Homework 6 GSI: Mary Knox
Due: Monday, October 15, 2007, at 5pm

Reading OWN Chapter 7.

Problem 1 *(Discrete-time Processing of Continuous-time Signals.)*

OWN Problem 7.29

Problem 2 *(Discrete-time Processing of Continuous-time Signals.)*

OWN Problem 7.43

Note: When finding the unit sample response h[n], please write an expression for it. You do not need to evaluate the expression.

Problem 3 *(Sampling Rates.)*

Suppose Jane is running out of space on her hard drive. In order to make sure she has the necessary memory to save her Matlab files, she is trying to reduce the storage space needed to store all of her audio files. Since the average human can hear 20 Hz to 20 kHz, she first put all of her music through a low pass filter with a cutoff frequency of 20 kHz in order to make sure unnecessary data is not stored. Assume that she can sample ideally and that all of her music contains all frequencies up to 20 kHz

(a) What is the lowest rate Jane could sample at in order to avoid aliasing and perfectly reconstruct her music for all frequencies that the average human can hear?

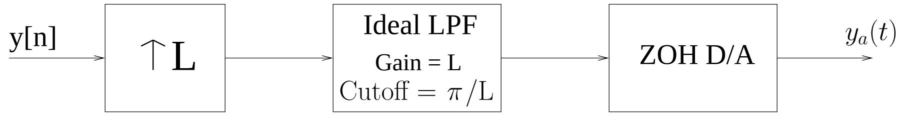
(b) After sampling at the rate found in (a) Jane realizes that she still does not have enough space to store all of her music and Matlab code. Jane has attended many concerts without hearing protection and cannot hear quite as well as the average human. In fact, her audible range is from 20 Hz to 15 kHz. What is the lowest rate she could sample at in order to avoid aliasing over all frequencies she can hear?

Problem 4 *(Discrete-time Zero-order and First-order Holds.)*

OWN 7.50

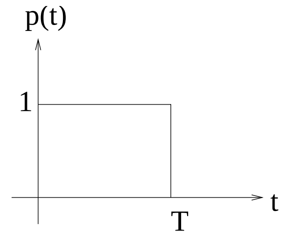
Problem 5 *(Oversampled D/A.)*

An oversampled D/A is implemented using a digital interpolator and a zero-order hold, as shown below.



The interpolator inserts $L-1$ zeros between each sample of $y[n]$. The zero-order hold has period

$T = 1/L$, i.e., the duration of the ZOH is dependent on the upsampling factor. As usual, the impulse response of the ZOH is given by



The discrete time signal $y[n]$ has DTFT $Y(e^{j\Omega}) = \begin{cases} 1 & |\Omega| < 3\pi/4 \\ 0 & 3\pi/4 < |\Omega| < \pi \end{cases}$

(a) Plot the magnitude spectrum $|Y_a(j\omega)|$ for three cases: $L = 1$, $L = 2$, and $L = 4$.

Plot each spectrum over the interval $-5\pi/T \leq \omega \leq 5\pi/T$ (remember that T is dependent on L). You may either use Matlab or sketch the plots by hand. Be sure to accurately label all important points on the ω-axis.

(b) Plot the magnitude spectrum $|Y_a(j\omega)|$ when the ZOH is replaced with an ideal D/A converter.

(c) For the four plots in parts (a) and (b), calculate the magnitude of the largest component of $Y_a(j\omega)$ *outside* the band $|\omega| \leq \pi/T$

Problem 6 *(Band-pass Sampling.)*

Suppose a continuous-time signal $x(t)$ has the spectrum shown in Figure 1. Such a signal is sometimes called a *band-pass* signal. The Nyquist rate (that is, the smallest sampling rate that avoids aliasing) for this signal is 6π, since the highest occupied frequency is 3π.

The *spectral support* of a band-pass signal is the amount of spectrum it uses. For the example given in Figure 1, the spectral support is π.[1] Clearly, the number of degrees of freedom of the signal is determined by the spectral support, rather than by the highest occupied frequency. So, the intuition is that the signal can be sampled at a rate equal twice its spectral support, which for the example shown in Figure 1 is 2π. This is true, but generally requires non-uniform sampling and involves certain forms of aliasing (overlapping replica of the original spectum), thus requiring involved reconstruction procedures.

For some lucky instances of band-pass signals, however, one can just go ahead and sample them at a sampling frequency equal to twice the spectral support. In this homework problem, we examine these lucky cases.

(a) Show that for the example given in Figure 1, it *is* true that sampling at $\omega_s = 2\pi$ enables perfect reconstruction. *Hint:* Draw the spectrum of the sampled signal.

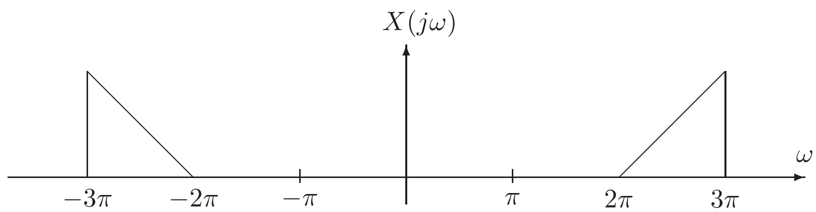


Figure 1: The spectrum for Problem 6, Part *(a)*.

[1]Since most interesting signals are real-valued, and hence their spectra conjugate symmetric, the spectral support is usually given for the positive frequencies only.

(b) Consider the signal whose spectrum is shown in Figure 2. This is the exact same figure as Figure 1, *except* that the two triangles are moved $\pi/4$ closer to the origin. What is the Nyquist sampling rate in this case? The spectral support is unchanged, but show that it is *not* true that this signal can be uniformly sampled at $\omega_s = 2\pi$ without introducing aliasing. We *could* sample it at the Nyquist frequency, but it can be shown that a smaller sampling frequency is already sufficient. What is the smallest sampling frequency that avoids aliasing? *Hint:* Draw again the spectrum of the sampled signal.

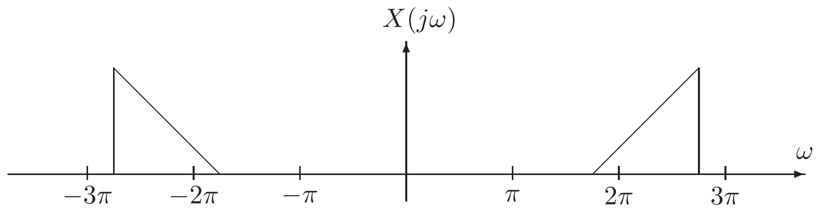


Figure 2: The spectrum for Problem 6, Part *(b)*.

(c) Suppose that you are allowed to process the continuous-time band-pass signal (using a suitable continuous-time system) *before* sampling it. Show that in this case, it is always possible to sample the signal at a sampling frequency equal to twice its spectral support.

Remark: Band-pass sampling is quite an interesting tool in digital communications. Suppose that the incoming continuous-time signal is at $1900MHz$ (the GSM band in the United States), but its spectral support is very small (it may be a speech signal with a spectral support of less than $50kHz$). Hence, instead of demodulating the signal in the continuous time domain, requiring expensive local oscillators, one can potentially sample it at a very low rate, corresponding to the spectral support (for the speech example, sampling at $100kHz$ may be enough, as we have shown in this problem). Clearly, this is a very attractive system design. The downside is that the implementation of precise sampling devices is known to be a rather challenging task.

Problem 7 *(Aliasing.)*

First, note that the signal

$$\begin{aligned} x(t) &= \frac{W}{4\pi}\text{sinc}^2\left(\frac{Wt}{4\pi}\right) \\ &= \frac{4\pi}{W}\left(\frac{\sin(\frac{W}{4}t)}{\pi t}\right)^2 \end{aligned}$$

Note: The above definition of $sinc(x)$ is consistent with Matlab's definition, which is $sinc(x) = sin(\pi x)/(\pi x)$.

has Fourier transform

$$X(j\omega) = \begin{cases} 1 - \left|\frac{2\omega}{W}\right| & \text{if} - \frac{W}{2} < \omega < \frac{W}{2} \\ 0 & \text{elsewhere} \end{cases}$$

which is bandlimited with one sided bandwidth $\omega_M = \frac{W}{2}$. From the sampling theorem we know that if the signal is sampled with sampling interval T satisfying $\frac{2\pi}{T} > W$, then we can reconstruct $x(t)$ perfectly.

For this problem, pick $W = 8$. If the sampling interval is $T = 0.5$ we would have perfect reconstruction, while with $T = 2$ we should expect to see aliasing.

Using Matlab:

On the parts of this problem that involve multiplication by impulses or time/frequency conversions, don't worry too much about getting the scale on the y-axis exactly right. The scale of the independent variable (t or ω) is very important, however.

(a) Plot $x(t)$ over $-10 < t < 10$. Also plot $X(j\omega)$.

(b) For $T = 0.5$, plot $x_p(t)$, the result of multiplying $x(t)$ by an impulse train of period T. Be sure your time axis is labeled correctly. Plot its Fourier transform. What is its period?

(c) Low pass filter $x_p(t)$ to retain only one period and plot the resulting time-domain waveform. You may do this filtering in the frequency domain (multiply by a box function) or in the time domain (convolve with a sinc).

(d) For $T = 2$, plot $x_p(t)$, the result of multiplying $x(t)$ by an impulse train of period T. Be sure your time axis is labeled correctly. Plot its Fourier transform. What is its period?

(e) Low pass filter $x_p(t)$ to retain only one period and plot the resulting time-domain waveform. You may do this filtering in the frequency domain (multiply by a box function) or in the time domain (convolve with a sinc). Don't worry about getting the y-axis scaling exactly right.