Homework 12: No due date
Homeowork GSI: Pulkit Grover.

Note: For the final examination, we do not expect you to have worked on the optional problems.

Problem 1 *(Properties of the z-Transform)*

OWN 10.44 (a), (b), (c). In each case, also determine the region of convergence (in terms of the region of convergence R of the original signal $x[n]$).

Remark. These properties are at the foundation of wavelets.

Problem 2 *(Discrete-time LTI system analysis.)*

OWN Problem 10.47

Problem 3 *(z-transform properties: Filter banks.)*

Consider the two-channel filterbank illustrated in Figure 1.

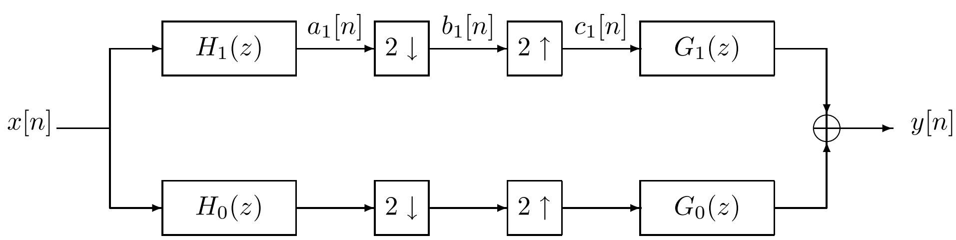

Figure 1: A two-channel filter bank.

The box labeled $2\downarrow$ denotes "downsampling by two", that is,

$$b_1[n] \quad = \quad a_1[2n]. \tag{1}$$

From Problem 1, it follows that $B_1(z) = \frac{1}{2}(A_1(z^{1/2}) + A_1(-z^{1/2}))$.

The box labeled $2\uparrow$ denotes "upsampling by two", that is,

$$c_1[n] \quad = \quad \begin{cases} b_1[n/2], & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

Again, from Problem 1, it follows that $C_1(z) = B_1(z^2)$.

(a) Determine $Y(z)$ in terms of $X(z)$ and $H_0(z), H_1(z), G_0(z), G_1(z)$.

(b) Find conditions on $H_0(z), H_1(z), G_0(z), G_1(z)$ such that $Y(z) = X(z)$. *Hint:* There are two conditions that must be satisfied.

Optional Problem *(Filter design.)*

Suppose we want to implement a (continuous-time) filter with a magnitude response that satisfies

$$|H(j\omega)|^2 \quad = \quad \frac{1}{(1+(j(\omega-4))^{2K})\,(1+(j(\omega+4))^{2K})}. \tag{3}$$

We want the filter to have a *real-valued impulse response.* As we have seen earlier, the Fourier transform of a real-valued signal satisfies $H^*(j\omega) = H(-j\omega)$. Hence, our filter must satisfy

$$|H(j\omega)|^2 \quad = \quad H(j\omega)H(-j\omega). \tag{4}$$

(a) Draw the pole/zero diagram of $H(s)H(-s)$, where $H(s)$ denotes the transfer function of our filter.

(b) For $K = 2$, determine the differential equation (with real-valued coefficients) of a stable and causal filter whose frequency response satisfies Equation (3).

(c) Describe in words and with a rough sketch of the magnitude response the operation that this filter performs.

Optional Problem *(Trapdoors and Fibonacci numbers)*

Consider the system shown in Fig. 2. Balls with numbers on them go in the box. At any time, there can be at most two balls in the box. At each time, the box releases one of the two balls, chosen arbitrarily. Let input $X[n]$ denote the number on the ball that goes in at time n, the state $S[n]$ be the sum of the numbers on the two balls inside the box (before one is released), and the output $Y[n]$ be the ball that is released. At time $n+1$, $X[n+1]$ enters the box, and $Y[n+1]$ is released, and so on.

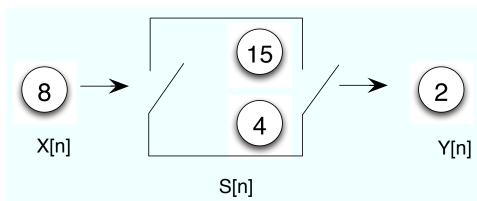

Figure 2: Problem 5f

a) Derive a difference equation connecting $X[\cdot]$, $Y[\cdot]$, and the output $S[\cdot]$. Represent it as a block-diagram.

b) Suppose the inputs $X[n]$'s are chosen such that $X[n] = Y[n] + S[n-1]$. Find an explicit formula for $S[n]$ (in terms of n).

Hint: Note that $S[n]$'s now satisfy a difference equation. Now use the unilateral z-transform.

Remark: The problem set-up and the first part of the problem are the same as that of the *Optional Problem* in Homework 2.