Homework 3 Solutions

(Send your grades to ee120.gsi@gmail.com. Check the course website for details)

Review Problem 1 *(Orthogonality.)*

(i) In order to find an orthormal basis, we follow the Gram-Schmidt algorithm. Since we have four vectors, we will have at most four basis vectors. Lets call them $\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3$, and $\widehat{\beta}_4$.

$$\widehat{\beta}_1 = \frac{\overrightarrow{v_1}}{\|\overrightarrow{v_1}\|} = \frac{\overrightarrow{v_1}}{\sqrt{46}} = [0.1474 \quad 0.5898 \quad 0.2949 \quad 0 \quad 0.7372]^\top$$

$$\widehat{\beta}_2 = \frac{\overrightarrow{v_2} - (\overrightarrow{v_2}^\top \widehat{\beta}_1)\widehat{\beta}_1}{\|\overrightarrow{v_2} - (\overrightarrow{v_2}^\top \widehat{\beta}_1)\widehat{\beta}_1\|} = [0.0130 \quad -0.6962 \quad -0.2733 \quad 0 \quad 0.6637]^\top$$

$$\widehat{\beta}_3 = \frac{\overrightarrow{v_3} - (\overrightarrow{v_3}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_3}^\top \widehat{\beta}_1)\widehat{\beta}_1}{\|\overrightarrow{v_3} - (\overrightarrow{v_3}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_3}^\top \widehat{\beta}_1)\widehat{\beta}_1\|} = [0.0090 \quad 0.4009 \quad -0.9151 \quad 0 \quad 0.0435]^\top$$

$$\widehat{\beta}_4 = \frac{\overrightarrow{v_4} - (\overrightarrow{v_4}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{v_4}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_4}^\top \widehat{\beta}_1)\widehat{\beta}_1}{\|\overrightarrow{v_4} - (\overrightarrow{v_4}^\top \widehat{\beta}_3)\widehat{\beta}_3 - (\overrightarrow{v_4}^\top \widehat{\beta}_2)\widehat{\beta}_2 - (\overrightarrow{v_4}^\top \widehat{\beta}_1)\widehat{\beta}_1\|} = [0 \quad 0 \quad 0 \quad 0 \quad 0]^\top$$

Therefore, we have only three basis vectors for $\mathbf{S}$ (not surprising since $\overrightarrow{v_4} = \overrightarrow{v_1} + 2\overrightarrow{v_2}$).

(ii)

$$\overrightarrow{v_1} = w_{11}\widehat{\beta}_1 + w_{12}\widehat{\beta}_2 + w_{13}\widehat{\beta}_3 = (\overrightarrow{v_1}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_1}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_1}^\top \widehat{\beta}_3)\widehat{\beta}_3 = \sqrt{46}\widehat{\beta}_1$$

$$\overrightarrow{v_2} = w_{21}\widehat{\beta}_1 + w_{22}\widehat{\beta}_2 + w_{23}\widehat{\beta}_3 = (\overrightarrow{v_2}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_2}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_2}^\top \widehat{\beta}_3)\widehat{\beta}_3 = 6.1926\widehat{\beta}_1 + 6.6822\widehat{\beta}_2$$

$$\overrightarrow{v_3} = w_{31}\widehat{\beta}_1 + w_{32}\widehat{\beta}_2 + w_{33}\widehat{\beta}_3 = (\overrightarrow{v_3}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_3}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_3}^\top \widehat{\beta}_3)\widehat{\beta}_3 = 12.0902\widehat{\beta}_1 + 10.0461\widehat{\beta}_2 + 9.6386\widehat{\beta}_3$$

$$\overrightarrow{v_4} = w_{41}\widehat{\beta}_1 + w_{42}\widehat{\beta}_2 + w_{43}\widehat{\beta}_3 = (\overrightarrow{v_4}^\top \widehat{\beta}_1)\widehat{\beta}_1 + (\overrightarrow{v_4}^\top \widehat{\beta}_2)\widehat{\beta}_2 + (\overrightarrow{v_4}^\top \widehat{\beta}_3)\widehat{\beta}_3 = 19.1675\widehat{\beta}_1 + 13.3645\widehat{\beta}_2$$

Note: the answer to this problem is *not* unique. However, all answers must satisfy the following:

$$\widehat{\beta}_i^\top \widehat{\beta}_j = \delta[i-j] \quad i,j = 1,2,3$$
$$\overrightarrow{v_i} - w_{i1}\widehat{\beta}_1 - w_{i2}\widehat{\beta}_2 - w_{i3}\widehat{\beta}_3 = 0 \quad i = 1,2,3,4$$
$$\text{where} \quad w_{ij} = \overrightarrow{v_i}^\top \widehat{\beta}_j$$

Also, if you are using matlab, you probably won't be getting the answers to be exactly what you expect due to finite precision.

Review Problem 2 *(Frequency responses.)*

The output of an LTI system when the input is a linear combination of complex exponentials has a simple form:

$$e^{j\omega t} * h(t) = H(j\omega)e^{j\omega}, \;\; H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

(a) $H(j\omega) = \frac{1}{2j\omega}, x(t) = 2e^{j2t} - \cos(-\pi t) = 2e^{j2t} - \frac{e^{j\pi t}}{2} - \frac{e^{-j\pi t}}{2}$

$$\Rightarrow y(t) = x(t) * y(t) = 2H(j2)e^{j2t} - H(j\pi)\frac{e^{j\pi t}}{2} - H(-j\pi)\frac{e^{-j\pi t}}{2}$$
$$= \frac{-j}{2}e^{j2t} - \frac{1}{j4\pi}e^{j\pi t} + \frac{1}{j4\pi}e^{-j\pi t} = \frac{-j}{2}e^{j2t} - \frac{1}{2\pi}sin(\pi t)$$

(b) In order to take advantage of the Eigenfunction property, we need to write $x(t)$ as a linear combination of complex exponentials (Fourier Series expansion). $x(t)$ is periodic with fundamental period $T = 10^{-4}s$.

$$\Rightarrow \omega_0 = \frac{2\pi}{T} = 2\pi * 10000 \approx 6.28 \times 10^4$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{\omega_0}{\pi}\int_0^{T_d} e^{-jk\omega_0 t}dt$$
$$= -\frac{e^{-jk\omega_0 t}}{jk\pi}\Big|_{t=0}^{t=T_d}$$

$$a_k = \begin{cases} \frac{\omega_0 T_d}{\pi} = \frac{1}{2} & \text{if } k = 0 \\ \frac{1-e^{-jk\omega_0 T_d}}{jk\pi} = \frac{1-e^{-jk\frac{\pi}{2}}}{jk\pi} & \text{otherwise} \end{cases}$$

Notice that $a_k = a^*_{-k}$, which is what we expect because $x(t)$ is a real signal. Also, $H(j\omega)$ rejects all frequencies $\omega \geqslant 1.5 \times 10^5 rad/s$. Therefore, all harmonics $|k| > 2$ will be gone.

$$H(j\omega) = \begin{cases} 8(1 - \frac{|\omega|}{150000}) & \text{if } |\omega| \leq 150000 \\ 0 & \text{otherwise} \end{cases}$$

$$y(t) = a_{-2}H(-j2\omega_0)e^{-j2\omega_0 t} + a_{-1}H(-j\omega_0)e^{-j\omega_0 t} + a_0 H(j0) + a_1 H(j\omega_0)e^{j\omega_0 t} + a_2 H(j2\omega_0)e^{j2\omega_0 t}$$
$$H(j0) = 8, \; H(-j\omega_0) = H(j\omega_0) \approx 4.649, \; H(-j2\omega_0) = H(j2\omega_0) \approx 1.298$$
$$a_0 = \frac{1}{2}, \; a_1 = \frac{1}{\pi}(1-j) = \frac{\sqrt{2}}{\pi}e^{-j\frac{\pi}{4}}, \; a_{-1} = \frac{1}{\pi}(1+j) = \frac{\sqrt{2}}{\pi}e^{j\frac{\pi}{4}}, \; a_2 = \frac{-j}{\pi}, \; a_{-2} = \frac{j}{\pi}$$
$$\Rightarrow y(t) = a_0 H(j0) + H(j\omega_0)(a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t}) + H(j2\omega_0)(a_2 e^{2j\omega_0 t} + a_2^* e^{-j2\omega_0 t})$$
$$\Rightarrow y(t) \approx 4 + \frac{13.15}{\pi}\cos(\omega_0 t - \frac{\pi}{4}) + \frac{2.59}{\pi}\sin(2\omega_0 t)$$

(c) $h[n] = (\frac{1}{4})^n u[n], x[n] = 3e^{j\frac{\pi}{4}(n-2)} - \sin(\frac{5\pi}{4}n)$. First we need to find the frequency response $H(e^{j\omega})$:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-jk\omega} = \sum_{k=0}^{\infty}(\frac{1}{4})^k e^{-jk\omega} = \sum_{k=0}^{\infty}(\frac{1}{4}e^{-j\omega})^k = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$x[n] = 3e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{4}n} + \frac{j}{2}e^{j\frac{5\pi}{4}n} - \frac{j}{2}e^{-j\frac{5\pi}{4}n} = j(-3e^{j\frac{\pi}{4}n} + \frac{1}{2}e^{j\frac{5\pi}{4}n} - \frac{1}{2}e^{-j\frac{5\pi}{4}n})$$

$$\Rightarrow y[n] = j((\frac{-3}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}})e^{j\frac{\pi}{4}n} + (\frac{\frac{1}{2}}{1-\frac{1}{4}e^{-j\frac{5\pi}{4}}})e^{j\frac{5\pi}{4}n} - (\frac{\frac{1}{2}}{1-\frac{1}{4}e^{j\frac{5\pi}{4}}})e^{-j\frac{5\pi}{4}n})$$

$$= j((\frac{-3}{1-\frac{1}{4}e^{-j\frac{\pi}{4}}})e^{j\frac{\pi}{4}n} + (\frac{\frac{1}{2}}{1+\frac{1}{4}e^{-j\frac{\pi}{4}}})e^{j\frac{5\pi}{4}n} - (\frac{\frac{1}{2}}{1+\frac{1}{4}e^{j\frac{\pi}{4}}})e^{-j\frac{5\pi}{4}n})$$

Problem 1 *(Noise suppression system for airplanes, continued.)*

(a) From Homework 2, the impulse response of the noise suppression filter is $g[n] = \frac{2}{3}\delta[n] + \frac{1}{3}\delta[n-1] + \frac{1}{3}\delta[n-2]$. Thus the frequency response is:

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n} = \frac{2}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}.$$

(b) See Figure 1. $H(e^{j\omega})$ is a "better" filter. Looking at the graph of the frequency responses, the magnitude of $H(e^{j\omega})$ decreases from $\omega = 0$ to $\omega = \pi$ thereby reducing the high frequency input of the system; whereas $|\frac{3}{4}G(e^{j\omega})|$ decreases and then increases back to 0.5 in that same interval. One could also compute the SNR for both systems and see that for the speech and noise given in Homework 2, the SNR is lower for $H(e^{j\omega})$ than for $G(e^{j\omega})$.

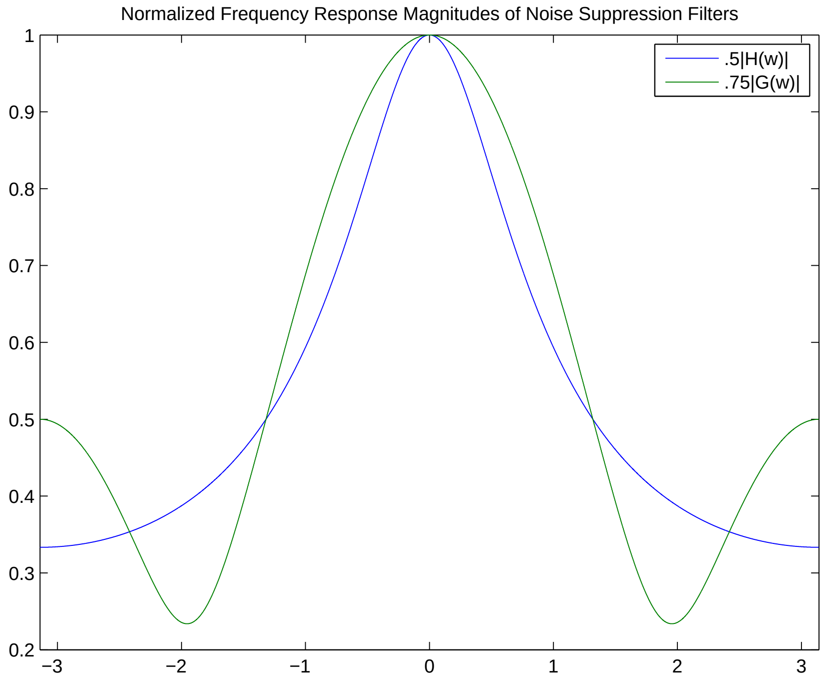

Figure 1: Problem 1b.

Problem 2 *(Continuous-time Fourier series.)*

(a) $x(t)$ is periodic with period $T=3$ and fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$, and over one period is defined as

$$x(t) = \begin{cases} 2, & 0 < t \leq 1 \\ 1, & 1 < t \leq 2 \\ 0, & 2 < t \leq 3 \end{cases}.$$

The Fourier series coefficients of $x(t)$ are

$$a_0 = \frac{1}{T}\int_T x(t)dt = \frac{1}{3}\int_0^3 x(t)dt = 1,$$

and for $k \neq 0$,

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{3}\int_0^1 2e^{-jk\frac{2\pi}{3}t}dt + \frac{1}{3}\int_1^2 e^{-jk\frac{2\pi}{3}t}dt \\
&= \frac{1}{-jk\pi}\left(e^{-jk\frac{2\pi}{3}} - 1\right) + \frac{1}{-jk2\pi}\left(e^{-jk\frac{4\pi}{3}} - e^{-jk\frac{2\pi}{3}}\right) \\
&= \frac{1}{-jk2\pi}\left(\left(e^{-jk\frac{2\pi}{3}} - 1\right) + \left(e^{-jk\frac{4\pi}{3}} - 1\right)\right) \\
&= \frac{1}{-jk2\pi}\left(e^{-jk\frac{\pi}{3}}\left(e^{-jk\frac{\pi}{3}} - e^{jk\frac{\pi}{3}}\right) + e^{-jk\frac{2\pi}{3}}\left(e^{-jk\frac{2\pi}{3}} - e^{jk\frac{2\pi}{3}}\right)\right) \\
&= \frac{e^{-jk\pi/3}\sin(k\pi/3) + e^{-jk2\pi/3}\sin(k2\pi/3)}{k\pi}.
\end{aligned}$$

Now $y(t) = 0.5x(t-1)$ is periodic with $T=3$. By the linearity and time shifting properties of the CTFS, $y(t)$ has FS coefficients $b_k = 0.5e^{-jk\omega_0}a_k = 0.5e^{-jk\frac{2\pi}{3}}a_k$.

(b) Let $x(t)$ be a periodic signal with fundamental period T and FS coefficients a_k. By the time shifting property of the CTFS, the FS coefficients of $x(t-t_0)$ are $b_k = a_k e^{-jk\frac{2\pi}{T}t_0}$. Similarly, the FS coefficients of $x(t+t_0)$ are $c_k = a_k e^{jk\frac{2\pi}{T}t_0}$. Therefore, the FS coefficients of $x(t-t_0) + x(t+t_0)$ are

$$d_k = b_k + c_k = \left(e^{-jk\frac{2\pi}{T}t_0} + e^{jk\frac{2\pi}{T}t_0}\right)a_k = 2\cos(k2\pi t_0/T)a_k.$$

Problem 3 *(CTFS Properties.)*

OWN 3.44

- From (1) and (2), $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}, \quad a_{-k} = a_k^*$
- From (3), $x(t) = a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t}$.
- From (4),

$$\begin{gathered}
x(t) = a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t} \\
x(t-3) = -a_1 e^{j\omega_0 t} - a_1^* e^{-j\omega_0 t} + a_2 e^{j2\omega_0 t} + a_2^* e^{-j2\omega_0 t} \\
x(t-3) = -x(t) \Leftrightarrow a_2 = a_2^* = 0 \\
\Rightarrow x(t) = a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t}
\end{gathered}$$

- From (5) and (6), $|x(t)|^2 = x(t)x^*(t) = (a_1 e^{j\omega_0 t} + a_1^* e^{-j\omega_0 t})(a_1^* e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}) = 2|a_1|^2 + a_1^2 e^{j2\omega_0 t} + {a_1^*}^2 e^{-j2\omega_0 t}$. When we integrate over a period, the last two terms will disappear.

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{6}\int_{-3}^{3} 2|a_1|^2 dt = 2|a_1|^2 = \frac{1}{2}$$

$$\Rightarrow |a_1| = \frac{1}{2}$$

Since a_1 is real and positive, $a_1 = a_1^* = \frac{1}{2}$.

$$\Rightarrow x(t) = \frac{1}{2}(e^{j\frac{\pi}{3}t} + e^{-j\frac{\pi}{3}t}) = \cos(\frac{\pi}{3}t)$$

$$\Rightarrow A = 1,\ \ B = \frac{\pi}{3},\ \ C = 0$$

Problem 4 *(CTFS Properties.)*

$y_1(t) = x(t - \frac{T}{2})$ has Fourier series coefficients b_k. From the time-shifting property, we know that $b_k = a_k e^{-jk\omega_0 \frac{T}{2}} = a_k e^{-jk\pi} = a_k(-1)^k$.

$y_2(t) = Odd\{y(t)\} = \frac{y(t)-y(-t)}{2}$ has Fourier series coefficients c_k. From the properties of Fourier series, we know that $c_k = j\Im m\{b_k\} = j(-1)^k \Im m\{a_k\}$. However, this property only holds when the signal is *real*. The question did not specify $x(t)$ to be real. If we assume that $x(t)$ is complex, we can still use the *Time Reversal* property.

$$y_2(t) = Odd\{y(t)\} = \frac{y(t)-y(-t)}{2} \Leftrightarrow c_k = \frac{b_k - b_{-k}}{2} = \frac{a_k(-1)^k - a_{-k}(-1)^{-k}}{2} = \frac{1}{2}(-1)^k(a_k - a_{-k})$$

Notice that when $x(t)$ is real, $a_k^* = a_{-k}$, which leads to $a_k - a_{-k} = a_k - a_k^* = 2j\Im m\{a_k\}$.

Problem 5 *(DTFS/Frequency responses.)*

OWN 3.16

(a) $x_1[n] = (-1)^n = e^{j\pi n}$. The output $y_1[n] = (x_1 * h)[n] = 0$, since $H(e^{j\pi}) = 0$.

(b) $x_2[n] = 1 + \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$. The DC component e^{0n} disappears while the remaining part $\sin(\frac{3\pi}{8}n + \frac{\pi}{4})$ passes without any distortion. Therefore, $y_2[n] = (x_2 * h)[n] = \sin(\frac{3\pi}{8}n + \frac{\pi}{4})$.

(c) $x_3[n] = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{n-4k}u[n-4k]$

$$x_3[n-4l] = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{n-4l-4k}u[n-4l-4k]$$

$$= \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{n-4(k+l)}u[n-4(k+l)] \quad \text{(replace } k \text{ by } m = k+l\text{)}$$

$$= \sum_{m=-\infty}^{\infty}(\frac{1}{2})^{n-4m}u[n-4m] = x_3[n]$$

Therefore, $x_3[n]$ is periodic with period $N = 4$.

$$x_3[n] = \sum_{k=0}^{3} a_k e^{jkn\frac{\pi}{2}} = a_0 + a_1 e^{j\frac{n\pi}{2}} + a_2 e^{jn\pi} + a_3 e^{jn\frac{3\pi}{2}}$$

However, notice that $H(e^{j0}) = 0, \quad H(e^{j\frac{\pi}{2}}) = 0, \quad H(e^{j\pi}) = 0, \quad H(e^{j\frac{3\pi}{2}}) = 0$. Therefore, $y_3[n] = (x_3 * h)[n] = 0$ (we don't need to compute the Fourier series coefficients).

Problem 6 *(Discrete-time Fourier series.)*

OWN 3.28 (b) Let's first write $x[n]$ as a sum of exponentials. By doing so, we will easily be able to determine the Fourier series coefficients.

$$x[n] = sin(2\pi n/3)cos(\pi n/2) = \frac{(e^{\frac{j2\pi n}{3}} - e^{\frac{-j2\pi n}{3}})(e^{\frac{j\pi n}{2}} + e^{\frac{-j\pi n}{2}})}{4j}$$

$$= \frac{e^{j\pi n(\frac{2}{3}+\frac{1}{2})} - e^{j\pi n(\frac{1}{2}-\frac{2}{3})} - e^{-j\pi n(\frac{2}{3}+\frac{1}{2})} + e^{j\pi n(\frac{2}{3}-\frac{1}{2})}}{4j}$$

$$= \frac{e^{j\pi n(\frac{7}{6})} - e^{-j\pi n(\frac{1}{6})} - e^{-j\pi n(\frac{7}{6})} + e^{j\pi n(\frac{1}{6})}}{4j}$$

$$= \frac{-j}{4}(e^{j\pi n(\frac{7}{6})} - e^{-j\pi n(\frac{1}{6})} - e^{-j\pi n(\frac{7}{6})} + e^{j\pi n(\frac{1}{6})})$$

The above signal is periodic with period 12. Since the Fourier series coefficients repeat every period, we know:

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{12}n} = \sum_{k=<N>} a_k e^{jk\frac{\pi}{6}n}$$

$$\Rightarrow a_1 = -j/4, a_{-7} = a_5 = j/4, a_7 = -j/4, a_{-1} = a_{11} = j/4$$

For $0 \le k \le 11$,

$$|a_k| = \begin{cases} \frac{1}{4} & \text{if } k = 1, 5, 7, 11 \\ 0 & \text{otherwise} \end{cases}$$

$$Phase(a_k) = \begin{cases} \frac{\pi}{2} & \text{if } k = 5, 11 \\ \frac{-\pi}{2} & \text{if } k = 1, 7 \\ 0 & \text{otherwise} \end{cases}$$

Problem 7 *(Parseval's Relation.)*

First let's consider the periodic signal $x(t) = \sum_{n=-\infty}^{\infty} f(t-4n)$ and derive its Fourier series coefficients a_k. We will then derive the Fourier series coefficients of $y(t)$ using the convolution property.

$x(t)$ is periodic with fundamental period $T = 4$. Therefore, $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$.

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt = \frac{-1}{4}\int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-jk\omega_0 t}dt$$

$$= \frac{1}{4jk\omega_0} e^{-jk\omega_0 t}|_{-1/2}^{1/2} = \frac{1}{4jk\omega_0}(e^{-jk\frac{\omega_0}{2}} - e^{jk\frac{\omega_0}{2}})$$

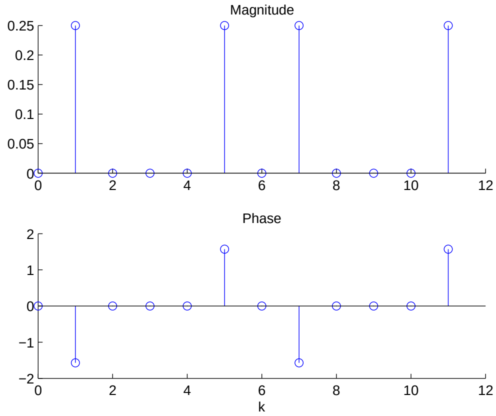

Figure 2: Problem 6.

$$= \frac{-1}{2k\omega_0}\left(\frac{1}{2j}\left(e^{jk\frac{\omega_0}{2}} - e^{-jk\frac{\omega_0}{2}}\right)\right) = \frac{-\sin(k\frac{\omega_0}{2})}{2k\omega_0} = \frac{-\sin(k\frac{\pi}{4})}{k\pi}$$

$$\Rightarrow a_k = \begin{cases} \frac{-1}{4} & \text{if } k = 0 \\ \frac{-\sin(k\frac{\pi}{4})}{k\pi} & \text{otherwise} \end{cases}$$

Let b_k be the Fourier series coefficients of $y(t)$. Since $y(t)$ is periodic with fundamental period T, then we know from the convolution property that $b_k = Ta_k^2$.

$$\Rightarrow y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}, \quad b_k = \begin{cases} \frac{1}{4} & \text{if } k = 0 \\ \frac{4\sin^2(k\frac{\pi}{4})}{(k\pi)^2} & \text{otherwise} \end{cases}$$

The power of the signal $y(t)$ is defined as $\frac{1}{T}\int_T |y(t)|^2 dt$. From Parseval's Relation, we know that $\frac{1}{T}\int_T |y(t)|^2 dt = \sum_{k=-\infty}^{\infty} |b_k|^2$. Therefore, in order to approximate $y(t)$ as a *finite* linear sum of complex exponentials, we need to retain the coefficients that contain most of the power. We also know that the Fourier series coefficients b_k are real, positive and even and strictly decreasing as $|k|$ increases.

$$\Rightarrow \hat{y}(t) = \sum_{k=-M_1}^{M_2} b_k e^{jk\omega_0 t}$$

$$P_y = \frac{1}{T}\int_T |y(t)|^2 dt = \frac{1}{4}\int_{-1}^{1} |g(t)|^2 dt$$

$$= \frac{1}{2}\int_{-1}^{0} (t+1)^2 dt = \frac{1}{2}\frac{(t+1)^3}{3}\Big|_{t=-1}^{t=0} = \frac{1}{6}$$

Therefore, we need to choose M_1 and M_2 such that $\sum_{k=-M}^{M} |b_k|^2 \geq \frac{.9}{6} = 0.15$.

$$b_0^2 = \frac{1}{16} = 0.0625, \quad b_1^2 = b_{-1}^2 = \left(\frac{2}{\pi^2}\right)^2 \approx 0.041, \quad b_2^2 = b_{-2}^2 = \left(\frac{4}{(2\pi)^2}\right)^2 \approx 0.1$$

Notice that $b_0^2 + b_1^2 + b_{-1}^2 + b_2^2 \geq 0.15$. Also, this sum is minimum (i.e. if we remove any of the terms, the inequaltiy no longer holds).

$$\hat{y}(t) = b_0 + b_1 e^{j\omega_0 t} + b_1 e^{-j\omega_0 t} + b_2 e^{j2\omega_0 t} = b_0 + 2b_1 \cos(\omega_0 t) + b_2 e^{j2\omega_0 t}$$

Problem 8 *(CTFS for LTI systems.)*

(a) $x(t)$ is periodic with period $\frac{2\pi}{200\pi} = \frac{1}{100}$. Thus, $x(t) = x(t + k/100)$. To show that $y(t)$ is periodic with period $1/100$, we must show that $y(t) = y(t + k/100)$.

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j200\pi(t-\tau)}h(\tau)d\tau$$

$$y(t + k/100) = \int_{-\infty}^{\infty} x(t + k/100 - \tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{j200\pi(t+k/100-\tau)}h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j2\pi k}e^{j200\pi(t-\tau)}h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{j200\pi(t-\tau)}h(\tau)d\tau \ = y(t)$$

We could also write this for an arbitrary $h(t)$ and periodic $x(t)$ with period T as follows.

$$y(t + kT) = \int_{-\infty}^{\infty} x(t + kT - \tau)h(\tau)d\tau \ = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = y(t)$$

(b) Since $y(t)$ is periodic with period $1/100$, we can write $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k e^{jk200\pi t}$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{j200\pi(t-\tau)}e^{-\tau}u(\tau)d\tau$$

$$= e^{j200\pi t}\int_{0}^{\infty} e^{-\tau(j200\pi+1)}d\tau = \frac{-e^{j200\pi t}(0-1)}{j200\pi + 1} = \frac{e^{j200\pi t}}{j200\pi + 1}$$

$$= \frac{1}{j200\pi + 1}x(t)$$

The Fourier series coefficients of $x(t)$ are $a_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$

By the linearity property the Fourier series coefficients for $y(t)$ are $b_k = \begin{cases} \frac{1}{j200\pi+1} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$

(c) In this case, $c_k = b_k$ since $a_k = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$ Again, we could solve this for an arbitrary $h(t)$ and peridic $x(t)$ with period T and discover the following:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(t-\tau)}h(\tau)d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-jk\omega_0\tau}d\tau$$

where $\int_{-\infty}^{\infty} h(\tau)e^{-jk\omega_0\tau}d\tau$ is just a complex function of k equal to c_k

Problem 9 *(Fourier Series and Gibbs phenomenon - Matlab.)*

(a)

$$\begin{aligned} c_k &= \int_0^1 p(t)e^{-j2\pi kt}dt \\ &= \int_0^{1/2} e^{-j2\pi kt}dt - \int_{1/2}^1 e^{-j2\pi kt}dt \\ &= \frac{1}{-j2\pi k}\left(e^{-j\pi k} - 1 - e^{-j2\pi k} + e^{-j\pi k}\right) \\ &= \frac{1 - e^{-j\pi k}}{j\pi k} \end{aligned}$$

$$c_0 = \int_0^1 p(t)dt = 0$$

(b) The following Matlab code generates Figure 3.

```
function [] = gibbs();
[t10, p10, y10] = FS(10);
[t100, p100, y100] = FS(100);
[t1000, p1000, y1000] = FS(1000);
figure;
plot(t1000,y1000,'g-');
hold on;
stairs(t1000,p1000,'k--');
plot(t100,y100,'k-');
plot(t10,y10,'b-');
title('Fourier series convergence and Gibbs phenomenon');
xlabel('t');
ylabel('p_N(t)');

function [t, p, y] = FS(N)
k = (-N:N);
t = linspace(-.5,.5,20*N+1);
p = (t>=0);
p = 2.*p - 1;
c = (1 - exp(-j*pi.*k))./(j*pi.*k);
c(N+1) = 0; % c_k at k=0
y = zeros(size(t));
for i=1:length(c)
 y = y + c(i)*exp(j*2*pi*k(i).*t);
end
y = real(y);
```

The partial sum approximations at $t = 0$ are $p_N(0) = 0$, which does not agree with the value of the function $p(0) = 1$.

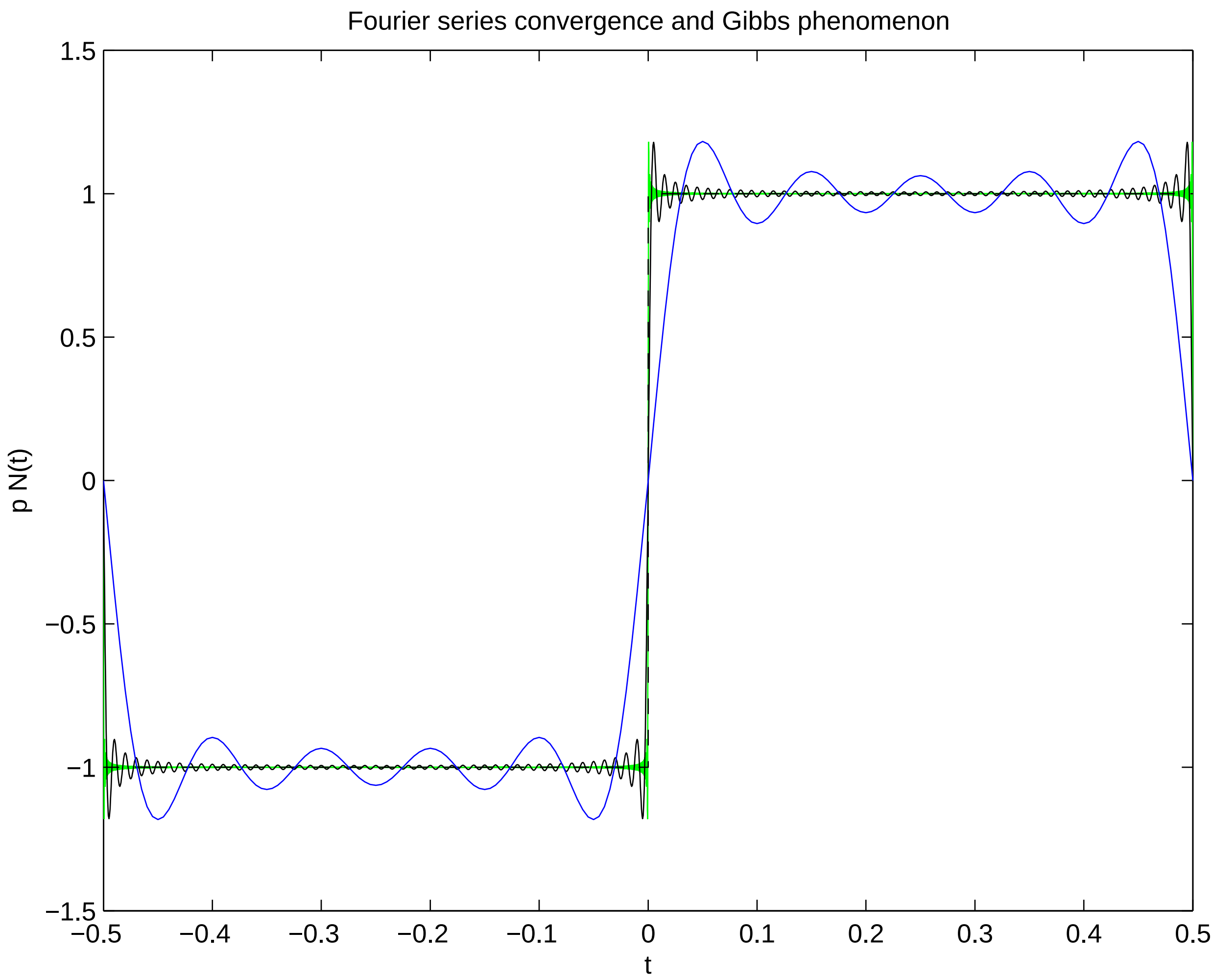

Figure 3: Problem 9b.

(c) The maximum overshoot stays constant as the number of terms in the partial sum approximation increases, $max|p(t) - p_N(t)| \approx 0.18$.

(d) As the number of terms in the partial sum approximation increases, the time-locations of the maximum overshoot gets closer and closer to the points of discontinuity at $t = 0, \pm 0.5$.