Problem 1

The system is not memoryless, linear, not time-invariant (counter example $x(t) = t$), non-causal and non-stable (counter example $x(t) = 1$ for all t).

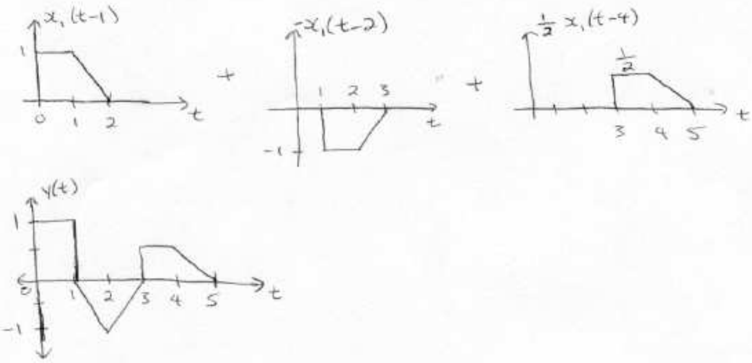
Problem 2

Problem 2

(i)

$$y(t) = x_1(t) * x_2(t) = x_1(t) * \left(\delta(t-1) - \delta(t-2) + \tfrac{1}{2}\delta(t-4)\right)$$

$$= x_1(t-1) - x_1(t-2) + \tfrac{1}{2} x_1(t-4)$$



(ii)

$$X(jw) = \begin{cases} |w|\, e^{-jw} & , \text{ if } |w| < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{-1}^{1} |w|\, e^{-jw} e^{jwt}\, dw \qquad \text{inverse FT}$$

$$= \frac{1}{2\pi}\int_{-1}^{0} -w\, e^{jw(t-1)}\, dw + \frac{1}{2\pi}\int_{0}^{1} w\, e^{jw(t-1)}\, dw$$

integration by parts: $dv = e^{jw(t-1)}$, $u = w$, $v = \dfrac{e^{jw(t-1)}}{j(t-1)}$, $du = 1$

$$= \frac{1}{2\pi}\left[-w\frac{e^{jw(t-1)}}{j(t-1)} - \frac{e^{jw(t-1)}}{(t-1)^2}\right]_{-1}^{0}$$

$$+ \frac{1}{2\pi}\left[-w\frac{e^{jw(t-1)}}{j(t-1)} + \frac{e^{jw(t-1)}}{(t-1)^2}\right]_{0}^{1}$$

$$= \frac{1}{2\pi}\left[-\frac{1}{(t-1)^2} - \frac{e^{-j(t-1)}}{j(t-1)} + \frac{e^{-j(t-1)}}{(t-1)^2} + \frac{e^{j(t-1)}}{j(t-1)} + \frac{e^{j(t-1)}}{(t-1)^2} - \frac{1}{(t-1)^2}\right]$$

$$= \frac{1}{j2\pi(t-1)}\left(e^{j(t-1)} - e^{-j(t-1)}\right) + \frac{1}{2\pi(t-1)^2}\left(e^{j(t-1)} + e^{-j(t-1)}\right) - \frac{1}{\pi(t-1)^2}$$

$$= \frac{\sin(t-1)}{\pi(t-1)} + \frac{\cos(t-1) - 1}{\pi(t-1)^2}$$

Problem 3

Problem 3

(i)

$$a^2 y(t) + 2a \frac{d}{dt} y(t) + \frac{d^2}{dt^2} y(t) = x(t)$$

Taking the Fourier transform of both sides

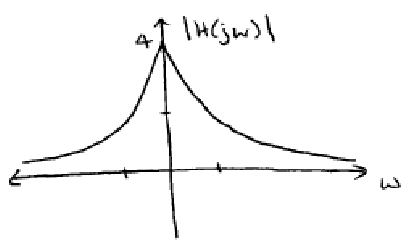
$$a^2 Y(j\omega) + 2a(j\omega) Y(j\omega) + (j\omega)^2 Y(j\omega) = X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a^2 + 2a(j\omega) + (j\omega)^2} = \frac{1}{(a+j\omega)^2}$$

(ii)

For $a = \frac{1}{2}$, $H(j\omega) = \frac{1}{(\frac{1}{2} + j\omega)^2}$

$$\Rightarrow |H(j\omega)| = \frac{1}{|\frac{1}{2} + j\omega|^2} = \frac{1}{\frac{1}{4} + \omega^2}$$



$\Rightarrow$ This system is a lowpass filter

as $|\omega| \to \infty$, $|H(j\omega)| \to 0$

(iii)

The system is stable iff $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$\boxed{\text{case 1: } a > 0}$

$$H(j\omega) = \frac{1}{(a+j\omega)^2} \xleftrightarrow{\text{FT table}} h(t) = t\, e^{-at} u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |t\, e^{-at} u(t)|\, dt = \int_{0}^{\infty} t\, e^{-at}\, dt$$

$$= \left[t \frac{e^{-at}}{-a} - \frac{e^{-at}}{a^2} \right]_0^{\infty} \qquad \text{integration by parts}$$

$$= \frac{1}{a^2}$$

$\Rightarrow$ system is $\boxed{\text{Stable}}$

$\boxed{\text{Case 2: } a = 0}$

$$H(j\omega) = \frac{1}{(j\omega)^2}$$

$$u(t) - \frac{1}{2} \xleftrightarrow{\text{FT table}} \frac{1}{j\omega}$$

By the differentiation in frequency property of FT

$$t\left(u(t) - \tfrac{1}{2}\right) \xleftrightarrow{\text{FT prop}} \frac{1}{(j\omega)^2}$$

$$\int_{-\infty}^{\infty} \left| t\left(u(t) - \tfrac{1}{2}\right)\right| dt = \infty$$

$\Rightarrow$ system is $\boxed{\text{unstable}}$

$\boxed{\text{case 3: } a < 0}$

$$h(t) = -t\, e^{-at} u(-t) \xleftrightarrow{\text{FT}} H(j\omega) = \frac{1}{(a+j\omega)^2}$$

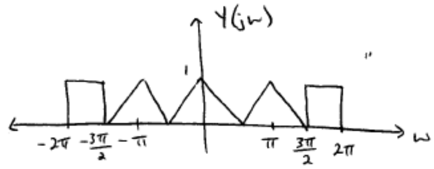
$$\int_{-\infty}^{\infty} |-t e^{-at} u(-t)|\, dt = \int_{-\infty}^{0} t\, e^{-at}\, dt = \frac{1}{a^2} \qquad \text{same as case 1}$$

$\Rightarrow$ system is $\boxed{\text{stable}}$

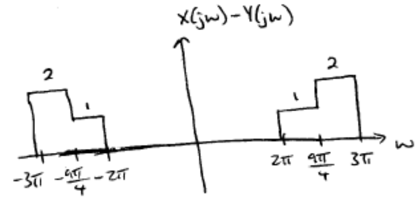
Problem 4

Problem 4 $h(t) = \dfrac{\sin(2\pi t)}{\pi t} \xleftrightarrow{\;FT\;} H(j\omega) = \begin{cases} 1, & |\omega| < 2\pi \\ 0, & |\omega| > 2\pi \end{cases}$

$Y(j\omega) = X(j\omega)H(j\omega)$



$$X(j\omega) - Y(j\omega) = \begin{cases} 2, & \frac{9\pi}{4} < |\omega| < 3\pi \\ 1, & 2\pi < |\omega| < \frac{9\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$



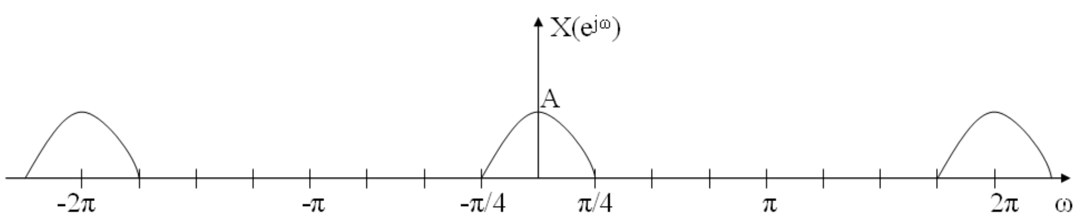
By Parseval's theorem

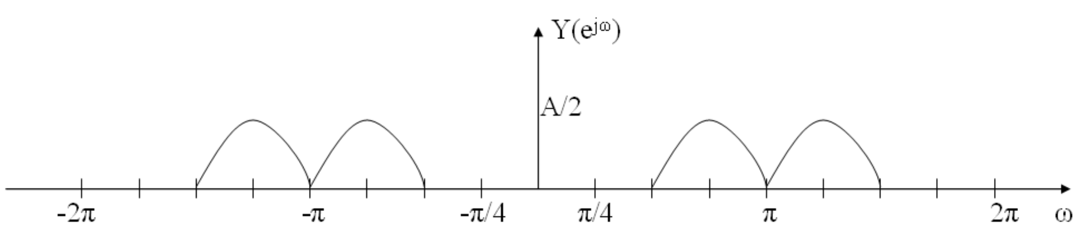
$$\int_{-\infty}^{\infty} |x(t) - y(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega) - Y(j\omega)|^2 \, d\omega$$

$$= \frac{2}{2\pi} \left[\int_{2\pi}^{9\pi/4} (1)^2 \, d\omega + \int_{9\pi/4}^{3\pi} (2)^2 \, d\omega \right]$$

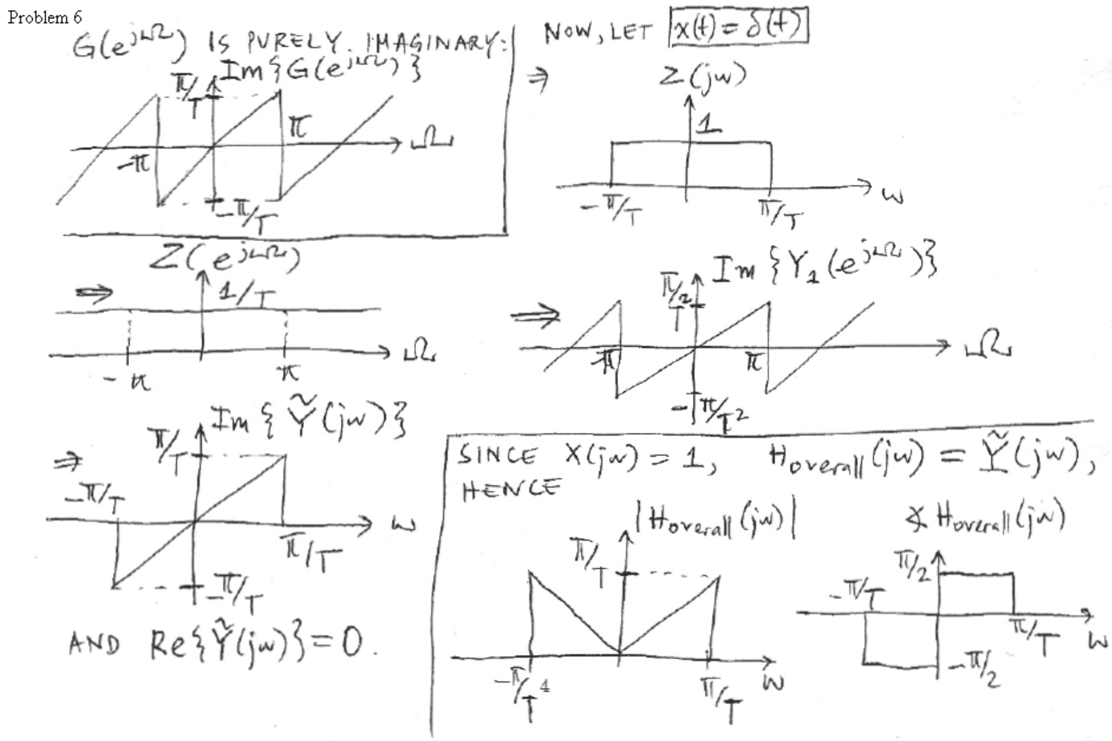
$$= \frac{13}{4}$$

Problem 5





Problem 6



$$y(t) = \frac{d}{dt}x(t) \Rightarrow Y(j\omega) = j\omega X(j\omega) = j\omega e^{-|\omega|}.$$

$$\hat{Y}(j\omega) = \hat{H}(j\omega)X(j\omega) = \begin{cases} j\omega e^{-|\omega|}, & |\omega| \le \pi/T \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow E = \int_{-\infty}^{\infty} |y(t) - \tilde{y}(t)|^2 \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega) - \tilde{Y}(j\omega)|^2 \, d\omega$$

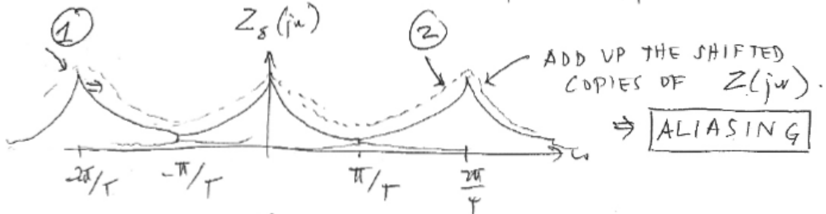
$$= \frac{1}{2\pi}\left[\int_{\pi/T}^{\infty} |j\omega e^{-|\omega|}|^2 d\omega + \int_{-\infty}^{-\pi/T} |j\omega e^{-|\omega|}|^2 d\omega \right]$$

$$= \frac{1}{2\pi} \cdot 2 \cdot \int_{\pi/T}^{\infty} \omega^2 e^{-2\omega} \, d\omega$$

This integral was actually provided in the exam.

$$= \frac{1}{2\pi} \frac{e^{-\frac{2\pi}{T}}}{4} \left(4\left(\frac{\pi}{T}\right)^2 + 4\frac{\pi}{T} + 2 \right)$$

As we increase the sampling frequency, $T \to 0 \Rightarrow E \to 0$, which is clear: We capture more and more of the signal spectrum.



$$Z_\delta(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Z\left(j\left(\omega - k\frac{2\pi}{T}\right)\right).$$

CONSIDER ω BETWEEN 0 AND π/T:

$$Z_\delta(j\omega) = \frac{1}{T}\left(e^{-\omega} + \underbrace{\varepsilon e^{-\left(\frac{2\pi}{T}+\omega\right)}}_{\text{contribution from (1)}} + \dots + \underbrace{\varepsilon e^{-\left(\frac{2\pi}{T}-\omega\right)}}_{\text{contribution from (2)}} + \dots \right)$$

$$= \frac{1}{T}\left(e^{-\omega} + \underbrace{\varepsilon \sum_{n=1}^{\infty} e^{-\left(n\frac{2\pi}{T}+\omega\right)}} + \underbrace{\varepsilon \sum_{n=1}^{\infty} e^{-\left(n\frac{2\pi}{T}-\omega\right)}} \right)$$

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