Homework 9 Solutions
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Problem 1 *(Image Suppression.)*

In this problem we are interested in extracting the desired signal $m_r(t)$ at frquency ω_{IF} at the output.

(a)

Let's write the output of each stage of the system:

$$x(t) = x(t) = m_r(t)\cos(\omega_{LO} + \omega_{IF})t + m_I(t)\cos(\omega_{LO} - \omega_{IF})t$$

$$A = x(t)\cos(\omega_{LO}t) = \frac{1}{2}\left\{ \begin{array}{c} m_r(t)(\cos(2\omega_{LO} + \omega_{IF})t + \cos\omega_{IF}t) \\ +m_I(t)(\cos(2\omega_{LO} - \omega_{IF})t + \cos\omega_{IF}t) \end{array} \right\}$$

$$B = x(t)\sin(\omega_{LO}t) = \frac{1}{2}\left\{ \begin{array}{c} m_r(t)(\sin(2\omega_{LO} + \omega_{IF})t - \sin\omega_{IF}t) \\ +m_I(t)(\sin(2\omega_{LO} - \omega_{IF})t + \sin\omega_{IF}t) \end{array} \right\}$$

$$C = \frac{1}{2}\left\{ \begin{array}{c} m_r(t)(-\cos(2\omega_{LO} + \omega_{IF})t + \cos\omega_{IF}t) \\ +m_I(t)(-\cos(2\omega_{LO} - \omega_{IF})t - \cos\omega_{IF}t) \end{array} \right\}$$

$$y(t) = A + C = m_r(t)\cos\omega_{IF}t$$

(b)

$$A = x(t)(1+\alpha)\cos(\omega_{LO}t + \frac{\phi}{2}) = \frac{1}{2}\left\{ \begin{array}{c} m_r(t)(1+\alpha)(\cos((2\omega_{LO} + \omega_{IF})t + \frac{\phi}{2}) + \cos((\omega_{IF}t - \frac{\phi}{2})) \\ +m_I(t)(1+\alpha)(\cos((2\omega_{LO} - \omega_{IF})t + \frac{\phi}{2}) + \cos((\omega_{IF})t + \frac{\phi}{2})) \end{array} \right\}$$

$$B = x(t)(1-\alpha)\sin(\omega_{LO}t - \frac{\phi}{2}) = \frac{1}{2}\left\{ \begin{array}{c} m_r(t)(1-\alpha)(\sin((2\omega_{LO} + \omega_{IF})t - \frac{\phi}{2}) - \sin(\omega_{IF}t + \frac{\phi}{2}) \\ +m_I(t)(1-\alpha)(\sin((2\omega_{LO} - \omega_{IF})t - \frac{\phi}{2}) + \sin(\omega_{IF}t - \frac{\phi}{2})) \end{array} \right\}$$

$$C = \frac{1}{2}\left\{ \begin{array}{c} m_r(t)(1-\alpha)(-\cos((2\omega_{LO} + \omega_{IF})t - \frac{\phi}{2}) + \cos(\omega_{IF}t + \frac{\phi}{2})) \\ +m_I(t)(1-\alpha)(-\cos((2\omega_{LO} - \omega_{IF})t - \frac{\phi}{2}) - \cos(\omega_{IF}t + \frac{\phi}{2})) \end{array} \right\}$$

We are only interested in the components of the output at ω_{IF}:

$$\begin{aligned} y(t) = A + C &= \frac{1}{2}m_r(t)[(1+\alpha)\cos(\omega_{IF}t - \frac{\phi}{2}) + (1-\alpha)\cos(\omega_{IF}t + \frac{\phi}{2})] \\ &+ \frac{1}{2}m_I(t)[(1+\alpha)\cos(\omega_{IF}t + \frac{\phi}{2}) - (1-\alpha)\cos(\omega_{IF}t - \frac{\phi}{2})] \\ &= m_r(t)[\cos(\omega_{IF}t)\cos(\frac{\phi}{2}) + \alpha\sin(\omega_{IF}t)\sin(\frac{\phi}{2})] + m_I(t)[\alpha\cos(\omega_{IF}t)\cos(\frac{\phi}{2}) - \sin(\omega_{IF}t)\sin(\frac{\phi}{2})] \end{aligned}$$

(c)

Let $M_r(j\omega) = \mathcal{FT}\{m_r(t)\}$, and $M_I(j\omega) = \mathcal{FT}\{m_I(t)\}$. Furthermore, since we assume that both the desired signal and the image have equal power at the input, then $\int_{-\infty}^{\infty} |M_r(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |M_I(j\omega)|^2 d\omega = P$. Assuming that the Bandwidth of both $m_r(t)$ and $m_I(t)$ is small relative to ω_{IF}:

$$\Rightarrow M_r^{IF}(j\omega) = \mathcal{FT}\{m_r(t)[\cos(\omega_{IF}t)\cos(\frac{\phi}{2}) + \alpha\sin(\omega_{IF}t)\sin(\frac{\phi}{2})]\}$$

$$= \frac{1}{2}(\cos(\frac{\phi}{2}) - j\alpha\sin(\frac{\phi}{2}))M_r(j(\omega-\omega_{IF})) + \frac{1}{2}(\cos(\frac{\phi}{2}) + j\alpha\sin(\frac{\phi}{2}))M_r(j(\omega+\omega_{IF}))$$

$$\Rightarrow \int_{-\infty}^{\infty} |M_r^{IF}(j\omega)|^2 d\omega = \frac{P}{2}(\cos^2(\frac{\phi}{2}) + \alpha^2\sin^2(\frac{\phi}{2}))$$

$$\Rightarrow M_I^{IF}(j\omega) = \mathcal{FT}\{m_I(t)[\alpha\cos(\omega_{IF}t)\cos(\frac{\phi}{2}) - \sin(\omega_{IF}t)\sin(\frac{\phi}{2})]\}$$

$$= \frac{1}{2}(\alpha\cos(\frac{\phi}{2}) + j\sin(\frac{\phi}{2}))M_I(j(\omega-\omega_{IF})) + \frac{1}{2}(\alpha\cos(\frac{\phi}{2}) - j\sin(\frac{\phi}{2}))M_I(j(\omega+\omega_{IF}))$$

$$\Rightarrow \int_{-\infty}^{\infty} |M_I^{IF}(j\omega)|^2 d\omega = \frac{P}{2}(\alpha^2\cos^2(\frac{\phi}{2}) + \sin^2(\frac{\phi}{2}))$$

$$\Rightarrow IR = \frac{\cos^2(\frac{\phi}{2}) + \alpha^2\sin^2(\frac{\phi}{2})}{\alpha^2\cos^2(\frac{\phi}{2}) + \sin^2(\frac{\phi}{2})}$$

Problem 2 *(FSK.)* OWN Problem 8.39.

(a)

There are two possible cases.

Case 0: $b(t) = m_0(t)$.

$$D_0 = \left|\int_0^T \cos^2(\omega_0 t)dt\right| - \left|\int_0^T \cos(\omega_0 t)\cos(\omega_1 t)dt\right|$$

Case 1: $b(t) = m_1(t)$.

$$D_1 = \left|\int_0^T \cos^2(\omega_1 t)dt\right| - \left|\int_0^T \cos(\omega_0 t)\cos(\omega_1 t)dt\right|$$

Both D_0 and D_1 are maximum when $\int_0^T \cos(\omega_0 t)\cos(\omega_1 t)dt = 0$.

(b)

$$\begin{aligned}
\int_0^T \cos(\omega_0 t)\cos(\omega_1 t)dt &= \int_0^T \frac{1}{2}(\cos((\omega_0+\omega_1)t) + \cos((\omega_0-\omega_1)t))dt \\
&= \left[\frac{\sin((\omega_0+\omega_1)t)}{2(\omega_0+\omega_1)} + \frac{\sin((\omega_0-\omega_1)t)}{2(\omega_0-\omega_1)}\right]_0^T \\
&= \frac{\sin((\omega_0+\omega_1)T)}{2(\omega_0+\omega_1)} + \frac{\sin((\omega_0-\omega_1)T)}{2(\omega_0-\omega_1)}
\end{aligned}$$

Thus for any choice of ω_0 and ω_1, $\omega_0 \neq \omega_1$, we can always find T so that $\int_0^T \cos(\omega_0 t)\cos(\omega_1 t)dt = 0$.

Problem 3 *(Angle Modulations.)*

The general form of an angle modulated signal:

$$A\cos(2\pi f_c t + k_p a m(t)) \quad \text{(PM)}$$
$$A\cos(2\pi f_c t + k_f a \int m(t)) \quad \text{(FM)}$$

The modulation index is defined as:

$$\beta_p = k_p a \max[|m(t)|] = \Delta\phi_{max} \quad \text{(PM)}$$
$$\beta_f = \frac{k_f a \max[|m(t)|]}{W} = \frac{\Delta f_{max}}{W} \quad \text{(FM)}$$

Where $\Delta\phi_{max}$ and Δf_{max} are the maximum phase and frequency deviations respectively. W is the bandwidth of the modulating signal in Hz (i.e. $m(t)$ is bandlimited to W). In this problem, the modulated signal:

$$u(t) = 100\cos(2\pi f_c t + 4\sin 2\pi f_m t)$$

The instantaneous phase $\phi(t) = 2\pi f_c t + 4\sin(2\pi f_m t)$, the instantaneous frequency $f(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = f_c + 4f_m\cos(2\pi f_m t)$, and $W = f_m$.

$$\Rightarrow \Delta\phi_{max} = 4, \Delta f_{max} = 4f_m$$
$$\Rightarrow \beta_p = \Delta\phi_{max} = 4, \beta_f = \frac{\Delta f_{max}}{f_m} = 4$$

We estimate the effective bandwidth B_c (same for both PM and FM) of the modulated signal using *Carson's rule*:

$$B_c \approx 2(\beta + 1)W$$

According to OWN, $B_c = 2\beta W$ (both answers are acceptable).

$$B_c = 2(4+1)f_m = 10\text{KHz}$$

For SSB, the bandwidth of the modulated signal is $f_m = 1$ KHz. I have only considered the positive side bands in this problem. If you wish to include the negative side bands, then simply scale B_c by a factor of 2. Also, notice that B_c is directly proportional to f_m. Therefore, when f_m is doubled:

$$B_c = 20\text{KHz}$$

Problem 4 *(Laplace Transforms.)*

(a) OWN Problem 9.21 (c).

$$x(t) = e^{2t}u(-t) + e^{3t}u(-t)$$

We can find the following Laplace transform pairs in Table 9.2 of OWN:

$$\begin{aligned} e^{2t}u(-t) \quad &\longleftrightarrow \quad -\frac{1}{s-2}\text{ , with ROC } Re\{s\} < 2 \\ e^{3t}u(-t) \quad &\longleftrightarrow \quad -\frac{1}{s-3}\text{ , with ROC } Re\{s\} < 3 \end{aligned}$$

By the linearity property of the Laplace transform (see Table 9.1 of OWN)

$$X(s) = -\frac{1}{s-2} - \frac{1}{s-3} = \frac{-2s+5}{(s-2)(s-3)}$$

with region of convergence (ROC) $Re\{s\} < 2$. Thus $X(s)$ has a pole at 2, another pole at 3, and a zero at -2.5.

(b) OWN Problem 9.21 (e).

$$x(t) = |t|e^{-2|t|} = te^{-2t}u(t) - te^{2t}u(-t)$$

Using the Laplace transform tables again, we find the following Laplace transform pairs in OWN Table 9.2

$$\begin{aligned} e^{-2t}u(t) \quad &\longleftrightarrow \quad \frac{1}{s+2}\text{ , with ROC } Re\{s\} > -2 \\ e^{2t}u(-t) \quad &\longleftrightarrow \quad -\frac{1}{s-2}\text{ , with ROC } Re\{s\} < 2 \end{aligned}$$

By the differentiation in the s-domain property, in OWN Table 9.1,

$$\begin{aligned} te^{-2t}u(t) \quad &\longleftrightarrow \quad -\frac{d}{ds}\left(\frac{1}{s+2}\right) = \frac{1}{(s+2)^2} \\ -te^{2t}u(-t) \quad &\longleftrightarrow \quad \frac{d}{ds}\left(-\frac{1}{s-2}\right) = \frac{1}{(s-2)^2} \end{aligned}$$

Finally by the linearity property,

$$X(s) = \frac{1}{(s+2)^2} + \frac{1}{(s-2)^2} = \frac{2s^2}{(s+2)^2(s-2)^2}$$

with ROC $-2 < Re\{s\} < 2$. Thus $X(s)$ has two poles at -2, two poles at 2, and two zeros at 0.

Problem 5 *(LT/LT Properties.)*

(a) OWN Problem 9.21 (g).

$$x(t) \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{1} e^{-st}dt = \frac{1-e^{-s}}{s}$$

Even though there is a pole at $s = 0$, there also is a zero at $s = 0$. Due to the pole-zero cancellation, the ROC of $x(t)$ is the entire s-plane.

(b) OWN Problem 9.26.

$$x_1(t) = e^{-2t}u(t), x_2(t) = e^{-3t}u(t)$$
$$y(t) = x_1(t-2) * x_2(3-t)$$

Using the time shifting, time scaling (reversal), convolution properties:

$$\begin{aligned}
x_1(t) = e^{-2t}u(t) &\longleftrightarrow X_1(s) = \mathcal{L}\{x_1(t)\} = \frac{1}{s+2}\text{ , with ROC } Re\{s\} > -2 \\
x_2(t) = e^{-3t}u(t) &\longleftrightarrow X_2(s) = \mathcal{L}\{x_2(t)\} = \frac{1}{s+3}\text{ , with ROC } Re\{s\} > -3 \\
x_1(t-2) &\longleftrightarrow \mathcal{L}\{x_1(t-2)\} = e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2}\text{ , with ROC } Re\{s\} > -2 \\
x_2(3-t) &\longleftrightarrow \mathcal{L}\{x_2(3-t)\} = e^{-3s}X_2(-s) = \frac{e^{-3s}}{3-s}\text{ , with ROC } Re\{s\} < 3 \\
y(t) &\longleftrightarrow Y(s) = \mathcal{L}\{y(t)\} = e^{-5s}X_1(s)X_2(-s) = \frac{e^{-5s}}{(3-s)(s+2)},\quad -2 < Re\{s\} < 3
\end{aligned}$$

The convolution property indicates that the ROC of $y(t)$ must contain the intersection of the ROCs of both $x_1(t-2)$ and $x_2(3-t)$. However, in this case we know that the ROC of $y(t)$ is exactly the intersection of the ROCs of $x_1(t-2)$ and $x_2(3-t)$ since there is no pole-zero cancellation. Notice that the ROC is preserved under time shifting, but not under time scaling and time reversal.

Problem 6 *(Inverse Laplace.)*

(a) OWN Problem 9.22 (a).

Using the Laplace transform pairs Table 9.2 of OWN, we can immediately find the inverse Laplace transform of $X(s)$ to be $x(t) = \frac{1}{3}\sin(3t)u(t)$. Alternatively we can use partial fraction expansions to find the inverse.

$$\begin{aligned}
X(s) &= \frac{1}{s^2+9} \\
&= \frac{j/6}{s+j3} + \frac{-j/6}{s-j3} \\
&\quad \text{for } Re(s) > 0 \\
x(t) &= \frac{j}{6}e^{-j3t}u(t) - \frac{j}{6}e^{j3t}u(t) \\
&= -\frac{j}{6}2j\sin(3t)u(t) \\
&= \frac{1}{3}\sin(3t)u(t)
\end{aligned}$$

(b) OWN Problem 9.22 (b).

$$
\begin{aligned}
X(s) &= \frac{s}{s^2+9} \\
&= \frac{1/2}{s+j3} + \frac{1/2}{s-j3} \\
& \quad \text{for } Re(s) < 0 \\
x(t) &= -\frac{1}{2}e^{-j3t}u(-t) - \frac{1}{2}e^{j3t}u(-t) \\
&= -\cos(3t)u(-t)
\end{aligned}
$$

Problem 7 *(Matlab - Synchronization in PAM.)*

(a)

Figure 1 shows that the signal-to-interference (SIR) ratio is 1 when the synchronization offset $\Delta t = 0$ and there is no intersymbol interference, and decreases to less than $\frac{1}{2}$ as the synchronization offset increases to half the sampling period $T_s = 1$. The sinc pulse is relatively sensitive to synchronization offsets because it dies off as $\frac{1}{|t|}$.

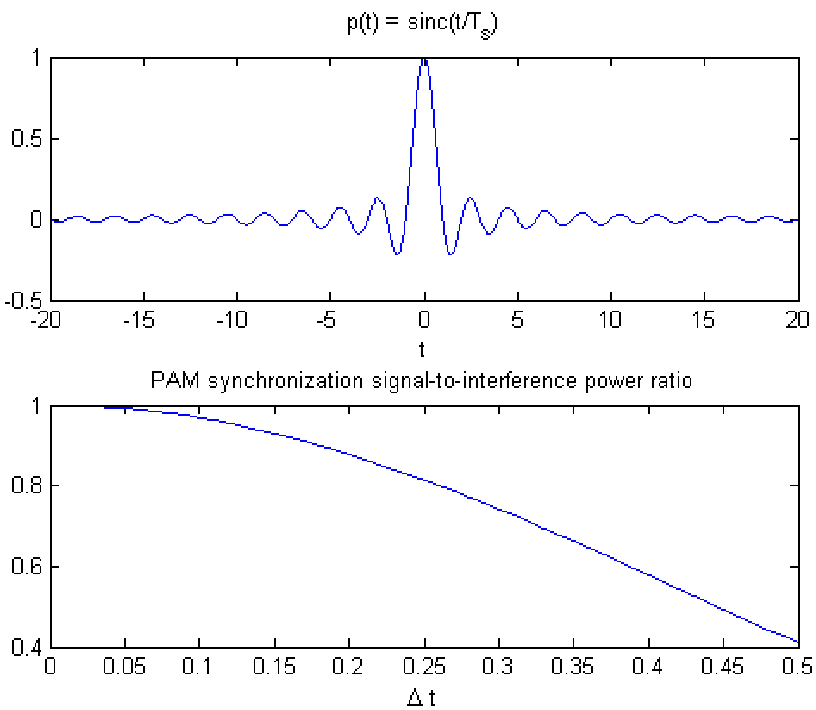

Figure 1: Problem 7 (a).

(b)

Figure 2 shows that when $\alpha = 0$, the SIR ratio is unchanged because the raised cosine pulse is equal to the sinc pulse. As α increases to 1, the SIR ratio for the raised cosine pulse decays more and more slowly for $\Delta t \in [0, .5]$. Thus the raised cosine pulse is less sensitive to synchronization offsets for larger values of α. At $\alpha = 1$, the SIR ratio decreases to half the max power when the synchronization offset is half the sampling period. As the synchronization offset increases to the sampling period, the SIR ratio goes to zero.

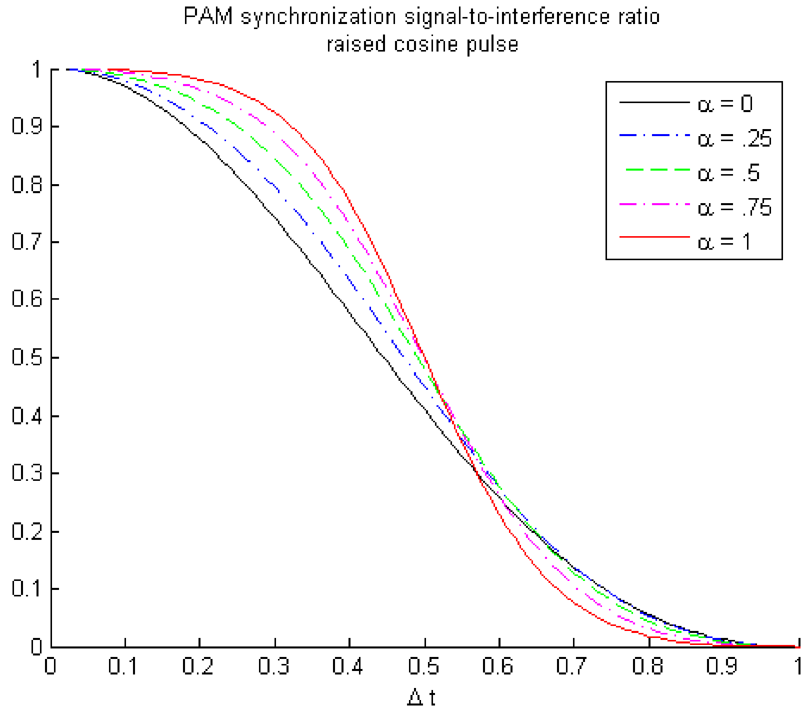

Figure 2: Problem 7 (b).

(c)

With $T_s = 1$, the bandwidth of the raised cosine pulse is $(1+\alpha)\pi$ (see Figure 3). The bandwidth is an increasing function of α, while the synchronization error is a decreasing function of α. The tradeoff is between bandwidth used by the PAM system and the sensitivity of the pulse to synchronization offsets.

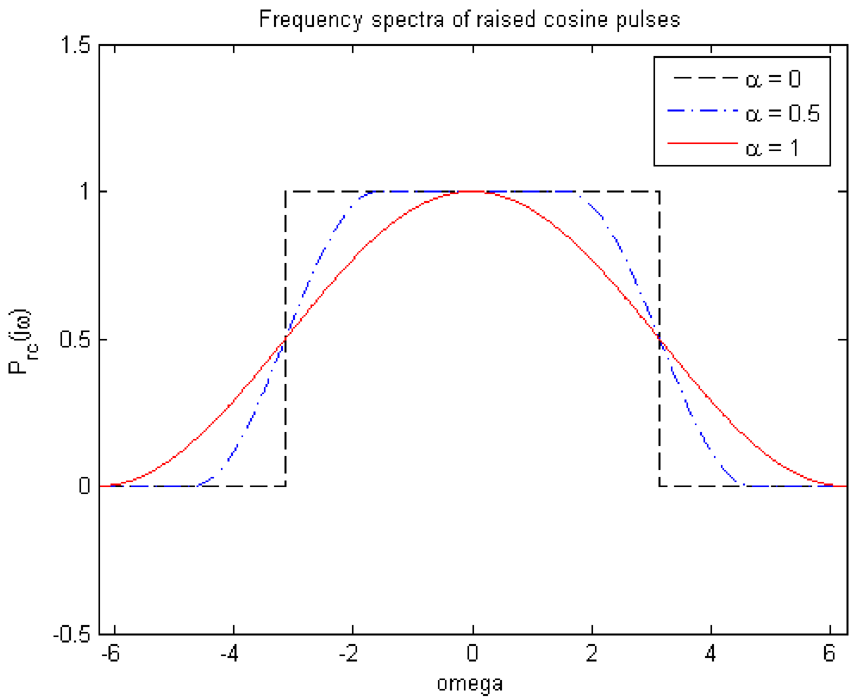

Figure 3: Problem 7 (c).