Homework 5 Solutions

Problem 1 *(DTFT.)*

- *(a)* OWN 5.21(e)
  $x[n] = \left(\frac{1}{2}\right)^{|n|}\cos(\frac{\pi}{8}(n-1))$

$$\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} (0.5)^{|n|}\cos[\pi(n-1)/8]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{-1} (0.5)^{-n}0.5(e^{j\pi(n-1)/8} + e^{-j\pi(n-1)/8})e^{-j\omega n} + \sum_{n=0}^{\infty}(0.5)^{n}0.5(e^{j\pi(n-1)/8} + e^{-j\pi(n-1)/8})e^{-j\omega n} \\
&= 0.5\sum_{n=-\infty}^{-1} e^{-j\pi/8}(0.5e^{-j\pi/8}e^{j\omega})^{-n} + 0.5\sum_{n=-\infty}^{-1} e^{j\pi/8}(0.5e^{j\pi/8}e^{j\omega})^{-n} + 0.5\sum_{n=0}^{\infty} e^{-j\pi/8}(0.5e^{j\pi/8}e^{-j\omega})^{n} + \\
&\quad 0.5\sum_{n=0}^{\infty} e^{j\pi/8}(0.5e^{-j\pi/8}e^{-j\omega})^{n} \\
&= 0.5\sum_{n=1}^{\infty} e^{-j\pi/8}(0.5e^{-j\pi/8}e^{j\omega})^{n} + 0.5\sum_{n=1}^{\infty} e^{j\pi/8}(0.5e^{j\pi/8}e^{j\omega})^{n} + 0.5\sum_{n=0}^{\infty} e^{-j\pi/8}(0.5e^{j\pi/8}e^{-j\omega})^{n} + \\
&\quad 0.5\sum_{n=0}^{\infty} e^{j\pi/8}(0.5e^{-j\pi/8}e^{-j\omega})^{n} \\
&= \frac{0.25e^{-j\pi/4}e^{j\omega}}{1-0.5e^{-j\pi/8}e^{j\omega}} + \frac{0.25e^{j\pi/4}e^{j\omega}}{1-0.5e^{j\pi/8}e^{j\omega}} + \frac{0.5e^{-j\pi/8}}{1-0.5e^{j\pi/8}e^{-j\omega}} + \frac{0.5e^{j\pi/8}}{1-0.5e^{-j\pi/8}e^{-j\omega}}
\end{aligned}$$

- *(b)* OWN 5.22(a)

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-3\pi/4}^{\pi/4} e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\pi/4}^{3\pi/4} e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{\omega=-3\pi/4}^{-\pi/4} + \frac{1}{2\pi}\left[\frac{1}{jn}e^{j\omega n}\right]_{\omega=pi/4}^{3\pi/4} \\
&= \frac{1}{j2\pi n}\left(e^{-jn\pi/4} - e^{-jn3\pi/4} + e^{jn3\pi/4} - e^{jn\pi/4}\right) \\
&= \frac{1}{\pi n}\left(\sin(3\pi n/4) - \sin(\pi n/4)\right)
\end{aligned}$$

Problem 2 *(More DTFT.)*

- *(c)* OWN 5.26(a)

  First, by looking at the plots in Fig. P5.26(a), we see that $X_1(e^{j\omega})$ possesses conjugate symmetry. This implies that $x[n]$ is a real valued signal.

  Next, we consider a signal $y[n]$ with DTFT $Y(e^{j\omega}) = \Re\{X_1(e^{j\omega})\}$

  By the even-odd decomposition property in Table 5.1, we have that $y[n] = \mathcal{E}\{x_1[n]\}$

  Then, we observe that we can express $X_2(e^{j\omega})$ as

  $$X_2(e^{j\omega}) = Y(e^{j\omega}) + Y(e^{j(\omega-2\pi/3)}) + Y(e^{j(\omega+2\pi/3)})$$

  Using the frequency shifting property of the DTFT and the linearity of the DTFT, both in Table 5.1, we have that

  $$\begin{aligned} x_2[n] &= \mathcal{E}\{x_1[n]\}\left[1 + e^{j2\pi/3} + e^{-j2\pi/3}\right] \\ &= \mathcal{E}\{x_1[n]\}\left[1 + 2\cos(2\pi/3)\right] \end{aligned}$$

- *(d)* OWN 5.50(a)

  Using the transform pairs in Table 5.2, and the time-shifting property in Table 5.1, we see that

  $$Y(e^{j\omega}) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}}$$

  $$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{1}{4}\frac{e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}} = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$

  We can then compute the frequency response

  $$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{\left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

  To find $h[n]$, we first find the partial fraction expansion of the frequency response

  $$H(e^{j\omega}) = \frac{A}{1 - \frac{1}{3}e^{-j\omega}} + \frac{B}{1 - \frac{1}{4}e^{-j\omega}}$$

  By setting the two expressions for $H(e^{j\omega})$ equal, cross-multiplying, and equating the constant terms and $e^{-j\omega}$ terms; we can solve for $A = -2$ and $B = 3$.

  Using the transform pairs in Table 5.2, we see that

  $$h[n] = 3\left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$$

  Now, to find the difference equation, we take the equation

  $$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - \frac{1}{2}e^{-j\omega}}{\left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$$

  and cross-multiply to show that

  $$Y(e^{j\omega}) - Y(e^{j\omega})\frac{7}{12}e^{-j\omega} + Y(e^{j\omega})\frac{1}{12}e^{-j2\omega} = X(e^{j\omega}) - X(e^{j\omega})\frac{1}{2}e^{-j\omega}$$

Taking the inverse Fourier transform, and using the time shifting property, we obtain

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

Problem 3 *(Fourier Representations and their interconnections.)*

- (a)
  We know from Table 4.2 that the inverse Fourier transform of a rectangular pulse is a sinc.

  $$x(t) = \frac{\sin(Wt)}{\pi t}$$

  The plot is shown at the end of this problem.

- (b)
  $Z(j\omega)$ is equal to $X(j\omega)$ convolved with a train of pulses.

  $$Z(\omega) = X(j\omega) \star \sum_{k=-\infty}^{\infty} \delta(\omega - 3Wk)$$

  In the time domain, letting $T = \frac{2\pi}{3W}$ and using the transform pair in Table 4.2, we have

  $$\begin{aligned} z(t) &= (2\pi)\,(x(t)) \left(\frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \delta(t - nT) \right) \\ &= T \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \\ &= T \sum_{n=-\infty}^{\infty} \frac{\sin(n \cdot 2\pi/3)}{\pi n T}\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin(n \cdot 2\pi/3)}{\pi n}\delta(t - nT) \end{aligned}$$

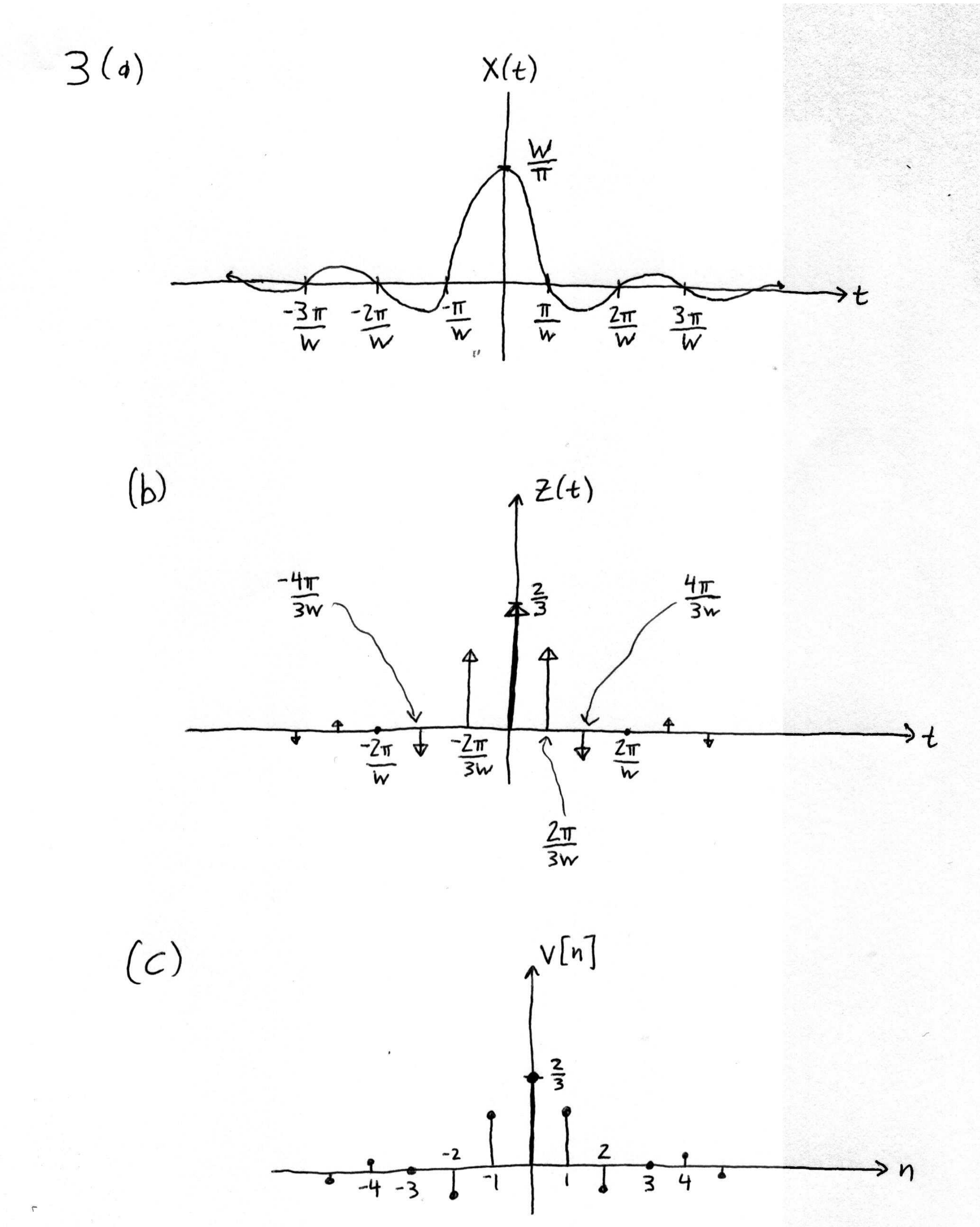
  Hence, we see that $z(t)$ is just $x(t)$ sampled with a sampling period of $T = \frac{2\pi}{3W}$. The plot is shown at the end of this problem.

- (c)
  From Table 5.2, we see that

  $$v[n] = \frac{\sin(n \cdot 2\pi/3)}{\pi n}$$

  Hence, $v[n]$ is the impulse train $z(t)$ converted to a discrete time sequence.

  The plots of $x(t)$, $z(t)$, and $v[n]$ are shown here.



Problem 4 *(Fourier via Matlab.)*

- (a)

  `function [F_OWN,F_M,F_u] = Fourier_matrix(N)`

```
for m=1:N,
for n=1:N,
F_OWN(m,n) = 1/N*exp(-j*2*pi/N*(m-1)*(n-1));
F_M(m,n) = exp(-j*2*pi/N*(m-1)*(n-1));
F_u(m,n) = 1/sqrt(N)*exp(-j*2*pi/N*(m-1)*(n-1));
end
end
```

- (b)

  For this problem, we will use the OWN indexing (the columns are numbered 0 through $N-1$, and the rows likewise) instead of the Matlab indexing in order to simplify the computations. Using the the OWN indexing convention, the entry in row m, column n of F_u is equal to

$$\frac{1}{\sqrt{N}}e^{-j2\pi mn/N}$$

If we take the dot product of column n and column k of F_u, we get the following (don't forget to take the conjugate of the first vector):

$$\sum_{m=0}^{N-1}\left(\frac{1}{\sqrt{N}}e^{-j2\pi mn/N}\right)^*\left(\frac{1}{\sqrt{N}}e^{-j2\pi mk/N}\right)=\frac{1}{N}\sum_{m=0}^{N-1}e^{-j2\pi m(k-n)/N}$$

If $k=n$ (which means that we are taking the dot product of a column with itself), then

$$\sum_{m=0}^{N-1}\frac{1}{N}e^{j0}=\sum_{m=0}^{N-1}\frac{1}{N}=1$$

Since the norm of a vector is equal to the squareroot of the dot product of the vector with itself (see Handout 2), the columns of F_u have unit length. If $\ell=k-n\neq 0$, then

$$\sum_{m=0}^{N-1}\frac{1}{N}e^{-j2\pi\ell/N\cdot m}=\frac{1}{N}\frac{1-e^{-j2\pi\ell}}{1-e^{-j2\pi\ell/N}}=0$$

Hence, the dot product of two unique columns of F_u is equal to 0. Combining these two facts, we see that the columns of F_u are orthonormal.

For the matrix F_M, again using the OWN indexing convention, the entry in row m, column n is

$$e^{-j2\pi mn/N}$$

Now, repeating the same steps as above, we find that the dot product of column k and column n of F_M is

$$\sum_{m=0}^{N-1}e^{-j2\pi/N\cdot(k-n)m}=N\cdot\delta(k-n)$$

So, the columns of F_M are orthogonal, but have length $\sqrt{N}$. For the matrix F_{OWN}, the entry in row m, column n is

$$\frac{1}{N}e^{-j2\pi mn/N}$$

So the dot product of column k and column n of F_{OWN} is

$$\sum_{m=0}^{N-1} \frac{1}{N^2} e^{-j2\pi/N\cdot(k-n)m} = \frac{1}{N}\cdot\delta(k-n)$$

The columns of F_{OWN} are orthogonal, but have length $1/\sqrt{N}$.

Because F_u is a symmetric matrix, the rows of F_u are also orthonormal. This means that when we multiply $F_u x$, we are projecting the vector x onto a new orthonormal basis. The DFT is expressing the signal in a new basis, the Fourier basis. We can think of the DFT as a change of basis operation. This notion extends to the other Fourier transforms, which can all be viewed as expressing the signal in terms of the Fourier basis.

- (c)

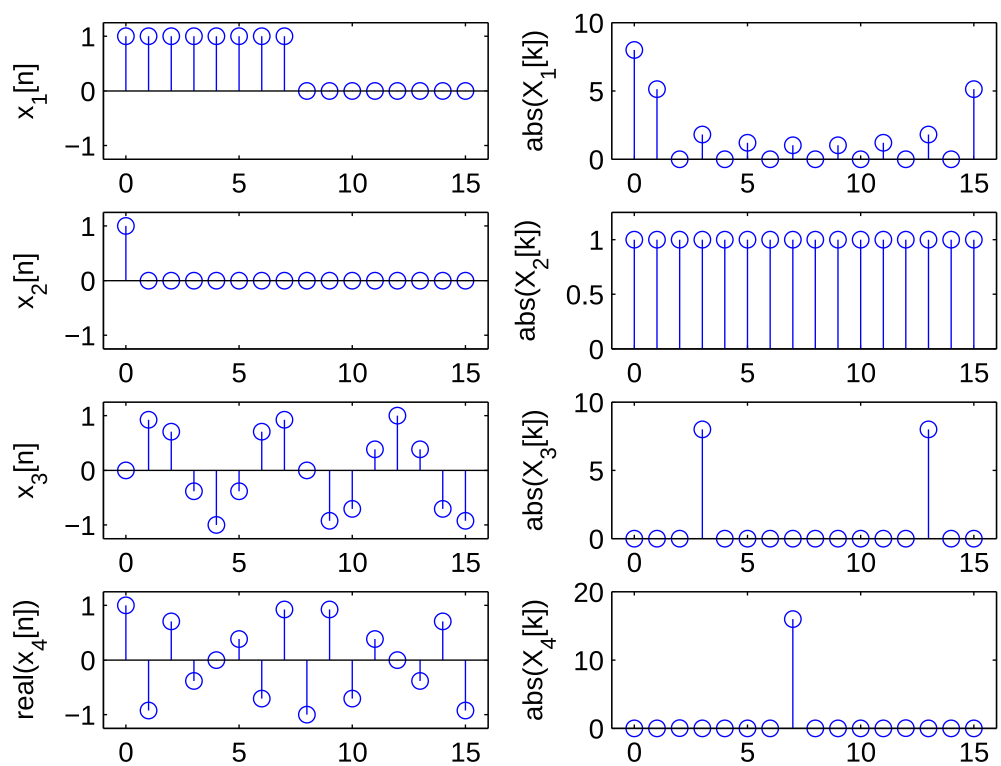
```
x1 = [1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0];
x2 = [1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0];
x3 = sin(3*pi*(0:15)/8);
x4 = exp(j*14*pi*(0:15)/16);
[ F_OWN,F_M,F_u ] = Fourier_matrix(16);
fftx1 = fft(x1.');
fft2x1 = F_M*(x1.');
fftx2 = fft(x2.');
fft2x2 = F_M*(x2.');
fftx3 = fft(x3.');
fft2x3 = F_M*(x3.');
fftx4 = fft(x4.');
fft2x4 = F_M*(x4.');
e1 = sum(abs(fftx1 - fft2x1))
e2 = sum(abs(fftx2 - fft2x2))
e3 = sum(abs(fftx3 - fft2x3))
e4 = sum(abs(fftx4 - fft2x4))
subplot(4,2,1), stem((0:15),x1)
axis([-1 16 -1.25 1.25])
ylabel('x_1[n]')
subplot(4,2,3), stem((0:15),x2)
axis([-1 16 -1.25 1.25])
ylabel('x_2[n]')
subplot(4,2,5), stem((0:15),x3)
axis([-1 16 -1.25 1.25])
ylabel('x_3[n]')
subplot(4,2,7), stem((0:15),real(x4))
axis([-1 16 -1.25 1.25])
ylabel('real(x_4[n])')
subplot(4,2,2), stem((0:15),abs(fftx1))
axis([-1 16 0 10])
ylabel('abs(X_1[k])')
subplot(4,2,4), stem((0:15),abs(fftx2))
axis([-1 16 0 1.25])
ylabel('abs(X_2[k])')
subplot(4,2,6), stem((0:15),abs(fftx3))
axis([-1 16 0 10])
ylabel('abs(X_3[k])')
subplot(4,2,8), stem((0:15),abs(fftx4))
axis([-1 16 0 20])
```

```
ylabel('abs(X_4[k])')
```

We note that the four error values $e1$, $e2$, $e3$, and $e4$ are all equal to 0 (or less than 10^{-13}, which is within the machine precision), so the `fft` function in Matlab is giving the same answers as our matrix multiplication.

The FFTs of signals 3 and 4 make sense because if a signal can be expressed as the sum of complex exponentials of period N, then the DTFS coefficients can be easily read off from the time-domain signal.

The FFT of signal 2 makes sense because we know that the DTFS of a (periodic) impulse is a constant function.

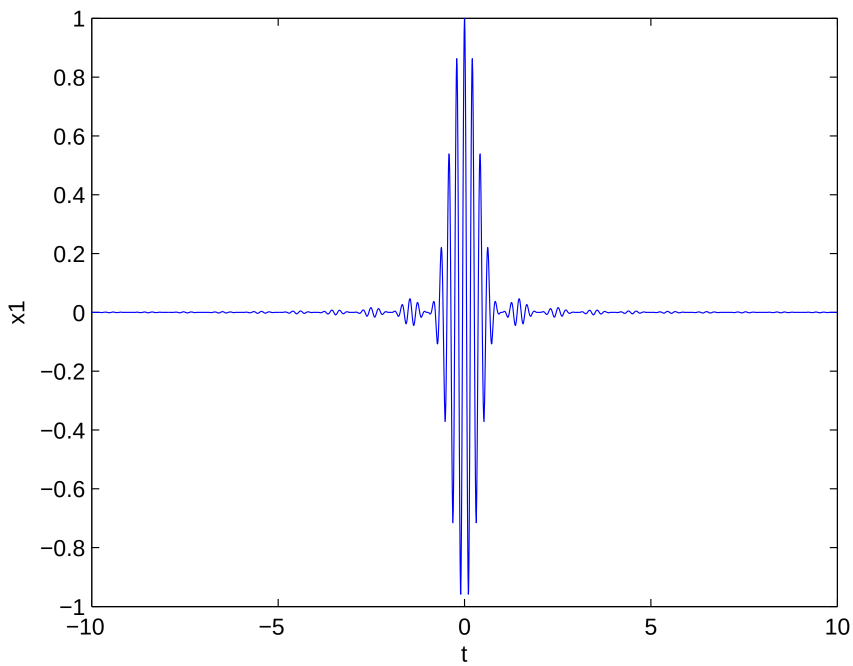


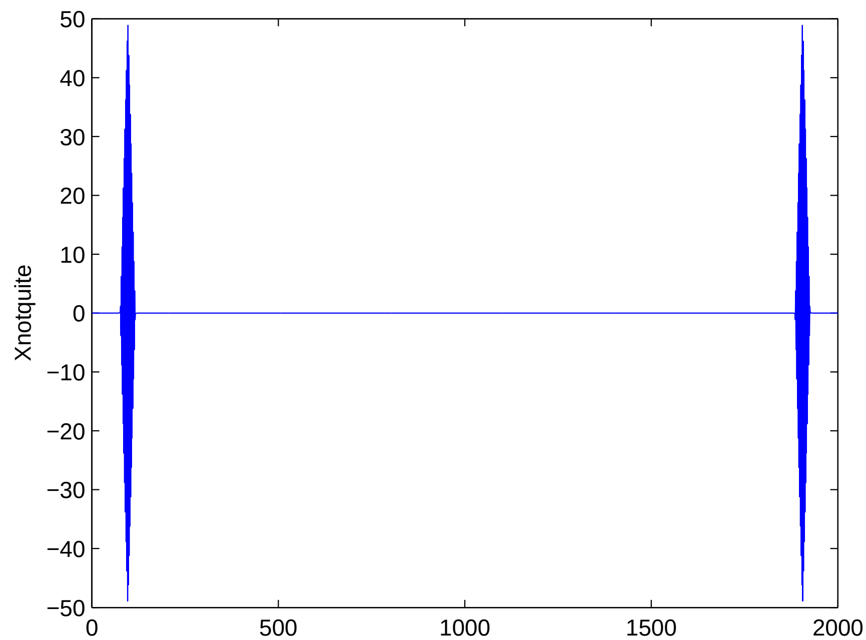
Problem 5 *(Fourier Transform via Matlab.)*

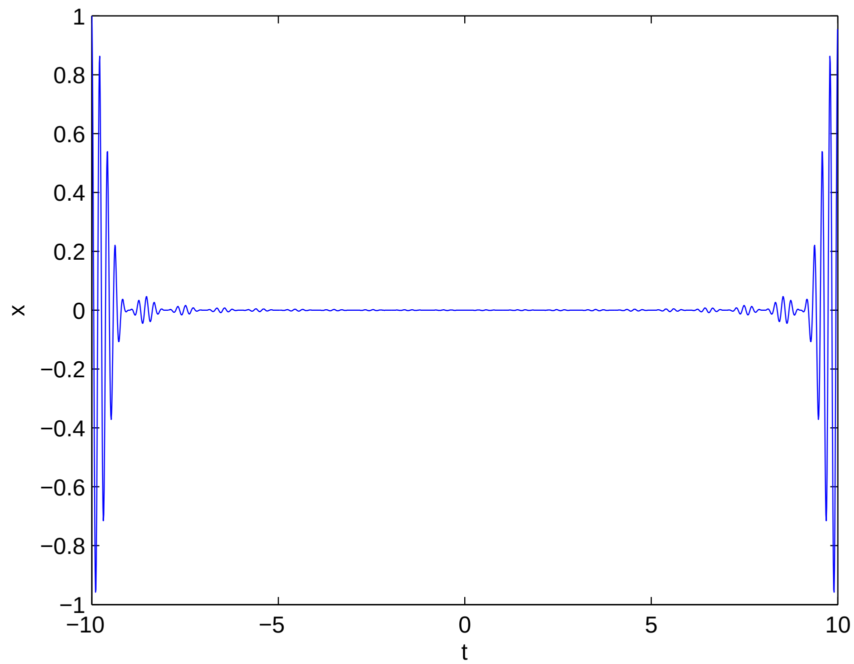
- Plots

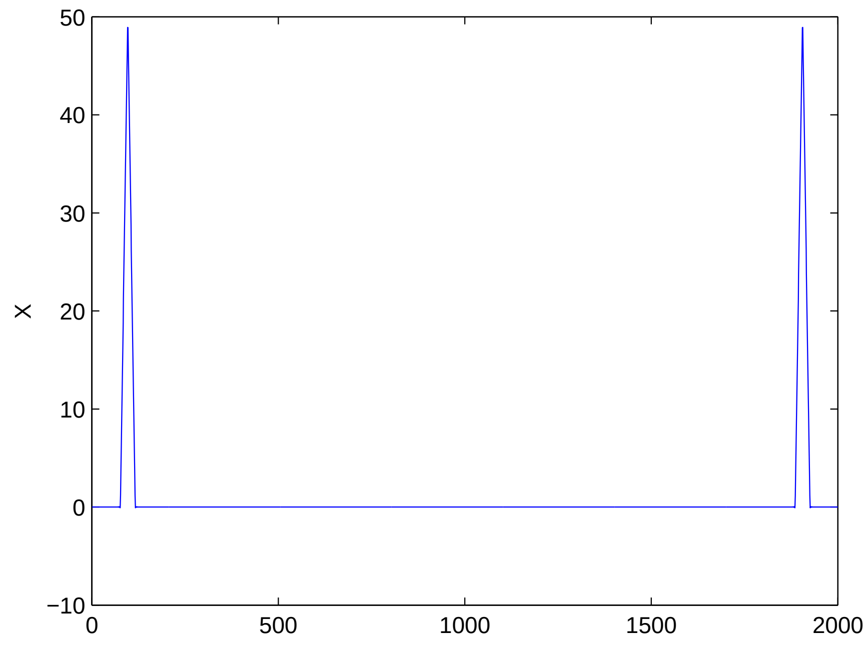
```
T = 10;
dt = 0.01;
t = [ -T : dt :  T-dt];
x1 = cos(30*t).*sinc(t).^2;
Xnotquite = fft(x1);
x = fftshift(x1);
X = fft(x);
```

```
figure(1); plot(t,x1); ylabel('x1'); xlabel('t')
figure(2); plot(real(Xnotquite)); ylabel('Xnotquite')
figure(3); plot(t,x); ylabel('x'); xlabel('t')
figure(4); plot(real(X)); ylabel('X')
```









- Explanation of `Xnotquite`

  We know that the Fourier transform of $(\sin(\pi t)/(\pi t))^2$ is a triangle. We also know that the Fourier transform of $\cos(30t)$ is two delta functions, and that multiplying in the time domain is equivalent to convolving in the frequency domain. Therefore, we would expect the spectrum of $x(t)$ to be two triangles.

  However, `Xnotquite` actually looks like two triangular shapes multiplied by a function that alternates between 1 and -1. If you zoom in on one of the triangular pulses, you can see the oscillatory nature of `Xnotquite` more clearly. The reason for this is that Matlab does not understand that the time origin is in the middle of `x1`. In fact, Matlab assumes that $t = 0$ is at the left edge of `x1`, which means that Matlab has essentially delayed the signal $x(t)$ by 10 time units. Time-shifting is equivalent to multiplying by a complex exponential in the frequency domain. This means that `Xnotquite` is the true spectrum multiplied by a complex exponential, which causes the oscillatory behavior.

- Explanation of `X(.)`

  When we use the `fftshift` function to produce `x`, Matlab swaps the left and right halves of `x1`. This means that the peak of $x(t)$ is now centered at $t = 0$. (Remember that `fft` computes the DTFS, which assumes that the input signal is periodic.)

  Now, the spectrum `X` looks just like we expected, because Matlab has the signal properly aligned with the origin.

  We see that `x` is the original continuous-time signal $x(t)$ sampled, where $\tau = 0.01$ is the sampling period. We saw in Section 7.4 that the spectrum of the sampled signal is $\frac{1}{\tau}$ times the spectrum of the continuous-time signal (see Figure 7.22). Therefore, we should multiply X by 0.01 to get the correct amplitude values.

  In addition, the two triangles should be centered at $\omega = 30$ and $\omega = -30$, because the spectrum of the term $\cos(30t)$ in $x(t)$ has a spectrum of two impulses at $\omega = \pm 30$

Problem 6 *(Sampling theorem.)*

- OWN 7.21(a)

  The Nyquist rate for the given signal is $2 \times 5000\pi = 10000\pi$. Therefore, to be able to recover $x(t)$ from $x_p(t)$, the sampling period must be at most $T_{max} = (2\pi)/(10000\pi) = 2 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} < T_{max}$, $x(t)$ can be recovered from $x_p(t)$

- OWN 7.21(b)

  The Nyquist rate for the given signal is $2 \times 15000\pi = 30000\pi$. Therefore, to be able to recover $x(t)$ from $x_p(t)$, the sampling period must be at most $T_{max} = (2\pi)/(30000\pi) = 0.66 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} > T_{max}$, $x(t)$ cannot be recovered from $x_p(t)$

- OWN 7.21(g)

  If $|X(j\omega)| = 0$ for $\omega > 5000\pi$, then $X(j\omega) = 0$ for $\omega > 5000\pi$. However, the question gives us no information about whether or not there exists some ω_M such that $X(j\omega) = 0$ for $\omega < -\omega_M$. Therefore, we cannot determine whether we can recover $x(t)$ from $x_p(t)$. (Full credit given for answering no, since you can construct an $x(t)$ matching the given properties that cannot be recovered from $x_p(t)$.)

Problem 7 *(Sampling.)*

- OWN 7.23(a) We can express $p(t)$ as $p(t) = p_1(t) - p_1(t - \Delta)$, where

$$p_1(t) = \sum_{n=-\infty}^{\infty} \delta(t - n(2\Delta))$$

  Now, using Table 4.2, the Fourier transform of $p_1(t)$ is given by

$$P_1(j\omega) = \frac{\pi}{\Delta} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi k}{\Delta}\right)$$

To find the Fourier transform of $q(t) = p_1(t - \Delta)$, we can find the Fourier series coefficients and use equation 4.22 in OWN. The Fourier series coefficients are given by

$$a_k = \frac{1}{T}\int_T q(t)e^{-jk(2\pi/T)t}dt = \frac{1}{2\Delta}\int_0^{2\Delta} \delta(t-\Delta)e^{-jk(\pi/\Delta)t}dt = \frac{1}{2\Delta}e^{-jk\pi}$$

Using equations 4.22 and 4.23, the Fourier transform of $q(t)$ is given by

$$\sum_{k=-\infty}^{\infty} \frac{\pi}{\Delta}e^{-jk\pi}\delta\left(\omega - \frac{\pi k}{\Delta}\right)$$

Finally, using the linearity of the Fourier transform, we find that

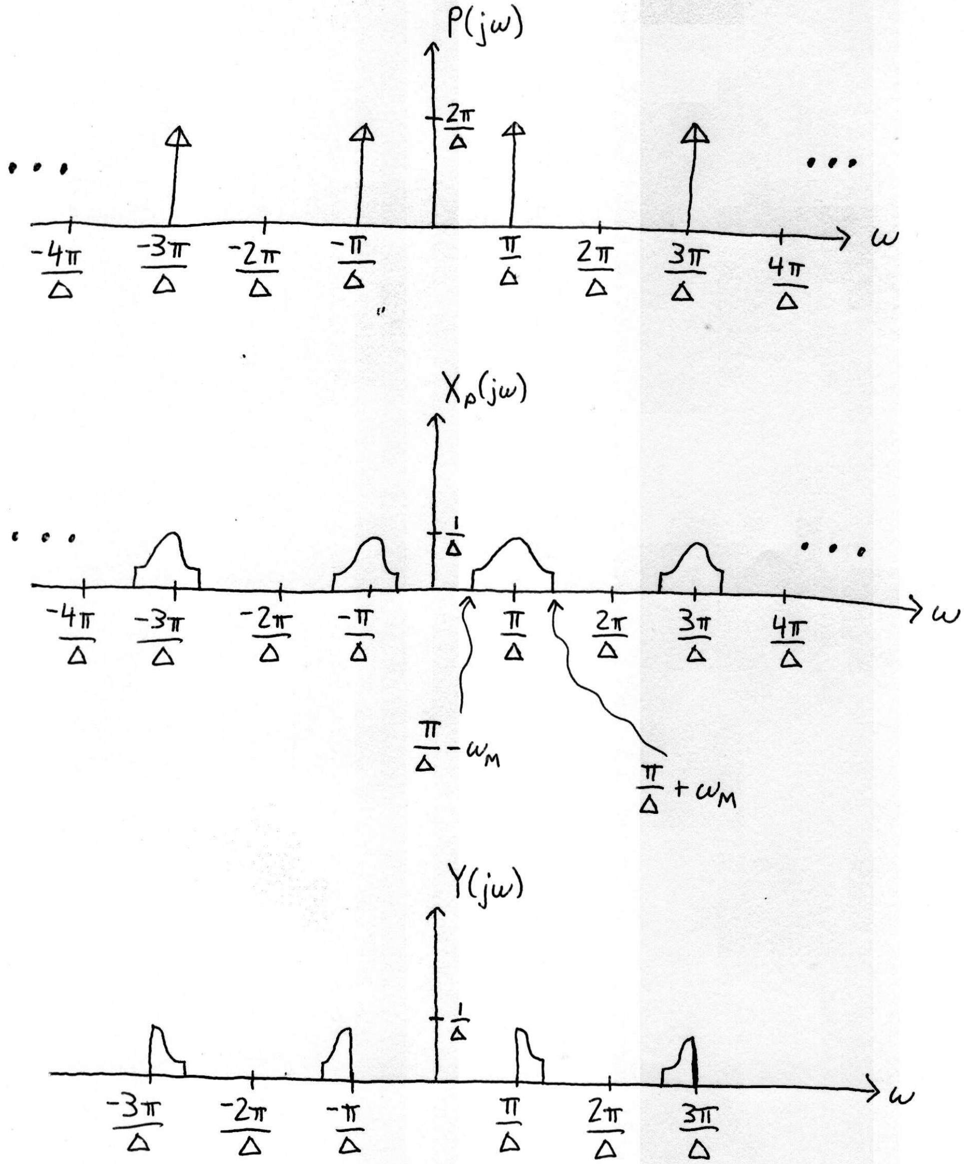
$$P(j\omega) = \frac{\pi}{\Delta}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{k\pi}{\Delta}\right) - \frac{\pi}{\Delta}\sum_{k=-\infty}^{\infty} e^{-jk\pi}\delta\left(\omega - \frac{k\pi}{\Delta}\right) = \frac{2\pi}{\Delta}\sum_{k\ odd}\delta\left(\omega - \frac{\pi k}{\Delta}\right)$$

Now, because $x_p(t) = x(t)p(t)$, we know that $X_p(j\omega) = \frac{1}{2\pi}X(j\omega) \star P(j\omega)$

$$X_p(j\omega) = \frac{1}{\Delta}\sum_{k\ odd} X\left(\omega - \frac{k\pi}{\Delta}\right)$$

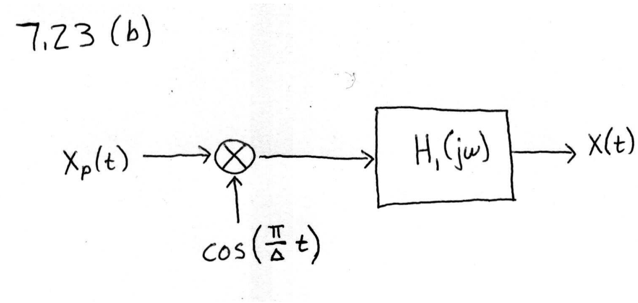
If $\Delta < \pi/(2\omega_M)$, then $\omega_M < \pi/(2\Delta)$. In this situation, the various copies of $X(j\omega)$ do not overlap in $X_p(j\omega)$. $P(j\omega)$, $X_p(j\omega)$, and $Y(j\omega)$ are shown in the following figure.

7.23 (a)



- OWN 7.23(b)

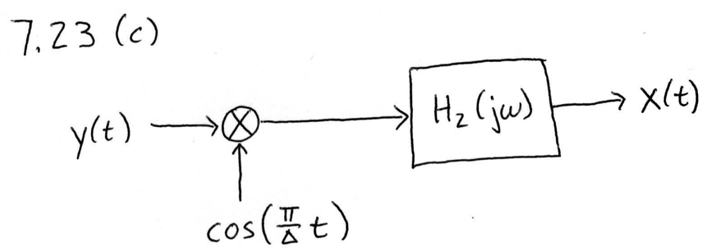
  To recover $x(t)$ from $x_p(t)$, we need to shift one copy of the original spectrum $X(j\omega)$ to the origin, and then filter out all of the other copies of $X(j\omega)$ in $X_p(j\omega)$. One simple way to shift the spectrum is to multiply $x_p(t)$ by $\cos(t\pi/\Delta)$. The following system will recover $x(t)$ from $x_p(t)$



$$H_1(j\omega) = \begin{cases} \Delta & |\omega| < \omega_M \\ 0 & else \end{cases}$$

- OWN 7.23(c)

  To recover $x(t)$ from $y(t)$, we need to shift the two parts of $X(j\omega)$ that are located near π/Δ and $-\pi/\Delta$ back to the origin, and then filter out the components of the spectrum at higher frequencies. The following system will recover $x(t)$ from $y(t)$



$$H_2(j\omega) = \begin{cases} 2\Delta & |\omega| < \omega_M \\ 0 & else \end{cases}$$

- OWN 7.23(d)

  Looking at the plot of $X_p(j\omega)$ in part *(a)*, we see that aliasing is avoided when $\omega_M \leq \pi/\Delta$. Therefore, the maximum value of Δ which allows $x(t)$ to be recovered is $\Delta_{max} = \pi/\omega_M$