Midterm II Review Problems

(4) (a)

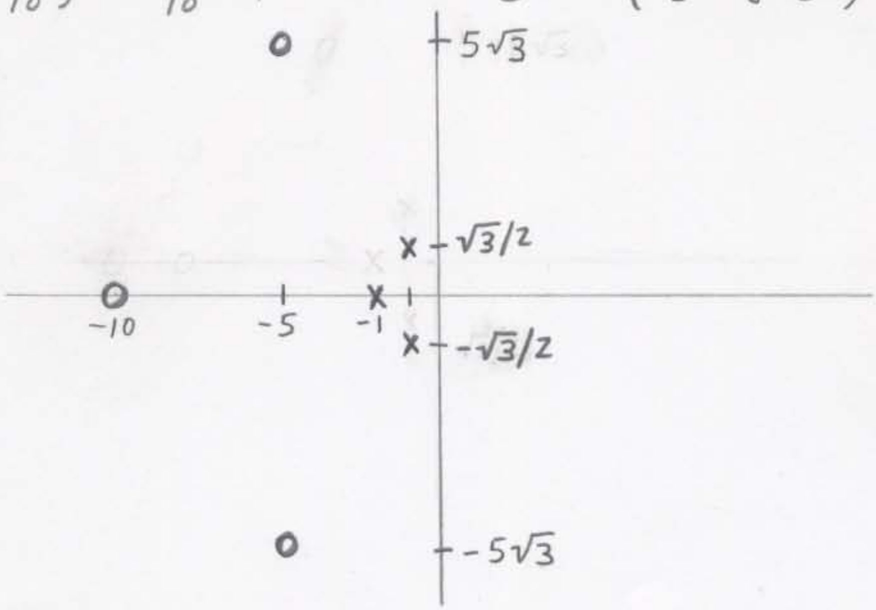
$$H(s) = \frac{\left(\frac{s}{10}+1\right)\left(\left(\frac{s}{10}\right)^2 + \frac{s}{10} + 1\right)}{(s+1)(s^2+s+1)}$$

poles: $s+1=0 \Rightarrow s=-1$

$s^2+s+1=0 \Rightarrow s = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$

zeros: $\frac{s}{10}+1=0 \Rightarrow s=-10$

$\left(\frac{s}{10}\right)^2 + \frac{s}{10} + 1 = 0 \Rightarrow s = 10\left(\frac{-1}{2} \pm j\frac{\sqrt{3}}{2}\right) = -5 \pm j5\sqrt{3}$



Three possible ROCs: $Re\{s\} < -1$

$-1 < Re\{s\} < \frac{-1}{2}$

$\frac{-1}{2} < Re\{s\}$

(b) Yes. If the system is causal, then the ROC is $Re\{s\} > \frac{-1}{2}$, which contains the $j\omega$-axis

(c) Yes. The poles of $H_I(s)$ are the zeros of $H(s)$, because $H_I(s) = \frac{1}{H(s)}$

If the inverse system is causal, then the ROC of the inverse is $Re\{s\} > -5$, which contains the $j\omega$-axis

(d)
$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{s^3}{1000} + \frac{s^2}{50} + \frac{s}{5} + 1}{s^3 + 2s^2 + 2s + 1}$$

$$s^3 Y(s) + 2 \cdot s^2 Y(s) + 2 \cdot s \cdot Y(s) + Y(s) = \frac{1}{1000} s^3 \cdot X(s) + \frac{1}{50} s^2 X(s) + \frac{1}{5} s \cdot X(s) + X(s)$$

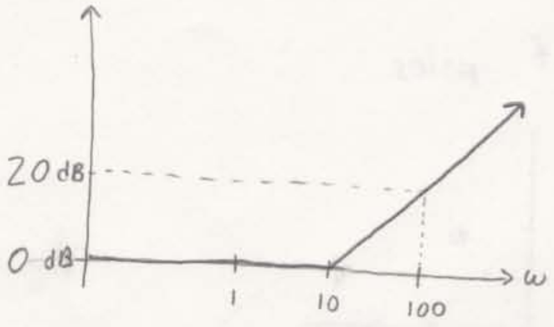
$$\frac{d^3 y(t)}{dt^3} + 2 \cdot \frac{d^2 y(t)}{dt^2} + 2 \cdot \frac{dy(t)}{dt} + y(t) = \frac{1}{1000} \cdot \frac{d^3 x(t)}{dt^3} + \frac{1}{50} \cdot \frac{d^2 x(t)}{dt^2} + \frac{1}{5} \cdot \frac{dx(t)}{dt} + x(t)$$

(e) $H(j\omega) = H(s)\big|_{s=j\omega}$

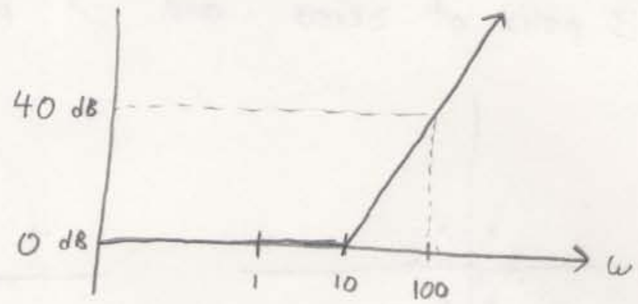
$$H(j\omega) = \frac{\left(1 + \frac{j\omega}{10}\right) \cdot \left(1 + \frac{j\omega}{10} + \left(\frac{j\omega}{10}\right)^2\right)}{(1 + j\omega) \cdot (1 + j\omega + (j\omega)^2)}$$

$$20 \cdot \log_{10} |H(j\omega)| = 20 \cdot \log_{10} \left|1 + \frac{j\omega}{10}\right| + 20 \cdot \log_{10} \left|1 + \frac{j\omega}{10} + \left(\frac{j\omega}{10}\right)^2\right| - 20 \cdot \log_{10} |1 + j\omega| - 20 \cdot \log_{10} \left|1 + j\omega + (j\omega)^2\right|$$

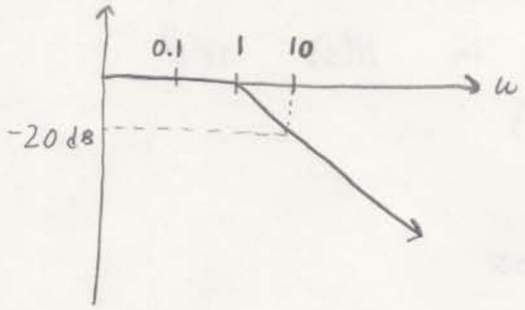
$$20 \cdot \log_{10} \left| 1 + \frac{j\omega}{10} \right|$$



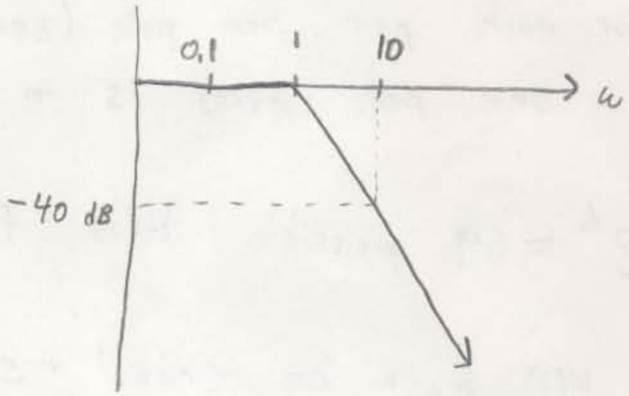
$$20 \cdot \log_{10} \left| 1 + \frac{j\omega}{10} + \left(\frac{j\omega}{10}\right)^2 \right|$$



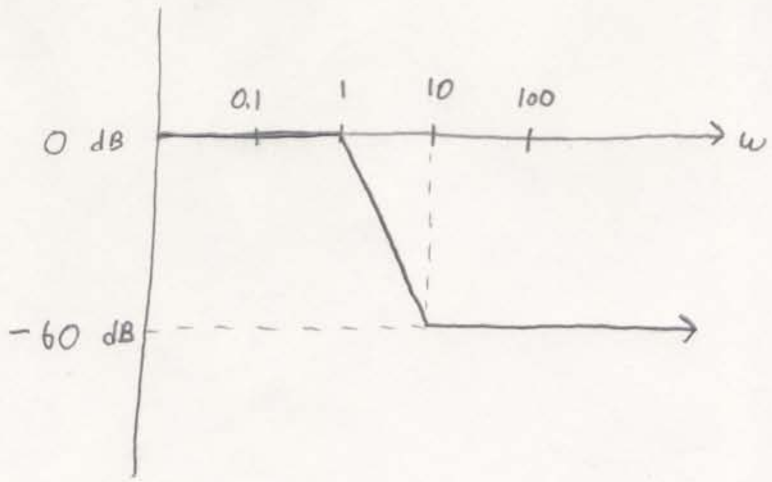
$$-20 \cdot \log_{10} \left| 1 + j\omega \right|$$



$$-20 \log_{10} \left| 1 + j\omega + (j\omega)^2 \right|$$



$$|H(j\omega)|_{dB}$$



The system implements a low pass filter.

Midterm II Review Problems

(5) (d)

$$F(z) = \frac{(1-\frac{1}{8}z^3)(1-az^3)}{z^4(1-\frac{1}{4}z^2)}$$

Poles: $z^4 = 0 \Rightarrow z = 0$ (4 poles at origin)

$$1-\frac{1}{4}z^2 = 0 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2$$

Zeros: $1-\frac{1}{8}z^3 = 0$

$$\Rightarrow e^{j2\pi n} - \frac{1}{8}z^3 = 0$$

$$\Rightarrow z^3 = 8e^{j2\pi n}$$

$$\Rightarrow z = 2e^{j\frac{2\pi}{3}n}, \quad n = 0, 1, 2$$

$1 - az^3 = 0$

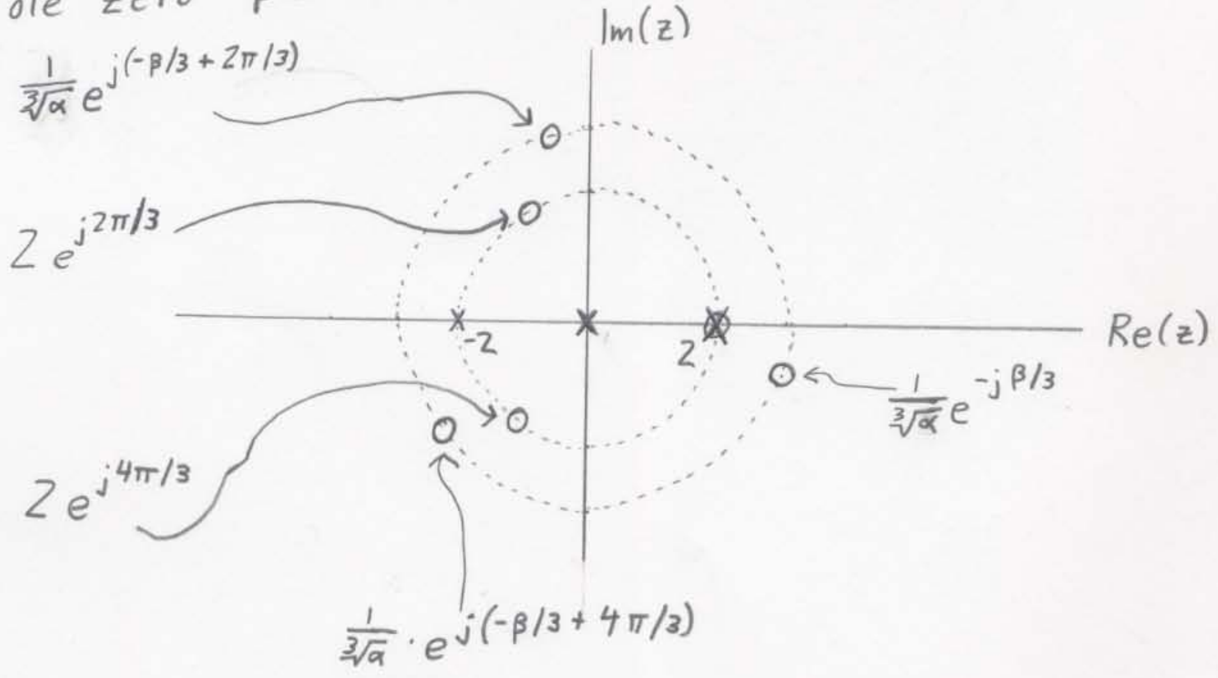
$\Rightarrow a$ is complex, so write $a = \alpha e^{j\beta}$

$$\Rightarrow e^{j2\pi n} - \alpha \cdot e^{j\beta} z^3 = 0$$

$$\Rightarrow z^3 = \frac{1}{\alpha} e^{j(-\beta + 2\pi n)}$$

$$\Rightarrow z = \frac{1}{\sqrt[3]{\alpha}} e^{j\left(-\frac{\beta}{3} + \frac{2\pi}{3} \cdot n\right)}$$

Pole-zero plot of $F(z)$:



(b) $F(z)$ is causal and rational, so the ROC is the region outside the pole with largest magnitude

System is stable iff ROC contains the unit circle

$\Rightarrow$ system is stable iff the poles outside the unit circle are cancelled with zeros

$\Rightarrow$ pole at $z = 2$ is already cancelled

$\Rightarrow$ choose a so that pole at $z = -2$ is also cancelled

$\Rightarrow$ let's cancel the pole at $z = -2$ with the zero at $z = \frac{1}{\sqrt[3]{a}} \cdot e^{j(4\pi/3 - \beta/3)}$

$$-2 = 2 \cdot e^{j\pi} = \frac{1}{\sqrt[3]{\alpha}} \cdot e^{j(4\pi/3 - \beta/3)}$$

$\Rightarrow$ set $\quad z = \frac{1}{\sqrt[3]{\alpha}} \quad \Rightarrow \quad \alpha = \frac{1}{8}$

set $\quad \pi = \frac{4\pi}{3} - \frac{\beta}{3} \quad \Rightarrow \quad \beta = \pi$

$\Rightarrow$ The system is stable if $a = \frac{1}{8} e^{j\pi}$

(c) If $d = \frac{3}{4}$ then

$$G(z) = \frac{1}{\frac{3}{4} + \frac{2}{3} - z} = \frac{1}{\frac{17}{12} - z}$$

The system has one pole at $z = \frac{17}{12}$

Because the system is causal, the ROC is $|z| > \frac{17}{12}$

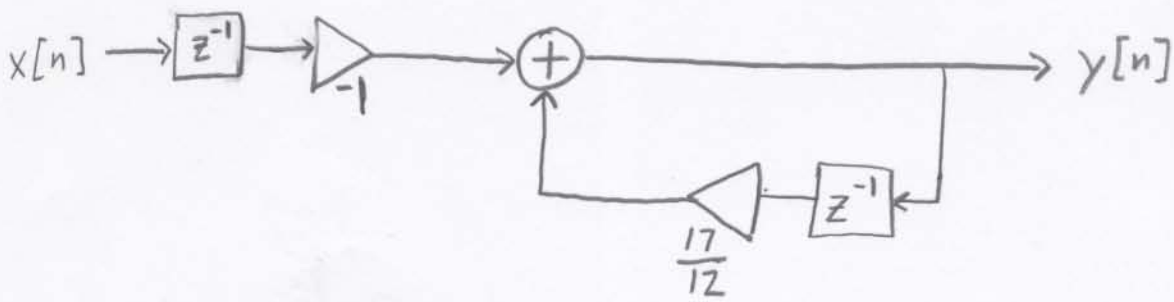
The ROC does not contain the unit circle, so the system is unstable

(d) $$G(z) = \frac{Y(z)}{X(z)} = \frac{1}{\frac{17}{12} - z}$$

$$\frac{17}{12} \cdot Y(z) - z \cdot Y(z) = X(z)$$

$$\frac{17}{12} \cdot z^{-1} \cdot Y(z) - Y(z) = z^{-1} \cdot X(z)$$

$$Y(z) = \frac{17}{12} \cdot z^{-1} \cdot Y(z) - z^{-1} \cdot X(z)$$



(e) $\begin{cases} Y(z) = G(z) \cdot E(z) \\ E(z) = X(z) - H(z) \cdot Y(z) \end{cases}$

$$Y(z) = G(z) \cdot X(z) - G(z) \cdot H(z) \cdot Y(z)$$

$$T(z) = \frac{Y(z)}{X(z)} = \frac{G(z)}{1 + G(z) H(z)}$$

(f)
$$T(z) = \frac{\frac{1}{d + \frac{1}{2d} - z}}{1 + \left(\frac{1}{d + \frac{1}{2d} - z}\right) \cdot \left(\frac{-1}{2} z^{-1}\right)}$$

multiply numerator and denominator by
$$\left(d + \frac{1}{2d} - z\right) \cdot (-2z)$$

$$T(z) = \frac{-2z}{-2z\left(d + \frac{1}{2d} - z\right) + 1}$$

poles of $T(z)$:
$$-2z\left(d + \frac{1}{2d} - z\right) + 1 = 0$$
$$2 \cdot z^2 - 2 \cdot z\left(d + \frac{1}{2d}\right) + 1 = 0$$
$$z^2 - z\left(d + \frac{1}{2d}\right) + \frac{1}{2} = 0$$
$$(z - d)\left(z - \frac{1}{2d}\right) = 0$$
$$z = d, \frac{1}{2d}$$

$T(z)$ is causal, so ROC is region outside pole with largest magnitude

$T(z)$ is stable if all poles are inside the unit circle

$T(z)$ is stable if $|d| < 1$ and $\left|\frac{1}{2d}\right| < 1$

$\Rightarrow T(z)$ is stable if $|d| < 1$ and $|d| > \frac{1}{2}$

$\Rightarrow T(z)$ is stable if $\frac{1}{2} < |d| < 1$

Yes, the overall system is stable if $d = \frac{3}{4}$