Homework 10 Solutions

(*Submit your grades to ee120.gsi@gmail.com*)

Problem 1 *(Laplace Transform Properties.)*

(a) OWN Problem 9.21 (g).

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_0^1 e^{-st}dt = \frac{1-e^{-s}}{s}$$

Observe that there is a pole at $s = 0$, and also a zero at $s = 0$. Due to pole-zero cancellation, the pole-zero plot does not have any poles or any zeros and the ROC of $x(t)$ is the entire s-plane.

(b) OWN Problem 9.26.

$$x_1(t) = e^{-2t}u(t), \qquad x_2(t) = e^{-3t}u(t)$$

$$y(t) = x_1(t-2) * x_2(3-t)$$

Using the time shifting, time scaling (reversal), and convolution properties:

$$\begin{aligned}
x_1(t) = e^{-2t}u(t) \quad &\longleftrightarrow \quad X_1(s) = \mathcal{L}\{x_1(t)\} = \frac{1}{s+2} \text{ , with ROC } Re\{s\} > -2 \\
x_2(t) = e^{-3t}u(t) \quad &\longleftrightarrow \quad X_2(s) = \mathcal{L}\{x_2(t)\} = \frac{1}{s+3} \text{ , with ROC } Re\{s\} > -3 \\
x_1(t-2) \quad &\longleftrightarrow \quad \mathcal{L}\{x_1(t-2)\} = e^{-2s}X_1(s) = \frac{e^{-2s}}{s+2} \text{ , with ROC } Re\{s\} > -2 \\
x_2(t+3) \quad &\longleftrightarrow \quad \mathcal{L}\{x_2(t+3)\} = e^{3s}X_2(s) = \frac{e^{3s}}{s+3} \text{ , with ROC } Re\{s\} > -3 \\
x_2(3-t) \quad &\longleftrightarrow \quad \mathcal{L}\{x_2(3-t)\} = e^{3(-s)}X_2(-s) = \frac{e^{-3s}}{3-s} \text{ , with ROC } Re\{s\} < 3 \\
y(t) \quad &\longleftrightarrow \quad Y(s) = \mathcal{L}\{y(t)\} = e^{-5s}X_1(s)X_2(-s) = \frac{e^{-5s}}{(3-s)(s+2)}, \quad -2 < Re\{s\} < 3
\end{aligned}$$

The convolution property indicates that the ROC of $y(t)$ must contain the intersection of the ROCs of $x_1(t-2)$ and $x_2(3-t)$. However, in this case we know that the ROC of $y(t)$ is exactly the intersection of the ROCs of $x_1(t-2)$ and $x_2(3-t)$ because there is no pole-zero cancellation. Notice that the ROC is preserved under time shifting, but not under time scaling or time reversal.

Problem 2 *(Inverse Laplace Transform.)*

(a) OWN 9.22 (c)

$$X(s) = \frac{s+1}{(s+1)^2+9}, \qquad \Re e\{s\} < -1$$

From entry 13 in Table 9.2, we know that

$$e^{t}\cos(3t)u(t) \longleftrightarrow \frac{s-1}{(s-1)^2+9}, \qquad \Re\{s\} > 1$$

Applying the time scaling property, we find

$$e^{-t}\cos(3t)u(-t) \longleftrightarrow -\frac{s+1}{(s+1)^2+9}, \qquad \Re\{s\} < -1$$

Therefore, the inverse Laplace transform of $X(s)$ is

$$x(t) = -e^{-t}\cos(3t)u(-t)$$

(b) OWN 9.22 (e)

We will find a partial fraction expansion of $X(s)$, and then use entries 6 and 7 of Table 9.2 in OWN.

$$X(s) = \frac{s+1}{s^2+5s+6}, \qquad -3 < \Re e\{s\} < -2$$

$$X(s) = \frac{s+1}{(s+3)(s+2)} = \frac{2}{s+3} + \frac{-1}{s+2}, \qquad -3 < \Re e\{s\} < -2$$

$$x(t) = 2e^{-3t}u(t) + e^{-2t}u(-t)$$

(c) OWN 9.22 (g)

We will first express $X(s)$ as a proper fraction, and then combine the time differentiation property with entry 8 in Table 9.2 to find the inverse transform.

$$X(s) = \frac{s^2-s+1}{(s+1)^2}, \qquad \Re e\{s\} > -1$$

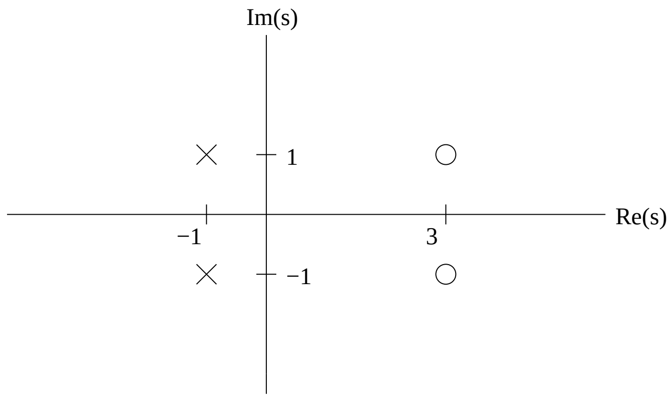
$$X(s) = \frac{s^2+2s+1-3s}{(s+1)^2} = 1 - \frac{3s}{(s+1)^2}, \qquad \Re e\{s\} > -1$$

$$x(t) = \delta(t) - 3\frac{d}{dt}\{te^{-t}u(t)\}$$

$$x(t) = \delta(t) - 3e^{-t}u(t) + 3te^{-t}u(t)$$

Problem 3 *(Stability and Causality.)*

Because $h(t)$ is real, its poles and zeros must occur in complex conjugate pairs. Therefore, the known poles and zeros of $H(s)$ are shown in the following figure.



Since $H(s)$ has exactly 2 zeros at infinity, $H(s)$ has *at least* two more unknown finite poles. If $H(s)$ has more than 4 poles, then it will have a zero at some finite location for every additional pole. Furthermore, because $h(t)$ is causal and stable, all poles of $H(s)$ must lie in the left half of the s-plane and the ROC must contain the $j\omega$ axis.

- OWN 9.51(a)

  This is **true**. Consider $g(t) = h(t)e^{-3t}$. The Laplace transform $G(s) = H(s+3)$ will have a ROC that is equal to the ROC of $H(s)$ shifted to the left by 3. Clearly, the ROC of $G(s)$ will still contain the $j\omega$ axis.

- OWN 9.51(b)

  There is **insufficient information**. As stated before, $H(s)$ has at least two more unknown poles. We do not know where the rightmost pole is, so we cannot determine the ROC of $H(s)$.

- OWN 9.51(c)

  This is **true**. Since $H(s)$ is rational, it may be expressed as the ratio of two polynomials in s. Furthermore, since $h(t)$ is real, the coefficients of these polynomials will be real. We can write

  $$\frac{Y(s)}{X(s)} = H(s) = \frac{P(s)}{Q(s)}$$

  where $P(s)$ and $Q(s)$ are polynomials in s. The differential equation relating $y(t)$ and $x(t)$ is found by taking the inverse Laplace transform of $Y(s)Q(s) = X(s)P(s)$. Clearly, this differential equation will have only real coefficients.

Problem 4 *(Stability and Causality (continued).)*

- OWN 9.51(d)

  This is **false**. Because $H(s)$ has two zeros at $s = \infty$, the limit of $H(s)$ as s goes to ∞ will be 0.

- OWN 9.51(e)

  This is **true**. As stated at the beginning of the problem, $H(s)$ has two zeros at infinity and at least two finite zeros, so $H(s)$ must have at least four poles.

- OWN 9.51(f)

  There is **insufficient information**. As stated earlier, $H(s)$ may have other zeros, including zeros on the $j\omega$ axis.

- OWN 9.51(g)

  This is **false**. We know that

$$e^{3t}\sin t = \frac{1}{j2}e^{(3+j)t} - \frac{1}{j2}e^{(3-j)t}$$

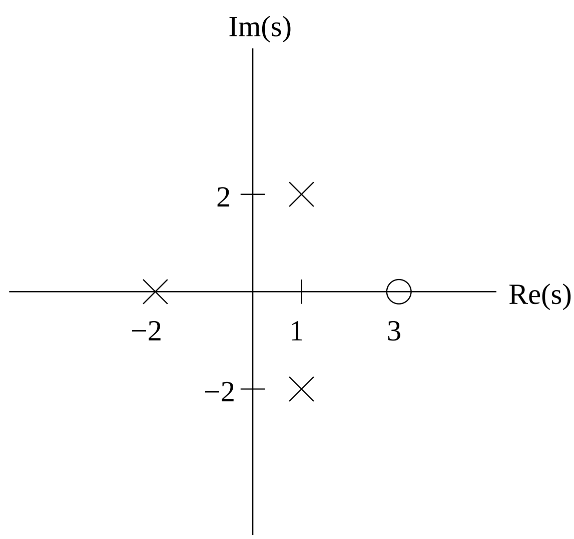
Both $e^{(3+j)t}$ and $e^{(3-j)t}$ are eigenfunctions of the LTI system. The response of the system to these two inputs will be $H(3+j)\cdot e^{(3+j)t}$ and $H(3-j)\cdot e^{(3-j)t}$, respectively. Because $H(s)$ has zeros at $3\pm j$, the output of the system in response to these two exponential inputs has to be zero. Therefore, the response of the system to the input $e^{3t}\sin t$ will be 0 for all t.

Problem 5 *(An LTI System.)*

- (a)

$$H(s) = \frac{5(s-3)}{(s+2)(s^2-2s+5)}$$

There is a zero at $s=3$, and poles at $s=-2$ and $s=1\pm j2$. The pole-zero diagram is shown in the following figure. We note that there are three possible ROCs.



- (b)

  Because $H(s) = Y(s)/X(s)$, we can write $Y(s)(s+2)(s^2-2s+5) = 5(s-3)X(s)$, which can be expanded as

$$s^3Y(s) + sY(s) + 10Y(s) = 5sX(s) - 15X(s)$$

The time-domain differential equation is given by

$$\frac{d^3y(t)}{dt^3} + \frac{dy(t)}{dt} + 10y(t) = 5\frac{dx(t)}{dt} - 15x(t)$$

- (c)

  First, we find the partial fraction expansion of the transfer function.

$$H(s) = \frac{5(s-3)}{(s+2)(s^2-2s+5)} = \frac{A}{s+2} + \frac{Bs+C}{s^2-2s+5}$$
$$5s - 15 = A(s^2-2s+5) + (Bs+C)(s+2)$$
$$0s^2 + 5s - 15 = (A+B)s^2 + (-2A+2B+C)s + (5A+2C)$$
$$A = \frac{-25}{13}, \quad B = \frac{25}{13}, \quad C = \frac{-35}{13}$$
$$H(s) = \frac{1}{13}\left(\frac{-25}{s+2} + \frac{25s-35}{(s-1)^2+(2)^2}\right)$$
$$H(s) = \frac{1}{13}\left(\frac{-25}{s+2} + \frac{25(s-1)}{(s-1)^2+(2)^2} - \frac{10}{(s-1)^2+(2)^2}\right)$$

  If the system is causal, then the ROC is the right half plane $\Re\{s\} > 1$. Using transform pairs 6, 13, and 14 in Table 9.2, we see that the impulse response is given by

$$h(t) = -\frac{25}{13}e^{-2t}u(t) + \frac{25}{13}e^{t}\cos(2t)u(t) - \frac{5}{13}e^{t}\sin(2t)u(t)$$

  The resulting system is not stable, since the ROC does not contain the $j\omega$ axis.

- (d)

  If the system is stable, then the ROC must contain the $j\omega$ axis, hence the ROC is $-2 < \Re\{s\} < 1$. To find the inverse transform, we first combine transform pair 11 in Table 9.2 with the time scaling property in Table 9.1, with $a = -1$, to see that the function $\cos(\omega_0 t)u(-t)$ has Laplace transform

$$\frac{(-s)}{(-s)^2+\omega_0^2} = \frac{-s}{s^2+\omega_0^2}$$

  and ROC $\Re\{s\} < 0$. Then, applying the shifting in the s-domain property from Table 9.1, with $s_0 = 1$, we see that the function $e^t\cos(\omega_0 t)u(-t)$ has Laplace transform

$$\frac{-(s-1)}{(s-1)^2+\omega_0^2}$$

  with ROC $\Re\{s\} < 1$. Using similar reasoning, we see that the function $-e^t\sin(\omega_0 t)u(-t)$ has Laplace transform

$$\frac{\omega_0}{(s-1)^2+\omega_0^2}$$

  with ROC $\Re\{s\} < 1$. Therefore, using transform pair 6 in Table 9.2 and the two transform pairs derived here, the impulse response of the system is given by

$$h(t) = -\frac{25}{13}e^{-2t}u(t) - \frac{25}{13}e^{t}\cos(2t)u(-t) + \frac{5}{13}e^{t}\sin(2t)u(-t)$$

  This is clearly not a causal system.

Problem 6 *(LTI System Analysis.)*

If $x(t) = e^{2t}$ produces $y(t) = (1/6)e^{2t}$, then $H(2) = 1/6$. By taking the Laplace transform of both sides of the differential equation, we find that

$$H(s) = \frac{\frac{1}{s+4} + b\frac{1}{s}}{s+2} = \frac{s + b(s+4)}{s(s+4)(s+2)}$$

Substituting in $H(2) = 1/6$, we find that $b = 1$. Therefore,

$$H(s) = \frac{2(s+2)}{s(s+4)(s+2)} = \frac{2}{s(s+4)}$$

Problem 7 *(LTI System Analysis.)*

(a) The overall system function $H(s)$ is equal to $H_1(s) \cdot H_2(s)$, because the two systems are cascaded together. Using Table 9.2 in OWN,

$$H_1(s) = \frac{1}{s+2} \qquad \Re\{s\} > -2$$

$H_2(s)$ is found by taking the Laplace transform of the differential equation characterizing System 2

$$sY(s) + Y(s) = sW(s) + \alpha W(s)$$

$$H_2(s) = \frac{Y(s)}{W(s)} = \frac{s+\alpha}{s+1} \qquad \Re\{s\} > -1$$

The ROC must be $\Re\{s\} > -1$, and not $\Re\{s\} < -1$, because we are given that System 2 is causal. Thus, the system function is given by

$$H(s) = \frac{s+\alpha}{(s+2)(s+1)} \qquad \Re\{s\} > -1$$

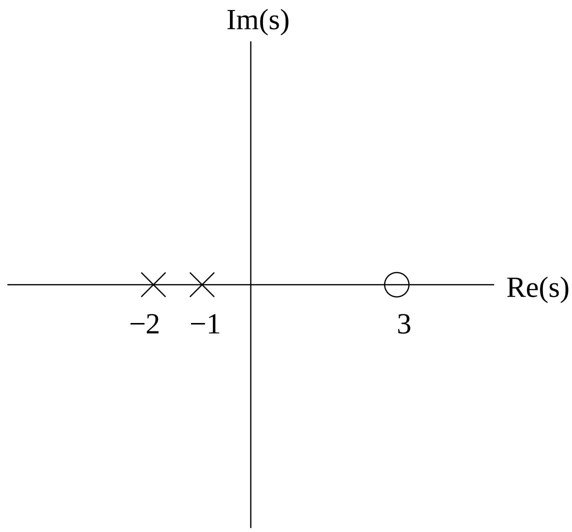
Finally, we can solve for α by using the fact that an input $x(t) = e^{3t}$ produces an output $y(t) = 0$. This means that $H(3) = 0$. Thus

$$\frac{3+\alpha}{(5)(4)} = 0$$

which means that $\alpha = -3$. The overall system function is given by

$$H(s) = \frac{s-3}{(s+2)(s+1)} \qquad \Re\{s\} > -1$$

The pole-zero plot is shown below.



(b) The differential equation relating $x(t)$ and $y(t)$ can be found by using the fact that $H(s) = \frac{Y(s)}{X(s)}$

$$Y(s)(s+2)(s+1) = X(s)(s-3)$$

Distributing on both sides of the equation gives

$$s^2Y(s) + 3sY(s) + 2Y(s) = sX(s) - 3X(s)$$

Taking the inverse Laplace transform, we find that

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} - 3x(t)$$