Homework 12 Solutions

Problem 1

Problem 5

OWN 10.44 (a)

By the time shifting and linearity properties in OWN Table 10.1, the z-transform of

$$x_a[n] = x[n] - x[n-1]$$

is

$$X_a(z) = X(z) - z^{-1}X(z) = \frac{z-1}{z}X(z)$$

with ROC R with the possible deletion of $z = 0$.

OWN 10.44 (b)

We can find the z-transform of

$$x_b[n] = \begin{cases} x[\frac{n}{2}] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

by using the time expansion property in OWN Table 10.1, as

$$X_b(z) = X(z^2) \quad \text{with ROC} \quad R^{1/2} = \{z : z^2 \in R\}.$$

Alternatively, we can find the z-transform by evaluating the definition

$$\begin{aligned} X_b(z) &= \sum_{n=-\infty}^{\infty} x_b[n]z^{-n} \\ &= \sum_{n \text{ even}} x[\frac{n}{2}]z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x[m]z^{-2m} \\ &= X(z^2). \end{aligned}$$

OWN 10.44 (c)

Define

$$g[n] = \frac{1}{2}(x[n] + (-1)^n x[n]).$$

Observe that $g[2n] = x[2n]$, and that $g[n] = 0$ for n odd. By the scaling in the z-domain property and the linearity property in OWN Table 10.1, the z-transform of $g[n]$ is $G(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$, with ROC R. Now we find the z-transform of $x_c[n] = x[2n]$ by evaluating the definition of the z-transform,

$$\begin{aligned}
X_c(z) &= \sum_{n=-\infty}^{\infty} x_c[n]z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x[2n]z^{-n} \\
&= \sum_{n=-\infty}^{\infty} g[2n]z^{-n} \\
&= \sum_{m \text{ even}} g[m]z^{-m/2} \\
&= \sum_{m=-\infty}^{\infty} g[m]z^{-m/2} \\
&= G(z^{1/2}) \\
&= \frac{1}{2}X(z^{1/2}) + \frac{1}{2}X(-z^{1/2})
\end{aligned}$$

and the ROC is R.

Problem 2 *(Discrete-time LTI system analysis)*

OWN 10.47

- *(a)*

  From Clue 1, we know that $H(-2) = 0$. From Clue 2, we know that when

  $$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad , \qquad |z| > \frac{1}{2}$$

  then

  $$Y(z) = 1 + \frac{a}{1 - \frac{1}{4}z^{-1}} = \frac{1 - \frac{1}{4}z^{-1} + a}{1 - \frac{1}{4}z^{-1}} \quad , \qquad |z| > \frac{1}{4}$$

  Therefore,

  $$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1 - \frac{1}{4}z^{-1} + a\right)\left(1 - \frac{1}{2}z^{-1}\right)}{1 - \frac{1}{4}z^{-1}} \quad , \qquad |z| > \frac{1}{4}$$

  Substituting $z = -2$ into this equation, and using the fact that $H(-2) = 0$, we find that

  $$a = -\frac{9}{8}$$

- *(b)*

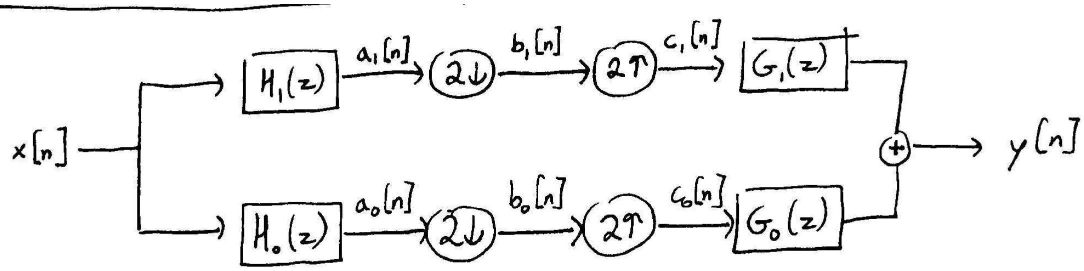
  The response to the signal $x[n] = 1 = 1^n$ will be $y[n] = H(1)x[n]$.

  $$y[n] = H(1) = -\frac{1}{4}$$

Problem 3 *(z-transform properties: Filter banks)*

Parts *(a)* and *(b)* are shown in the following figure.

3 a)



Recall from HW 10 that $b_i[n] = a_i[2n]$ so $B_i(z) = \frac{1}{2}A_i(z^{\frac{1}{2}}) + \frac{1}{2}A_i(-z^{\frac{1}{2}})$

and $c_i[n] = \begin{cases} b_i[\frac{n}{2}] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$ so $C_i(z) = B_i(z^2)$ for $i = 0, 1$.

Combining these two equations, we get that $C_i(z) = \frac{1}{2}A_i((z^2)^{\frac{1}{2}}) + \frac{1}{2}A_i(-(z^2)^{\frac{1}{2}})$

$$C_i(z) = \frac{1}{2}A_i(z) + \frac{1}{2}A_i(-z)$$

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$$A_1(z) = H_1(z)\,X(z) \qquad\Rightarrow\qquad C_1(z) = \tfrac{1}{2} H_1(z)X(z) + \tfrac{1}{2} H_1(-z)X(-z)$$

$$A_0(z) = H_0(z)\,X(z) \qquad\qquad C_0(z) = \tfrac{1}{2} H_0(z)X(z) + \tfrac{1}{2} H_0(-z)X(-z)$$

$$Y(z) = G_0(z)\,C_0(z) + G_1(z)\,C_1(z)$$

$$Y(z) = \tfrac{1}{2} H_0(z)G_0(z)X(z) + \tfrac{1}{2} H_0(-z)G_0(z)X(-z) + \tfrac{1}{2} H_1(z)G_1(z)X(z) + \tfrac{1}{2} H_1(-z)G_1(z)X(-z)$$

Let's group terms with respect to $X(z)$ and $X(-z)$.

$$Y(z) = \tfrac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]X(z) + \tfrac{1}{2}\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]X(-z)$$

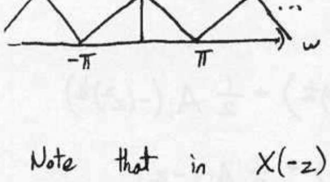
b) We would like to find conditions on H_0, G_0, H_1, and G_1 such that $Y(z)$ is exactly equal to $X(z)$. Well, if we use the following two conditions, we will get $Y(z) = X(z)$:

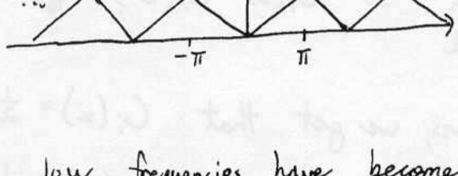
1) $H_0(z)\,G_0(z) + H_1(z)\,G_1(z) = 2$

2) $H_0(-z)\,G_0(z) + H_1(-z)\,G_1(z) = 0$

<u>aside</u>: We could, of course, make stricter conditions for each filter individually. However, that would exclude possible solutions that our more general conditions allow. Also, note that $X(-z)$ is the "aliased" version of $X(z)$. This is easier to see in a plot of the DTFT of $X(z)$ and $X(-z)$ for some sample signal x.

$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$ $\qquad$ $X(e^{j(\omega+\pi)}) = X(-z)|_{z=e^{j\omega}}$ $\quad$ (since $-1 = e^{j\pi}$)





Note that in $X(-z)$ the low frequencies have become high frequencies and vice versa. Mixing $X(z)$ and $X(-z)$ will result in an extremely distorted version of our desired signal, $x(-z)$. That is why we try to zero out $X(-z)$.

Problem 4 *(Filter design.)*

(a)

$$\begin{aligned}
H(j\omega)H(-j\omega) &= |H(j\omega)|^2 = \frac{1}{(1+(j(\omega-4))^{2K})\,(1+(j(\omega+4))^{2K})} \\
H(s)H(-s) &= \frac{1}{(1+(s-4j)^{2K})\,(1+(s+4j)^{2K})}
\end{aligned}$$

We want to find the poles of $H(s)H(-s)$. To do so, we first define $p = s-4j$ and $q = s+4j$, and solve for the poles in terms of p and q.

$$H(s)H(-s) = \frac{1}{(1+p^{2K})\,(1+q^{2K})}$$

Since the roots of $1+p^{2K}=0$ and $1+q^{2K}=0$ are the same, it is sufficient to solve for the roots of one of these polynomials.

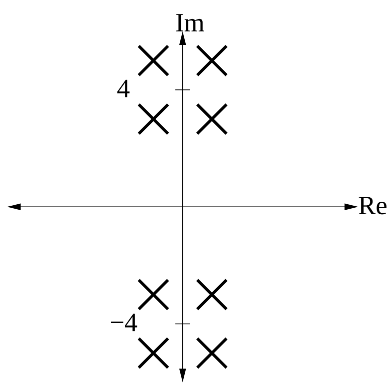
$$\begin{aligned}
1+p^{2K} &= 0 \\
e^{j2\pi n}+p^{2K} &= 0 \\
p^{2K} &= -e^{j2\pi n} = e^{j(2\pi n+\pi)} \\
p &= e^{j\left(\frac{\pi n}{K}+\frac{\pi}{2K}\right)} \text{ for } n = 0,1,\ldots,2K-1
\end{aligned}$$

We plug the $2K$ unique roots at p and q back into $p = s-4j$ and $q = s+4j$. Therefore the roots of $H(s)H(-s)$ are at

$$s = e^{j\left(\frac{\pi n}{K}+\frac{\pi}{2K}\right)} \pm 4j$$

for $n = 0,1,\ldots,2K-1$.

For $K=2$, $H(s)H(-s)$ has poles at $s = e^{j\pi/4} \pm 4j$, $s = e^{j3\pi/4} \pm 4j$, $s = e^{j5\pi/4} \pm 4j = e^{-j3\pi/4} \pm 4j$, and $s = e^{j7\pi/4} \pm 4j = e^{-j\pi/4} \pm 4j$.



(b)

For $K=2$, we would like to find a real-valued, stable and causal filter that satisfies $|H(j\omega)|^2$. In other words $H(s)$ must contain poles such that, when combined with their complex conjugates form exactly $|H(j\omega)|^2$. A rational $H(s)$ is causal iff its ROC is the right-half plane, to the right of the pole largest in magnitude. $H(s)$ is stable iff its ROC includes the $j\omega$-axis. Thus all the poles of $H(s)$ must be left of the $j\omega$-axis (e.g. $Re(s) < 0$). So we make the poles of $H(s)$: $s = e^{j3\pi/4} \pm 4j$ and $s = e^{-j3\pi/4} \pm 4j$.

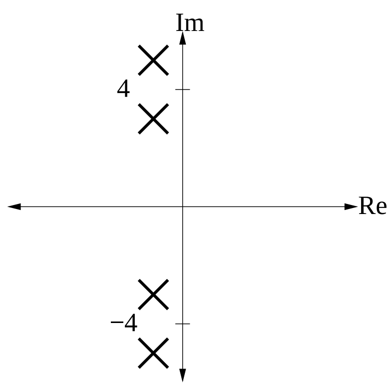
$$
\begin{aligned}
H(s) &= \frac{1}{(s - e^{j\frac{3\pi}{4}} - 4j)(s - e^{j\frac{3\pi}{4}} + 4j)(s - e^{-j\frac{3\pi}{4}} - 4j)(s - e^{-j\frac{3\pi}{4}} + 4j)} \\
\frac{1}{H(s)} &= ((s - e^{j\frac{3\pi}{4}})^2 + 16)((s - e^{-j\frac{3\pi}{4}})^2 + 16) \\
&= (s - e^{j\frac{3\pi}{4}})^2(s - e^{-j\frac{3\pi}{4}})^2 + 16(s - e^{j\frac{3\pi}{4}})^2 + 16(s - e^{-j\frac{3\pi}{4}})^2 + 256 \\
&= (s^2 - 2e^{j\frac{3\pi}{4}}s + e^{j\frac{3\pi}{2}})(s^2 - 2e^{-j\frac{3\pi}{4}}s + e^{-j\frac{3\pi}{2}}) + 16(s^2 - 2e^{j\frac{3\pi}{4}}s + e^{j\frac{3\pi}{2}}) + 16(s^2 - 2e^{-j\frac{3\pi}{4}}s + e^{-j\frac{3\pi}{2}}) + 256 \\
&= (s^4 - 2e^{-j\frac{3\pi}{4}}s^3 + js^2 - 2e^{j\frac{3\pi}{4}}s^3 + 4s^2 - 2e^{-j\frac{3\pi}{4}}s - js^2 - 2e^{j\frac{3\pi}{4}}s + 1) + 32s^2 + 32\sqrt{2}s + 256 \\
&= s^4 + 2\sqrt{2}s^3 + 36s^2 + 34\sqrt{2}s + 257
\end{aligned}
$$

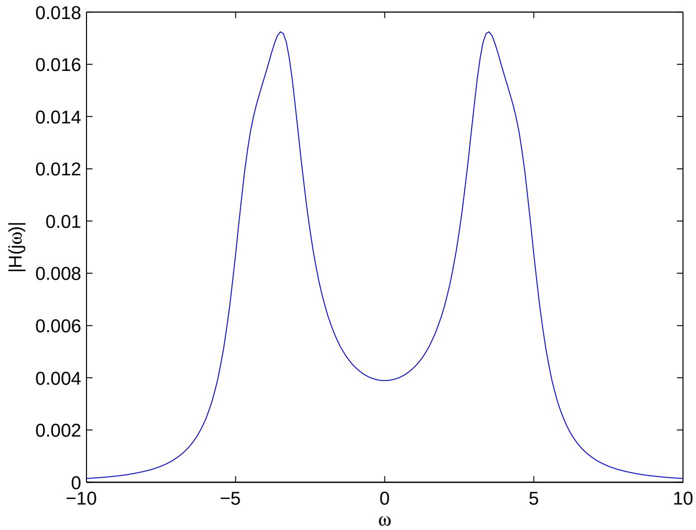
Now we use the Laplace transform properties (see OWN Table 9.1) to find the difference equation that describes this filter.

$$
\begin{aligned}
Y(s) &= H(s)X(s) \\
(s^4 + 2\sqrt{2}s^3 + 36s^2 + 34\sqrt{2}s + 257)Y(s) &= X(s) \\
\frac{d^4}{dt^4}y(t) + 2\sqrt{2}\frac{d^3}{dt^3}y(t) + 36\frac{d^2}{dt^2}y(t) + 34\sqrt{2}\frac{d}{dt}y(t) + 257y(t) &= x(t)
\end{aligned}
$$

(c)

The magnitude response of the filter $H(s)$ will have a peak at $\omega = 4$ and at $\omega = -4$, because that is where we are closest to the poles as we traverse the $j\omega$-axis.





Problem 5 *(Trapdoor and Fibonacci Numbers)*

Use the solutions of Optional problem in Homework 2, and what follows for finding the value for each n. Parts *(a)* and *(b)* are shown in the following figure.

⑥

[redacted] a) $F_1 = 1 \quad F_2 = 1 \quad F_{n+2} = F_{n+1} + F_n \quad \forall\ n \geq 1$

We would like to find an explicit formula for F_n. We can use the unilateral z-transform to find it. One consequence of this method is that our function will be $0 \ \forall\ n < 0$. We would like a function that can generate F_1 onwards so let's set $x[n] = F_{n+1}$ so that $x[0] = F_1$.

$$x[n+1] = x[n] + x[n-1]$$

$\Big\downarrow$ UZ

Recall from Table 10.3:

$$x[n-1] \xrightarrow{UZ} z^{-1}X(z) + x[-1]$$

$$x[n+1] \xrightarrow{UZ} z\,X(z) \quad z\,x[0]$$

$$z\,X(z) - z\,x[0] = X(z) + z^{-1}X(z) + x[-1]$$

Note that $x[-1] = F_0 = 0$ since

$$F_0 + F_1 = F_2$$

$$F_0 + 1 = 1 \Rightarrow F_0 = 0$$

$$z\,X(z) - z = X(z) + z^{-1}X(z)$$

$$X(z) = \frac{z}{z - 1 - z^{-1}} = \frac{z}{z(1 - z^{-1} - z^{-2})}$$

$$X(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{\left(1 - \frac{1-\sqrt5}{2}z^{-1}\right)\left(1 - \frac{1+\sqrt5}{2}z^{-1}\right)} = \frac{A}{1 - \frac{1+\sqrt5}{2}z^{-1}} + \frac{B}{1 - \frac{1-\sqrt5}{2}z^{-1}}$$

$$A\left(1 - \frac{1-\sqrt5}{2}z^{-1}\right) + B\left(1 - \frac{1+\sqrt5}{2}z^{-1}\right) = 1$$

$$A + B = 1 \Rightarrow A = 1 - B$$

$$A\left(-\frac{1-\sqrt5}{2}\right) + B\left(-\frac{1+\sqrt5}{2}\right) = 0$$

$$(1-B)\left(-\frac{1-\sqrt5}{2}\right) + B\left(-\frac{1+\sqrt5}{2}\right) = 0$$

$$\frac{B \quad B\sqrt5 - B \quad B\sqrt5}{2} = \frac{1-\sqrt5}{2}$$

$$-B\sqrt5 = \frac{1-\sqrt5}{2}$$

$$B = \frac{-1+\sqrt5}{2\sqrt5} = \frac{5-\sqrt5}{10}$$

$$A = 1 - \frac{5-\sqrt5}{10} = \frac{5+\sqrt5}{10}$$

$$\therefore\ X(z) = \frac{5+\sqrt5}{10}\left(\frac{1}{1 - \frac{1+\sqrt5}{2}z^{-1}}\right) + \frac{5-\sqrt5}{10}\left(\frac{1}{1 - \frac{1-\sqrt5}{2}z^{-1}}\right)$$

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$$X(z) = \frac{5+\sqrt{5}}{10}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}z^{-1}}\right) + \frac{5-\sqrt{5}}{10}\left(\frac{1}{1-\frac{1-\sqrt{5}}{2}z^{-1}}\right)$$

Recall: $\frac{1}{1-az^{-1}} \xrightarrow{uz^{-1}} a^n u[n]$

$\downarrow uz^{-1}$

$$x[n] = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n u[n] + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n u[n]$$

$F_n = x[n-1]$ by definition.

$$\boxed{F_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \qquad n \geq 1}$$

Note that there are many different possible formulas for $x[n]$ and thus for F_n. They differ based on what we choose for our "starting point" at $n=0$.

b) $\boxed{F_{21} = 10946}$