Homework 6 Solutions

Problem 1 *(Discrete-time Processing of Continuous-time Signals.)* OWN Problem 7.29

The outputs of each block of the overall system for filtering a continous-time signal using a discrete-time filter are given and plotted in Figure 1. Following OWN notation, in this problem Ω is used to denote the frequency variable of the discrete-time signal.

$$\begin{aligned}
x_p(t) &= x_c(t)p(t) \\
X_p(j\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega - \frac{2\pi k}{T})) \\
x[n] &= x_c(nT) \\
X(e^{j\Omega}) &= X_p(j\frac{\Omega}{T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\Omega - 2\pi k}{T})) \\
y[n] &= x[n] * h[n] \\
Y(e^{j\Omega}) &= X(e^{j\Omega})H(e^{j\Omega}) \\
y_p(t) &= \sum_{n=-\infty}^{\infty} y[n]\delta(t - nT) \\
Y_p(j\omega) &= \sum_{k=-\infty}^{\infty} y[n]e^{-j\omega nT} \\
Y_p(j\omega) &= Y(e^{j\omega T}) \\
y_c(t) &= y_p(t) * h(t) \\
Y_c(j\omega) &= Y_p(j\omega)H(j\omega)
\end{aligned}$$

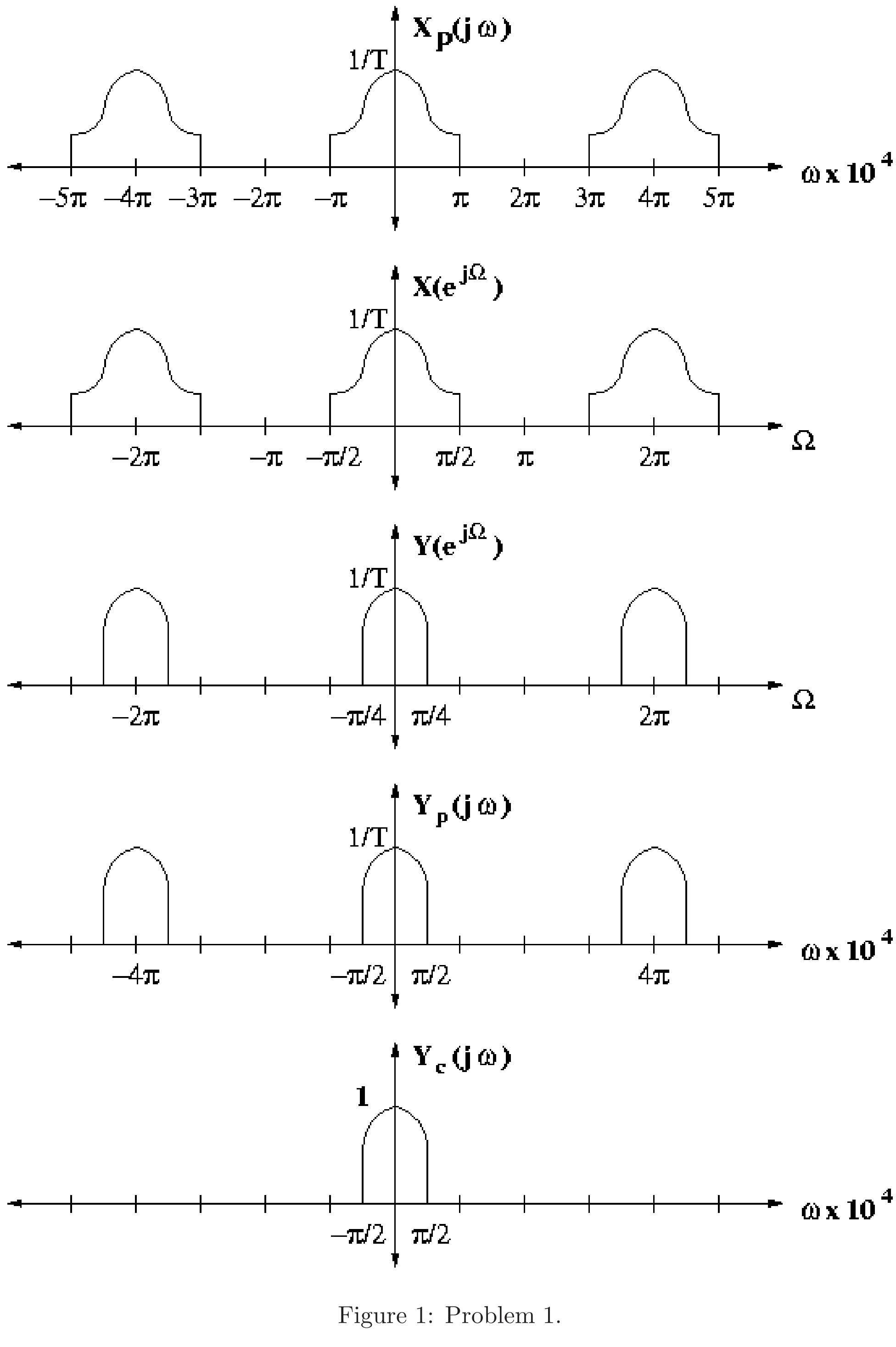


Figure 1: Problem 1.

Problem 2 *(Discrete-time Processing of Continuous-time Signals.)* OWN Problem 7.43

First let's determine the frequency response of the LTI system that has input $x_c(t)$ and output $y_c(t)$ by converting the following time domain equation to the frequency domain.

$$\frac{d^2y_c(t)}{dt^2} + 4\frac{dy_c(t)}{dt} + 3y_c(t) = x_c(t)$$

In the frequency domain, this is

$$\begin{aligned}
(j\omega)^2Y_c(j\omega) + 4(j\omega)Y_c(j\omega) + 3Y_c(j\omega) &= X_c(j\omega) \\
Y_c(j\omega)[(j\omega)^2 + 4(j\omega) + 3] &= X_c(j\omega) \\
H_c(j\omega) = \frac{Y_c(j\omega)}{X_c(j\omega)} = \frac{1}{(j\omega+3)(j\omega+1)} \\
H_c(j\omega) = \frac{0.5}{j\omega+1} - \frac{0.5}{j\omega+3}
\end{aligned}$$

Taking the inverse Fourier transform, we get that

$$h(t) = (0.5e^{-t} - 0.5e^{-3t})u(t)$$

Going through the block diagram shown in Figure P7.43(a) of OWN, we see that:

$$\begin{aligned}
x_p(t) &= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \\
X_p(j\omega) &= X(e^{j\omega T}) \\
X_c(j\omega) &= \begin{cases} TX_p(j\omega) = TX(e^{j\omega T}) & |\omega| \leq \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases} \\
Y_c(j\omega) &= \begin{cases} H_c(j\omega)X_c(j\omega) = H_c(j\omega)TX(e^{j\omega T}) & |\omega| \leq \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases} \\
Y_p(j\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c(j(\omega - \frac{k2\pi}{T}))
\end{aligned}$$

One period of $Y_p(j\omega) = \frac{1}{T}Y_c(j\omega) = H(j\omega)X(e^{j\omega T})$ for $|\omega| \leq \frac{\pi}{T}$.

Therefore, one period of $Y(e^{j\Omega}) = H(j\frac{\Omega}{T})X(e^{j\Omega})$ for $|\Omega| \leq \pi$.

Thus, the equivalent LTI system $H(e^{j\omega}) = H(j\frac{\Omega}{T})$ for $|\Omega| \leq \pi$. Note that $H(e^{j\omega})$ is Fourier transform of $h[n]$ which can be obtained by low-pass filtering $h(t)$ (with a filter of height T and cutoff frequency of $\frac{\pi}{T}$) and sampling the result every T. Therefore,

$$h[n] = \Big[h(t) * \frac{sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}\Big]_{t=nT} = \Big[\frac{T}{2}\int_{\tau=0}^{\infty}(e^{-\tau} - e^{-3\tau})\frac{sin(\frac{\pi(t-\tau)}{T})}{\pi(t-\tau)}d\tau\Big]_{t=nT}$$

Problem 3 *(Sampling Rates.)*

(a) In order to avoid aliasing we should sample at twice the maximum frequency. Thus, we should sample at 40kHz.

(b) Since we do not care if there is aliasing between 15 - 20 kHz, we should sample at 35 kHz. At this sampling rate there is no aliasing for frequencies less than 15 kHz.

Problem 4 *(Discrete-time Zero-order and First-order Holds.)* OWN Problem 7.50

(a)

$$h_0[n] = \begin{cases} 1 & n = 0, 1, ..., N-1 \\ 0 & else \end{cases}$$

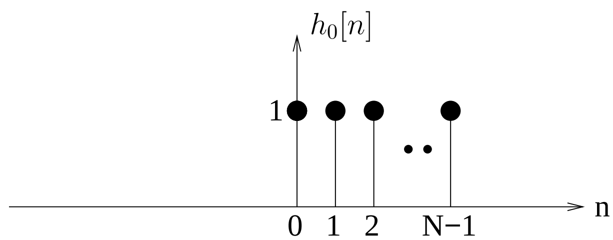


Figure 2: Problem 4 (a)

(b) The condition $\omega_S = \frac{2\pi}{N} > 2\omega_M$ is equivalent to $\omega_M < \frac{\pi}{N}$. Because

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_S)})$$

the signals $X(e^{j\omega})$ and $X_p(e^{j\omega})$ can be sketched as follows.

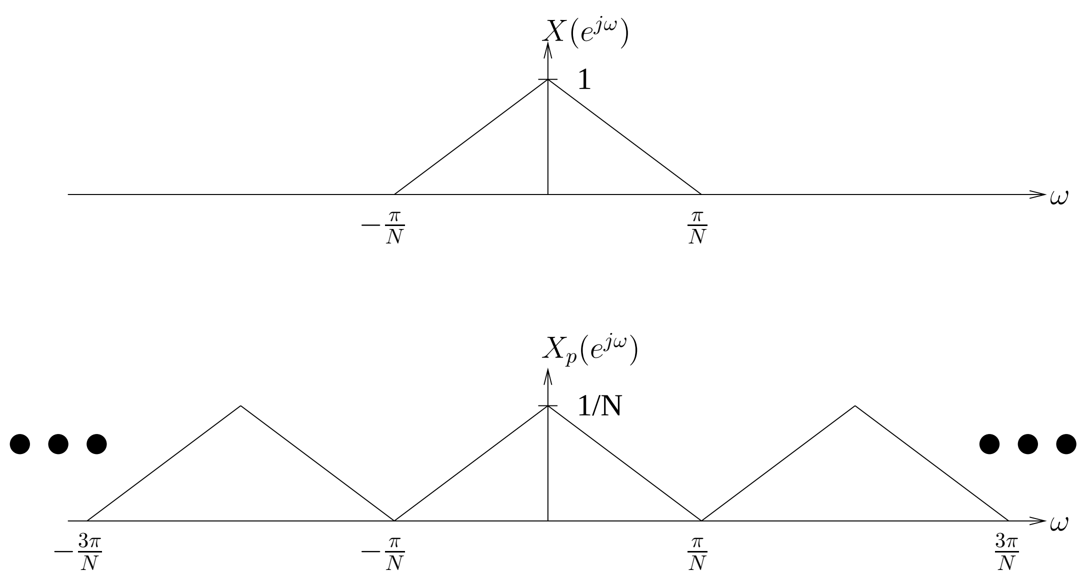


Figure 3: Problem 4 (b)

In order to perfectly recover $x[n]$, we require that

$$H_L(e^{j\omega}) = H_0(e^{j\omega})H(e^{j\omega}) = \begin{cases} N & |\omega| < \pi/N \\ 0 & else \end{cases}$$

The frequency response of the ZOH is given by

$$H_0(e^{j\omega}) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = e^{(-j\omega(N-1)/2)} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Therefore,

$$H(e^{j\omega}) = \frac{H_L(e^{j\omega})}{H_0(e^{j\omega})} = \begin{cases} Ne^{(j\omega(N-1)/2)} \cdot \frac{\sin(\omega/2)}{\sin(\omega N/2)} & |\omega| < \pi/N \\ 0 & else \end{cases}$$

A plot of $|H(e^{j\omega})|$ for $N = 2$ is given in Figure 4.

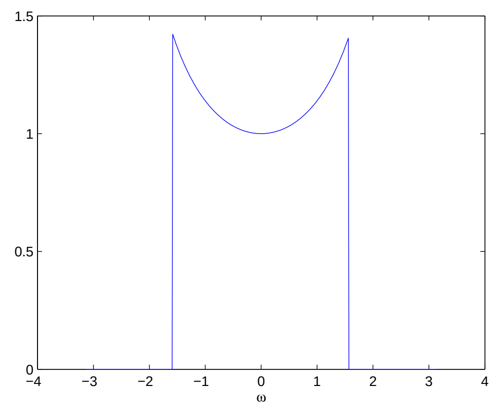


Figure 4: Problem 4 (b)

(c)

$$h_1[n] = \begin{cases} 1 - \left|\frac{n}{N}\right| & n = -N, ..., N \\ 0 & else \end{cases}$$

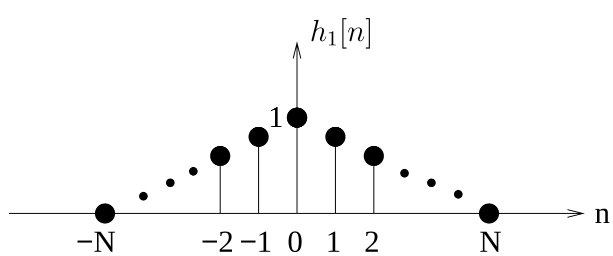


Figure 5: Problem 4 (c)

(d) First, we observe that $h_1[n] = \frac{1}{N}h_0[n] * h_0[-n]$. Therefore,

$$H_1(e^{j\omega}) = \frac{1}{N}H_0(e^{j\omega})H_0(e^{-j\omega}) = \frac{1}{N}\frac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$$

In order to perfectly recover $x[n]$, we require $H_L(e^{j\omega}) = H_1(e^{j\omega})H(e^{j\omega})$ to be the same as in part (b). Following the same reasoning as in part (b), we see that

$$H(e^{j\omega}) = \begin{cases} N^2 \frac{\sin^2(\omega/2)}{\sin^2(\omega N/2)} & |\omega| < \pi/N \\ 0 & else \end{cases}$$

A plot of $|H(e^{j\omega})|$ for $N = 2$ is given in Figure 6.

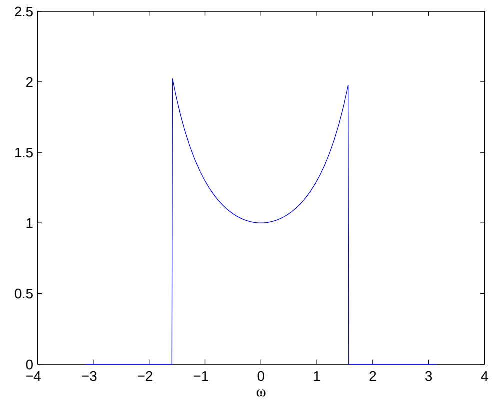


Figure 6: Problem 4 (d)

Problem 5 *(Oversampled D/A.)*

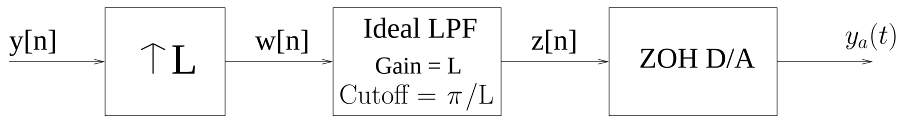


Figure 7: Problem 5. Block diagram of oversampled D/A

(a) We know that that $W(e^{j\Omega}) = Y(e^{jL\Omega})$. Applying a LPF to $w[n]$ gives a signal $z[n]$ with spectrum

$$Z(e^{j\Omega}) = \begin{cases} L & |\Omega| < \frac{3\pi}{4L} \\ 0 & \frac{3\pi}{4L} < |\Omega| < \pi \end{cases}$$

The spectra $W(e^{j\Omega})$ and $Z(e^{j\Omega})$ are plotted in the following figure.

The spectrum of the output of the ZOH is given by $Z(e^{j\omega T})$ multiplied by the frequency response of $p(t)$

$$\begin{aligned} Y_a(j\omega) &= Z(e^{j\omega T}) \cdot e^{-j\omega T/2} \cdot T \cdot \text{sinc}(\omega T/2) \\ &= Z(e^{j\omega/L}) \cdot e^{-j\omega/(2L)} \cdot \frac{1}{L} \cdot \text{sinc}(\omega/(2L)) \end{aligned}$$

where we are using the definition $\text{sinc}(x) = \sin(x)/x$. The magnitude of the output is given by

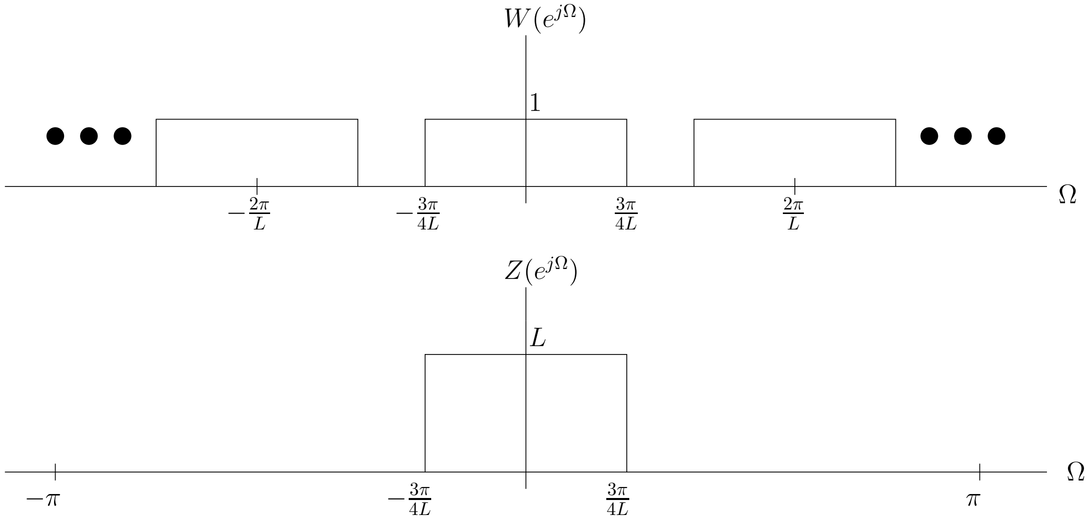


Figure 8: Problem 5 (a)

$$|Y_a(j\omega)| = |Z(e^{j\omega/L})| \cdot \frac{1}{L} \cdot |\text{sinc}(\omega/(2L))|$$

Plots of $|Y_a(j\omega)|$ in the interval $-5\pi L \leq \omega \leq 5\pi L$ are shown in the following figure.

(b) The ideal D/A converter is a LPF, which removes all of the high frequency images in the spectrum. Therefore, the magnitude spectrum $|Y(j\omega)|$ is as shown in Figure 10.

(c) The largest component of $|Y_a(j\omega)|$ outside of $|\omega| \leq \pi L$ is the left edge of the copy of the spectrum centered at $2\pi L$. At that point, the magnitude is equal to

$$\text{sinc}\left(\frac{2\pi L - 3\pi/4}{2L}\right) = \text{sinc}\left(\pi - \frac{3\pi}{8L}\right)$$

For $L = 1$, the magnitude of the largest out of band component is 0.4705

For $L = 2$, the magnitude of the largest out of band component is 0.2177

For $L = 4$, the magnitude of the largest out of band component is 0.1020

Problem 6 *(Band-pass Sampling.)*

(a) $x(t)$ has a Nyquist rate 6π. Sampling $x(t)$ at frequency $w_s = 2\pi$ produces

$$X_p(jw) = \sum_{k=-\infty}^{\infty} X(k(\omega - 2\pi k)).$$

As we can see from Figure 11, the shifted replicas of $X(j\omega)$ do not overlap. Since there is no aliasing, we can perfectly reconstruct $x(t)$ by bandpass filtering the sampled signal with $H(jw)$.

(b) $x(t)$ has a Nyquist rate $\frac{11\pi}{2}$. If we sample $x(t)$ at frequency $\omega_s = 2\pi$, the shifted replicas of $X(j\omega)$ would overlap and we get aliasing, as seen in Figure 12.

The smallest sampling frequency that avoids aliasing is $\omega_s = \frac{11\pi}{4}$. The samped signal $x_p(t)$ has Fourier transform shown in Figure 13.

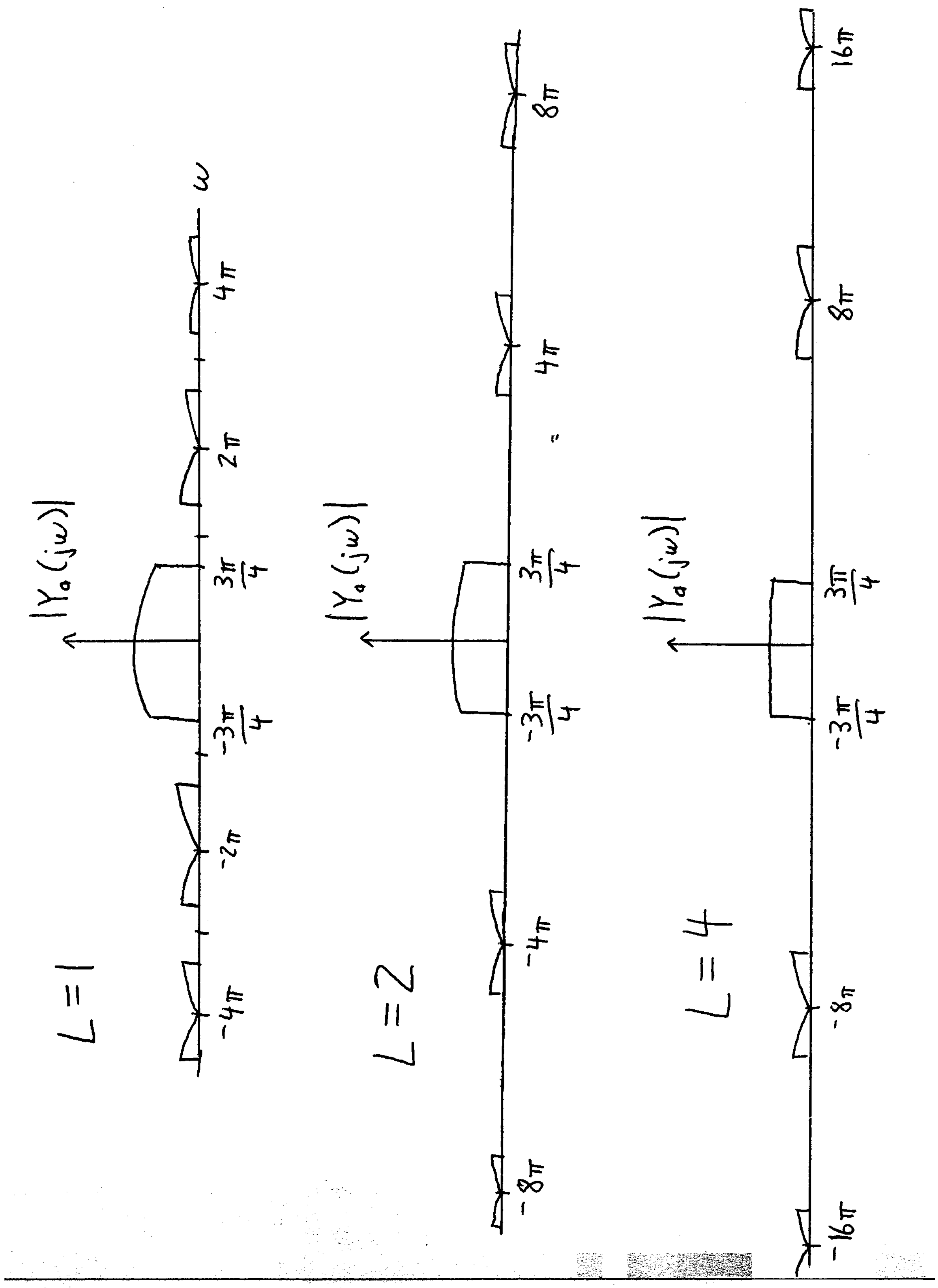


Figure 9: Problem 5 (a)

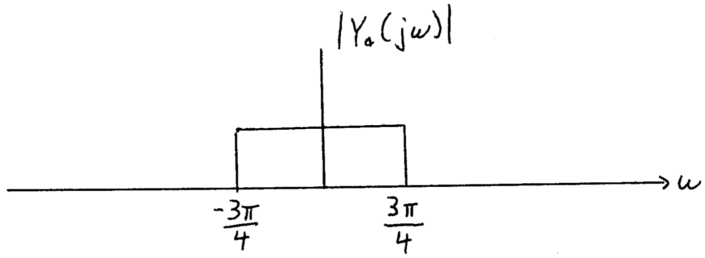


Figure 10: Problem 5 (b)

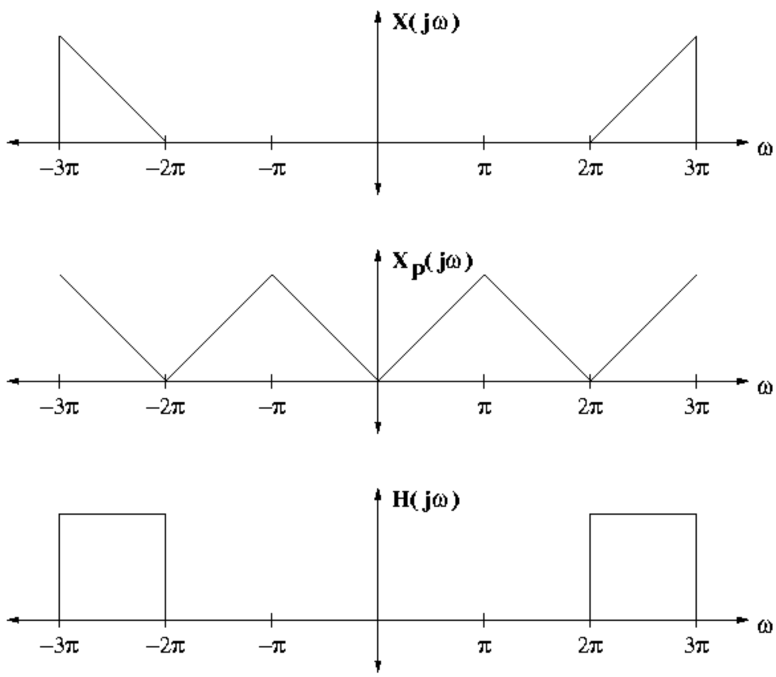


Figure 11: Problem 6 (a)

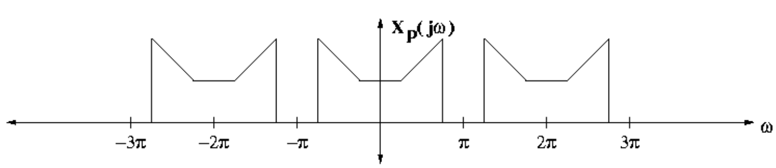


Figure 12: Problem 6 (b)

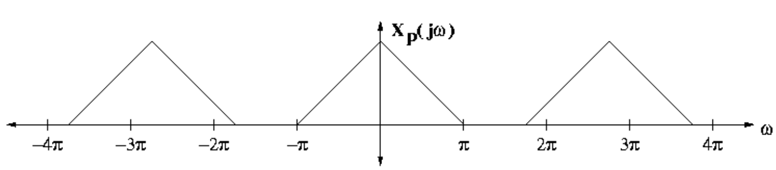


Figure 13: Problem 6 (b)

(c) If we pre-process the signal before sampling by making the bandpass signal a lowpass signal, then we can sample the lowpass singal at the Nyquist rate, which is twice the spectral support of the bandpass signal. We can then recover the original bandpass signal with reconstruction and post-processing. See Figure 14.

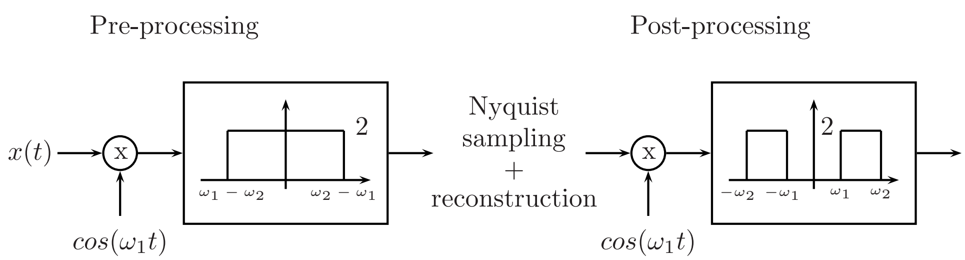


Figure 14: Problem 6 (c)

Problem 7 *(Aliasing.)*

(a)

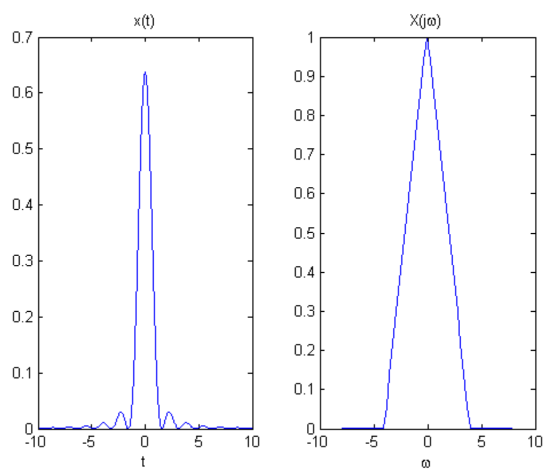


Figure 15: Problem 7 (a)

(b) The period is $2\pi/T = 4\pi$.

(d) The period is $2\pi/T = \pi$.

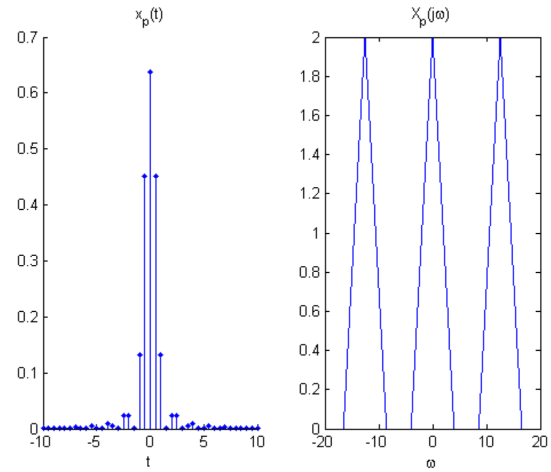
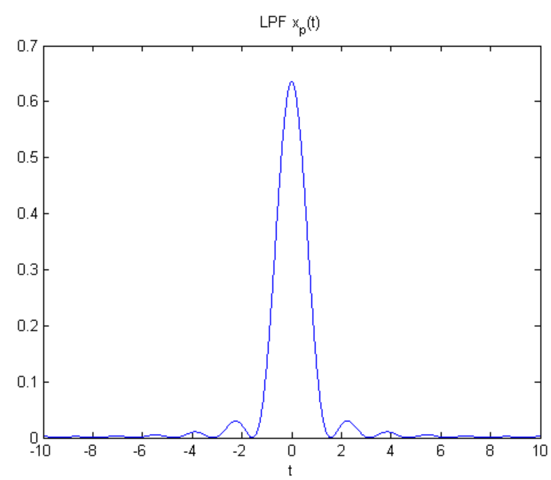


Figure 16: Problem 7 (b)



(c)

Figure 17: Problem 7 (c)

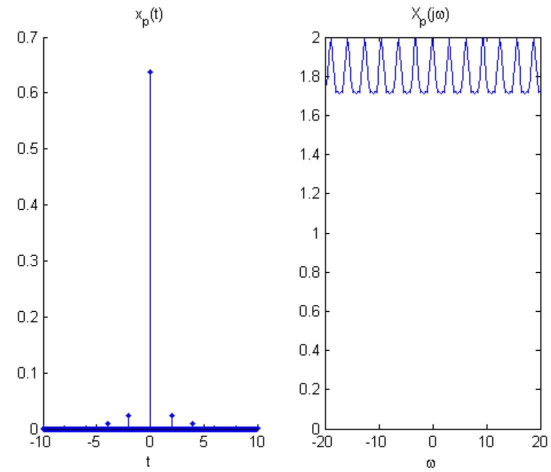


Figure 18: Problem 7 (d)

(e)

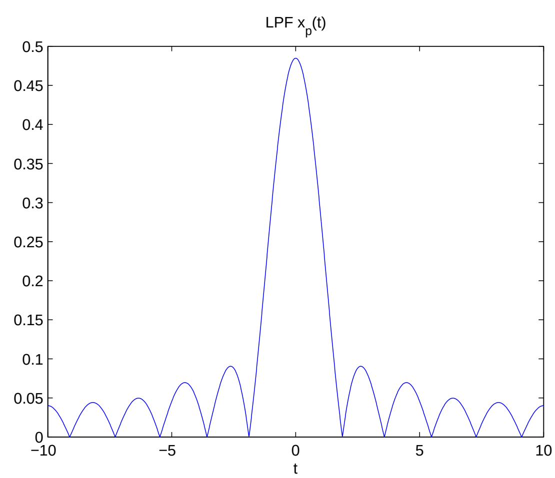


Figure 19: Problem 7 (e)