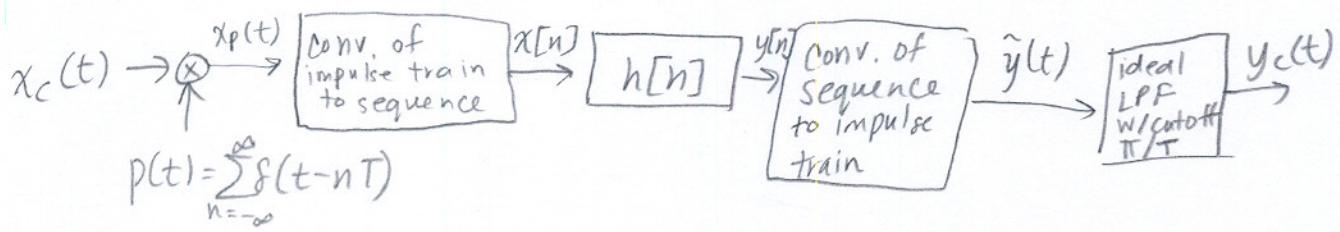


$$7.31 \quad y[n] = \frac{1}{2} y[n-1] + x[n]$$

Given: $X_c(j\omega) = 0$ for $|w| < \frac{\pi}{T}$

Find $H_c(j\omega)$, which is equivalent frequency response to below system with input $x_c(t)$ and output $y_c(t)$.



Let's solve this incrementally by finding $X_p(j\omega), X(e^{j\omega}), Y(e^{j\omega}), \tilde{Y}(j\omega), Y_c(j\omega)$

From sampling theorem,

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - \frac{2\pi k}{T})) \quad \boxed{\text{Eq. 1}}$$

Now, let's find a relationship between $X_p(j\omega)$ and $X(e^{j\omega})$

$$X_p(t) = \sum_{n=-\infty}^{\infty} x_c(t) \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT) \quad \boxed{\text{Eq. 2}}$$

by properties of impulses

Now, in the frequency domain $\delta(t - nT) \leftrightarrow e^{-j\omega n T}$ so

$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\omega n T} \quad \boxed{\text{Eq. 3}}$$

Let's rewrite $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-jn\omega T} \quad \boxed{\text{Eq. 4}}$$

The equation for $X(e^{j\omega})$ and $X(j\omega)$ look very similar. In fact

$$X(e^{j\omega}) = X_p(j\omega) \Big|_{\omega = \frac{\Omega}{T}} \quad \boxed{\text{Eq. 5}}$$

let's plug it in to check

$$X_p(j\omega) \Big|_{\omega = \frac{\Omega}{T}} = X_p(j\frac{\Omega}{T}) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\frac{\Omega}{T}nT} = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-jn\omega} \\ = X(e^{j\omega}) !$$

Summarizing,

$$X(e^{j\omega}) = X_p(j\frac{\Omega}{T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\Omega - 2\pi k}{T})) \quad \boxed{\text{Eq. 6}}$$

* Note that we wrote both $x_p(t)$ and $x[n]$ in terms of $x_c(t)$

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) \quad \boxed{\text{Eq. 7}}$$

Using the equation given at the beginning of the problem, let's solve for $H(e^{j\omega})$

We were given

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

Converting to the frequency domain

$$Y(e^{j\omega}) = \frac{1}{2} e^{-j\omega} Y(e^{j\omega}) + X(e^{j\omega})$$

$$Y(e^{j\omega}) \left[1 - \frac{1}{2} e^{-j\omega} \right] = X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} \quad \boxed{\text{Eq. 8}}$$

$$\text{Thus, } Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}) = H(e^{j\omega}) X_p(j \frac{\omega}{T}) \quad \boxed{\text{Eq. 9}}$$

↑
plugging in Eq. 6

To determine the relationship

between $\tilde{Y}(jw)$ and $Y(e^{j\omega})$, let's

use what we found in Eq. 5: namely

when we go from discrete to continuous time we see that $X(e^{j\omega}) = X_{\text{cont}}(j \frac{\omega}{T})$

Now, let $w = \frac{\omega}{T}$ and we see that

$$X_{\text{cont}}(jw) = X_{\text{discrete}}(e^{jwT}) \quad \boxed{\text{Eq. 10}}$$

Using Eq.10 for our example problem,
we see that

$$\begin{aligned}\tilde{Y}(jw) &= Y(e^{jwT}) = H(e^{jwT})X(e^{jwT}) \quad \boxed{\text{Eq.11}} \\ &= H(e^{jwT})X_p(jw) = \left(\frac{1}{1-\frac{1}{2}e^{-jwT}}\right)X_p(jw)\end{aligned}$$

from Eq.9

The last step is the LPF w/cutoff frequency $\frac{\pi}{T}$. Let's focus on the range from $-\frac{\pi}{T} \leq w \leq \frac{\pi}{T}$ and examine Eq.11.

Eq.11 states that $\tilde{Y}(jw) = H(e^{jwT})X_p(jw)$ and $X_p(jw) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(w - \frac{2\pi}{T}k))$

so from $-\frac{\pi}{T} \leq w \leq \frac{\pi}{T}$, $X_p(jw) = \underbrace{\frac{1}{T}X_c(jw)}$
and $\tilde{Y}(jw) = Y_c(jw)$.

Thus from $-\frac{\pi}{T} \leq w \leq \frac{\pi}{T}$,

$$Y_c(jw) = \tilde{Y}(jw) = H(e^{jwT}) \frac{1}{T} X_c(jw)$$

$$\text{and } H_c(jw) = \frac{Y_c(jw)}{X_c(jw)} = \frac{1}{T} H(e^{jwT}) = \boxed{\frac{V_T}{1 - \frac{1}{2}e^{-jwT}}}$$

Down sampling

Reduce the sampling rate of a DT signal

$$x[n] \rightarrow \boxed{\downarrow M} \rightarrow x_d[n] = x[nM]$$

Let's figure things out in the frequency domain,

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[nM] e^{-j\omega n}$$

Let $k = nM$

$$X_d(e^{j\omega}) = \sum_{k=0, \pm M, \pm 2M, \text{etc}} x[k] e^{-j\omega k/M}$$

Let's figure out how to rewrite $x[k]$

so that we can sum over all integer values of k (and not just multiples of M).

Let's do this for some example values of M to see if we see a pattern.

Let's examine the case when $M=2$

In that case,

$$X_d(e^{jw}) = \sum_{k=0, \pm 2, \pm 4, \text{etc.}} x[k] e^{-jwk/2} = \sum_{k=-\infty}^{\infty} \underbrace{\frac{(x[k] + (-1)^k x[k])}{2}}_{\uparrow \begin{array}{l} \text{for even } k = x[k] \\ \text{for odd } k = 0 \end{array}} e^{-jwk/2}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \left(\frac{1+e^{j\pi k}}{2} \right) e^{-jwk/2}$$

$$= \frac{1}{2} [X(e^{jw/2}) + X(e^{j(\frac{w}{2}-\pi)})]$$

GOOD!

Now, let's examine the case when $M=3$.

$$X_d(e^{jw}) = \sum_{k=0, \pm 3, \pm 6, \text{etc.}} x[k] e^{-jwk/3} = \sum_{k=-\infty}^{\infty} x[k] \left(\underbrace{\frac{1+e^{\frac{2\pi}{3}k} + e^{j\frac{2\pi}{3}2k}}{3}}_{\uparrow \begin{array}{l} \text{when } k \text{ is a multiple of 3} \\ \text{this is 1 and 0} \\ \text{otherwise!} \end{array}} \right) e^{-jwk/3}$$

$$= \frac{1}{3} \left[X(e^{jw/3}) + X(e^{j(\frac{w}{3}-\frac{2\pi}{3})}) + X(e^{j(\frac{w}{3}-\frac{4\pi}{3})}) \right]$$

\uparrow
Do you question this???

Check using geometric series.

$$\frac{1+e^{\frac{2\pi}{3}k} + e^{j\frac{2\pi}{3}2k}}{3} = \frac{1}{3} \sum_{i=0}^2 (e^{j\frac{2\pi}{3}k})^i$$

$$= \frac{1}{3} \frac{1 - (e^{j\frac{2\pi}{3}k})^3}{1 - e^{j\frac{2\pi}{3}k}}$$

$$= \begin{cases} 1 & \text{if } k \bmod 3 = 0 \\ 0 & \text{o.w.} \end{cases}$$

zero for all k
only if k mult of 3.

Now find the general

for an arbitrary M !