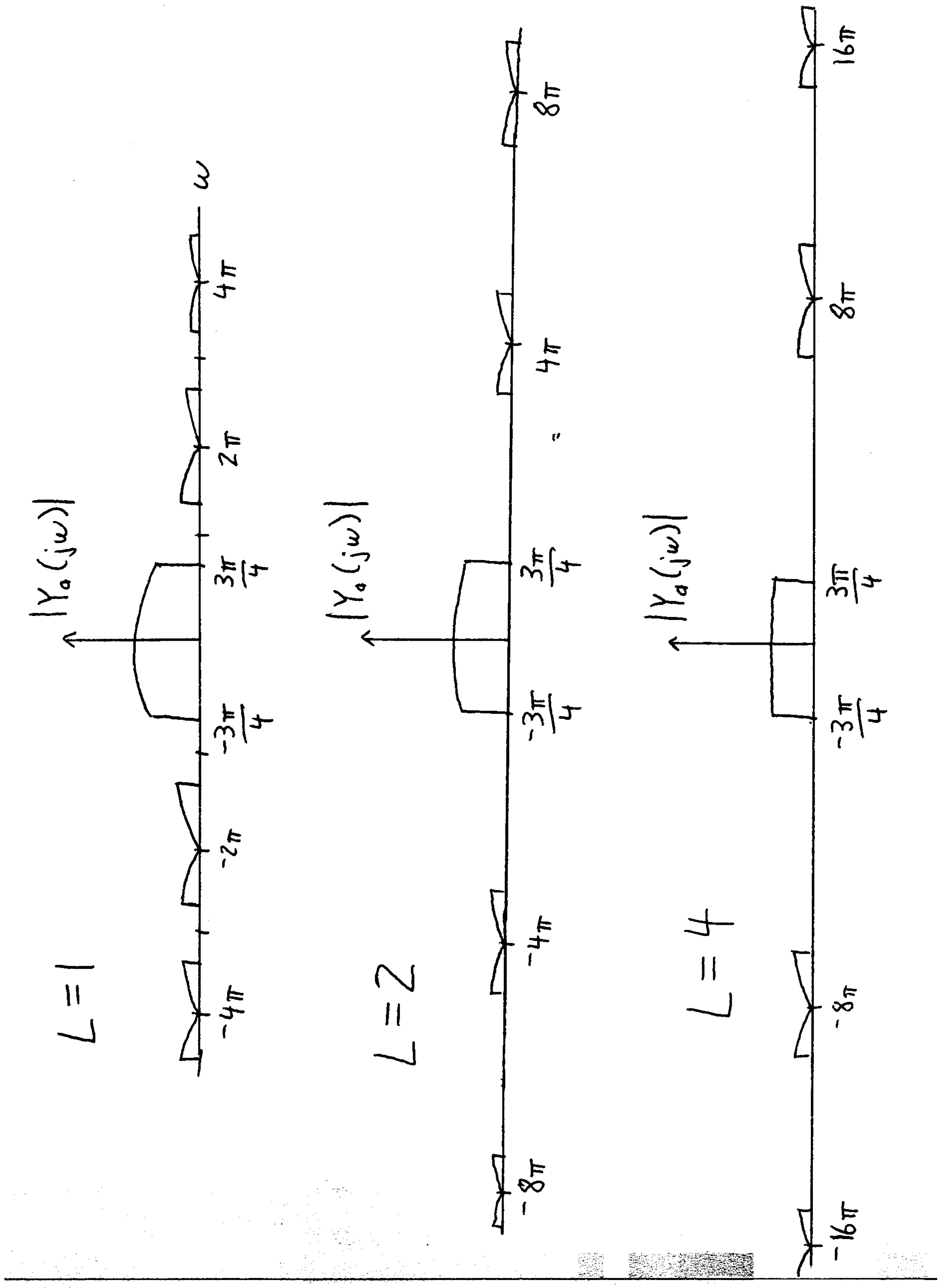


Figure 9: Problem 4 (a)

- (b)

  The ideal D/A converter is a LPF, which removes all of the high frequency images in the spectrum. Therefore, the magnitude spectrum $|Y(j\omega)|$ is as shown in the following figure.

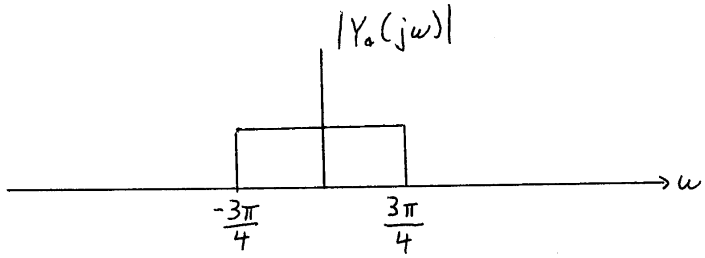


Figure 10: Problem 4 (b)

- (c)

  The largest component of $|Y_a(j\omega)|$ outside of $|\omega| \leq \pi L$ is the left edge of the copy of the spectrum centered at $2\pi L$. At that point, the magnitude is equal to

$$\operatorname{sinc}\left(\frac{2\pi L - 3\pi/4}{2L}\right) = \operatorname{sinc}\left(\pi - \frac{3\pi}{8L}\right)$$

  For $L = 1$, the magnitude of the largest out of band component is 0.4705

  For $L = 2$, the magnitude of the largest out of band component is 0.2177

  For $L = 4$, the magnitude of the largest out of band component is 0.1020

Problem 5 *(Aliasing.)*

- (a)

  The plot of $X_T(e^{j\Omega})$ is shown in the following figure.

- (b)

  The shifted copies of $X(j\omega)$ that make up $\tilde{X}_T(e^{j\Omega})$ are shown in the following figure.

  Aliasing occurs because each copy of $X(j\omega)$ has non-zero components outside the interval $[-\frac{\pi}{T}, \frac{\pi}{T}]$. In fact, because $X(j\omega)$ extends infinitely far in both directions, every copy of $X(j\omega)$ overlaps with every other copy.

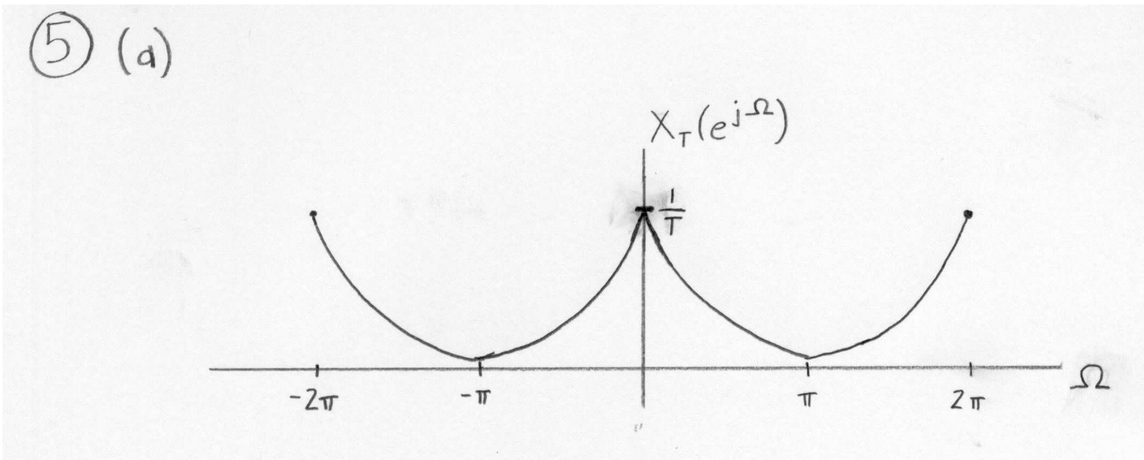


Figure 11: Problem 5 (a)

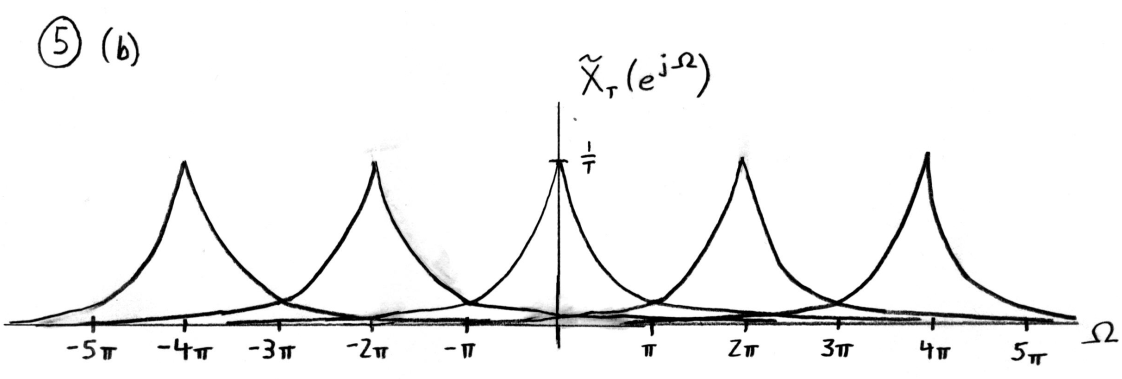


Figure 12: Problem 5 (b)

- (c)

$$\begin{aligned}
\tilde{X}_T(e^{j\Omega}) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-\left|\frac{\Omega-n2\pi}{T}\right|} \\
&= \frac{1}{T}\sum_{n=-\infty}^{-1} e^{-\left|\frac{\Omega-n2\pi}{T}\right|} + \frac{1}{T}e^{-\left|\frac{\Omega}{T}\right|} + \frac{1}{T}\sum_{n=1}^{\infty} e^{-\left|\frac{\Omega-n2\pi}{T}\right|} \\
&= \frac{1}{T}\sum_{m=1}^{\infty} e^{-\left|\frac{\Omega+m2\pi}{T}\right|} + \frac{1}{T}e^{-\left|\frac{\Omega}{T}\right|} + \frac{1}{T}\sum_{n=1}^{\infty} e^{-\left|\frac{\Omega-n2\pi}{T}\right|} \\
&= \frac{1}{T}\sum_{m=1}^{\infty} e^{-\left(\frac{\Omega+m2\pi}{T}\right)} + \frac{1}{T}e^{-\left|\frac{\Omega}{T}\right|} + \frac{1}{T}\sum_{n=1}^{\infty} e^{\left(\frac{\Omega-n2\pi}{T}\right)} \\
&= \frac{1}{T}e^{-\frac{\Omega}{T}}\sum_{m=1}^{\infty} \left(e^{-\frac{2\pi}{T}}\right)^m + \frac{1}{T}e^{-\left|\frac{\Omega}{T}\right|} + \frac{1}{T}e^{\frac{\Omega}{T}}\sum_{n=1}^{\infty} \left(e^{-\frac{2\pi}{T}}\right)^n \\
&= \frac{1}{T}e^{-\frac{\Omega}{T}}\frac{e^{-2\pi/T}}{1-e^{-2\pi/T}} + \frac{1}{T}e^{-\left|\frac{\Omega}{T}\right|} + \frac{1}{T}e^{\frac{\Omega}{T}}\frac{e^{-2\pi/T}}{1-e^{-2\pi/T}} \\
&= \frac{1}{T}e^{-\frac{\Omega}{T}}\frac{1}{e^{2\pi/T}-1} + \frac{1}{T}e^{-\left|\frac{\Omega}{T}\right|} + \frac{1}{T}e^{\frac{\Omega}{T}}\frac{1}{e^{2\pi/T}-1}
\end{aligned}$$

Problem 6 *(Aliasing - continued.)*

- (a)

  We first define a new signal $z[n] = \tilde{x}_T[n] - x_T[n]$. Now, we use Parseval's relation to show that

$$\begin{aligned}
E &= \sum_{n=-\infty}^{\infty} \left|\tilde{x}_T[n] - x_T[n]\right|^2 \\
&= \sum_{n=-\infty}^{\infty} \left|z[n]\right|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Z(e^{j\Omega})\right|^2 d\Omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\tilde{X}_T(e^{j\Omega}) - X_T(e^{j\Omega})\right|^2 d\Omega
\end{aligned}$$

  Now, from Problem 5, part (c), we see that we can write

$$\tilde{X}_T(e^{j\Omega}) - X_T(e^{j\Omega}) = K\left(e^{-\frac{\Omega}{T}} + e^{\frac{\Omega}{T}}\right)$$

  where

$$K = \frac{1}{T}\frac{1}{e^{2\pi/T}-1}$$

  Now, we evaluate the integral as

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\tilde{X}_T(e^{j\Omega}) - X_T(e^{j\Omega})\right|^2 d\Omega &= \frac{1}{2\pi}K^2\int_{-\pi}^{\pi}\left(e^{-\frac{\Omega}{T}} + e^{\frac{\Omega}{T}}\right)^2 d\Omega \\
&= \frac{1}{2\pi}K^2\int_{-\pi}^{\pi} e^{-\frac{2\Omega}{T}} + 2 + e^{\frac{2\Omega}{T}} d\Omega \\
&= \frac{1}{2\pi}K^2\left[-\frac{T}{2}e^{-\frac{2\Omega}{T}} + 2\Omega + \frac{T}{2}e^{\frac{2\Omega}{T}}\right]_{\Omega=-\pi}^{\pi} \\
&= \frac{1}{2\pi}K^2\left[-\frac{T}{2}e^{-\frac{2\pi}{T}} + \frac{T}{2}e^{\frac{2\pi}{T}} + 4\pi + \frac{T}{2}e^{\frac{2\pi}{T}} - \frac{T}{2}e^{\frac{-2\pi}{T}}\right] \\
&= \frac{1}{2\pi}K^2\left[4\pi + Te^{\frac{2\pi}{T}} - Te^{\frac{-2\pi}{T}}\right] \\
&= \frac{2}{T^2}\frac{1}{(e^{2\pi/T}-1)^2} + \frac{1}{2\pi T}\frac{e^{2\pi/T}}{(e^{2\pi/T}-1)^2} - \frac{1}{2\pi T}\frac{e^{-2\pi/T}}{(e^{2\pi/T}-1)^2}
\end{aligned}
$$

- (b)

  As T tends to zero, E tends to 0. This can be seen by examining the last equation, and remembering that as T goes to zero, $e^{2\pi/T}$ increases more quickly than T decreases. (This can be verified by plotting E as a function of T in Matlab.)

  This agrees with our intuition. We know that when we multiply $x(t)$ by the impulse train $p(t)$, the copies of $X(j\omega)$ in $X_p(j\omega)$ are centered at every multiple of $2\pi/T$. As T tends to zero, the copies of $X(j\omega)$ get very far apart. Because $X(j\omega) = e^{-|\omega|}$, when the copies of $X(j\omega)$ get very far apart, the amount of aliasing becomes negligible.

Problem 7 *(AM Communication Systems.)*

OWN Problem 8.22

The spectrum $Y(j\omega)$ is sketched in the following figure.

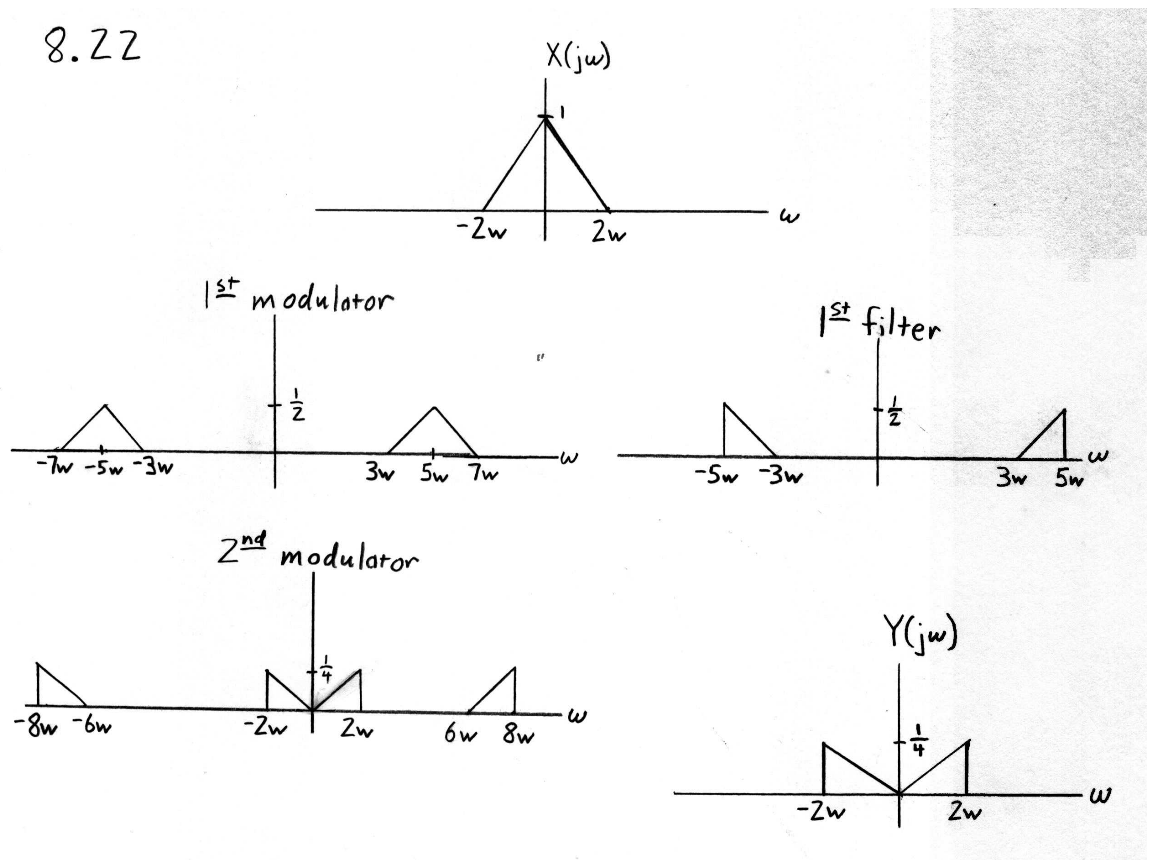


Figure 13: Problem 7