Homework 9 Solutions
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Problem 1 *(Angle Modulations.)*

The general form of an angle modulated signal:

$$A\cos(2\pi f_c t + k_p a m(t)) \quad \text{(PM)}$$
$$A\cos(2\pi f_c t + k_f a \int m(t)) \quad \text{(FM)}$$

The modulation index is defined as:

$$\beta_p = k_p a \max[|m(t)|] = \Delta\phi_{max} \quad \text{(PM)}$$
$$\beta_f = \frac{k_f a \max[|m(t)|]}{W} = \frac{\Delta f_{max}}{W} \quad \text{(FM)}$$

Where $\Delta\phi_{max}$ and Δf_{max} are the maximum phase and frequency deviations respectively. W is the bandwidth of the modulating signal in Hz (i.e. $m(t)$ is bandlimited to W). In this problem, the modulated signal:

$$u(t) = 100\cos(2\pi f_c t + 4\sin 2\pi f_m t)$$

The instantaneous phase $\phi(t) = 2\pi f_c t + 4\sin(2\pi f_m t)$, the instantaneous frequency $f(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = f_c + 4f_m\cos(2\pi f_m t)$, and $W = f_m$.

$$\Rightarrow \Delta\phi_{max} = 4, \Delta f_{max} = 4f_m$$
$$\Rightarrow \beta_p = \Delta\phi_{max} = 4, \beta_f = \frac{\Delta f_{max}}{f_m} = 4$$

We estimate the effective bandwidth B_c (same for both PM and FM) of the modulated signal using *Carson's rule* (look on wikipedia for this!):

$$B_c \approx 2(\beta + 1)W$$

According to OWN, $B_c = 2\beta W$ (both answers are acceptable; we don't expect you to wiki to solve problems :-)).

$$B_c = 2(4+1)f_m = 10\text{KHz}$$

For SSB, the bandwidth of the modulated signal is $f_m = 1$ KHz. I have only considered the positive side bands in this problem. If you wish to include the negative side bands, then simply scale B_c by a factor of 2. Also, notice that B_c is directly proportional to f_m. Therefore, when f_m is doubled:

$$B_c = 20\text{KHz}$$

Problem 2 *(Nyquist Pulses.)*

(a)

$$\begin{aligned}
\delta[n] &= p(nT_s) \\
\sum_{n=-\infty}^{\infty} \delta[n]\delta(t-nT_s) &= \sum_{n=-\infty}^{\infty} p(nT_s)\delta(t-nT_s) \\
\delta(t) &= \sum_{n=-\infty}^{\infty} p(nT_s)\delta(t-nT_s) \\
&= \sum_{n=-\infty}^{\infty} p(t)\cdot\delta(t-nT_s) \\
&= p(t)\sum_{n=-\infty}^{\infty} \delta(t-nT_s) \\
\mathcal{FT}\{\delta(t)\} &= \mathcal{FT}\{p(t)\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\} \\
1 &= \frac{1}{2\pi}\mathcal{FT}\{p(t)\} * \mathcal{FT}\{\sum_{n=-\infty}^{\infty} \delta(t-nT_s)\} \\
&= \frac{1}{2\pi}P(j\omega) * (\frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty} \delta(j(\omega - n\frac{2\pi}{T_s})) \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} P(j(\omega - n\frac{2\pi}{T_s})) \\
T_s &= \sum_{n=-\infty}^{\infty} P(j(\omega - n\frac{2\pi}{T_s}))
\end{aligned}$$

Problem 3

(Narrowband FM.) OWN Problem 8.45.

(a)

$$\begin{aligned}
y(t) &= \cos(\omega_c t + m\int_{-\infty}^{t} x(\tau)d\tau) \\
\theta(t) &= \omega_c t + m\int_{-\infty}^{t} x(\tau)d\tau \\
\omega_i(t) &= \frac{d\theta(t)}{dt} = \omega_c + mx(t)
\end{aligned}$$

(b)

Expanding $y(t)$, we get

$$y(t) = \cos(\omega_c t)\cos\left(m\int_{-\infty}^{t} x(\tau)d\tau\right) - \sin(\omega_c t)\sin\left(m\int_{-\infty}^{t} x(\tau)d\tau\right).$$

Let $z(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Thus $Z(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) = \frac{1}{j\omega}X(j\omega)$. Using Parseval's and the fact that $x(t)$ is band-limited and bounded, $\int_{-\infty}^{\infty} |z(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{X(j\omega)}{j\omega}\right|^2 d\omega = \frac{1}{2\pi}\int_{-\omega_m}^{\omega_m} \left|\frac{X(j\omega)}{j\omega}\right|^2 d\omega = M < \infty$. Thus $z(t)$ is bounded almost all of the time. Now applying the narrowband assumption, we assume that m is small enough to make $m\int_{-\infty}^{t} x(\tau)d\tau$ satisfy the small angle approximation. Therefore,

$$\begin{aligned}
\cos\left(m\int_{-\infty}^{t} x(\tau)d\tau\right) &\approx 1 \\
\sin\left(m\int_{-\infty}^{t} x(\tau)d\tau\right) &\approx m\int_{-\infty}^{t} x(\tau)d\tau
\end{aligned}$$

This implies that

$$y(t) \approx \cos(\omega_c t) - \left(m\int_{-\infty}^{t} x(\tau)d\tau\right)\sin(\omega_c t).$$

(c)

Let $y(t) \approx \cos(\omega_c t) - \left(m\int_{-\infty}^{t} x(\tau)d\tau\right)\sin(\omega_c t)$ and $z(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Thus $Z(j\omega) = \frac{1}{j\omega}X(j\omega)$. Since $X(j\omega) = 0$ for $|\omega| > \omega_m$ is band-limited, then $Z(j\omega) = 0$ for $|\omega| > \omega_m$.

$$\begin{aligned}
Y(j\omega) &\approx \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] - \frac{m}{2\pi}Z(j\omega) * \frac{\pi}{j}[\delta(\omega-\omega_c)-\delta(\omega+\omega_c)] \\
&\approx \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] - \frac{m}{2j}(Z(j(\omega-\omega_c)) - Z(j(\omega+\omega_c)))
\end{aligned}$$

Thus if the bandwidth of $x(t)$ is ω_m, then the bandwidth of $y(t)$ is $2\omega_m$.

Problem 4 *(Laplace Transforms.)*

(a) OWN Problem 9.21 (c).

$$x(t) = e^{2t}u(-t) + e^{3t}u(-t)$$

We can find the following Laplace transform pairs in Table 9.2 of OWN:

$$\begin{aligned}
e^{2t}u(-t) &\longleftrightarrow -\frac{1}{s-2}\text{, with ROC } Re\{s\} < 2 \\
e^{3t}u(-t) &\longleftrightarrow -\frac{1}{s-3}\text{, with ROC } Re\{s\} < 3
\end{aligned}$$

By the linearity property of the Laplace transform (see Table 9.1 of OWN)

$$X(s) = -\frac{1}{s-2} - \frac{1}{s-3} = \frac{-2s+5}{(s-2)(s-3)}$$

with region of convergence (ROC) $Re\{s\} < 2$. Thus $X(s)$ has a pole at 2, another pole at 3, and a zero at 2.5.

(b) OWN Problem 9.21 (e).

$$x(t) = |t|e^{-2|t|} = te^{-2t}u(t) - te^{2t}u(-t)$$

Using the Laplace transform tables again, we find the following Laplace transform pairs in OWN Table 9.2

$$\begin{aligned} e^{-2t}u(t) &\longleftrightarrow \frac{1}{s+2}\text{ , with ROC } Re\{s\} > -2 \\ e^{2t}u(-t) &\longleftrightarrow -\frac{1}{s-2}\text{ , with ROC } Re\{s\} < 2 \end{aligned}$$

By the differentiation in the s-domain property, in OWN Table 9.1,

$$\begin{aligned} te^{-2t}u(t) &\longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s+2}\right) = \frac{1}{(s+2)^2} \\ -te^{2t}u(-t) &\longleftrightarrow \frac{d}{ds}\left(-\frac{1}{s-2}\right) = \frac{1}{(s-2)^2} \end{aligned}$$

Finally by the linearity property,

$$X(s) = \frac{1}{(s+2)^2} + \frac{1}{(s-2)^2} = \frac{2(s^2+4)}{(s+2)^2(s-2)^2}$$

with ROC $-2 < Re\{s\} < 2$. Thus $X(s)$ has two poles at -2, two poles at 2, and two zeros at 0.

Problem 5 *(Region of convergence.)*

OWN 9.23. The four pole-zero plots shown may have the following possible ROCs:

- Top Left (TL): $\Re\{s\} < -2$ or $-2 < \Re\{s\} < 2$ or $\Re\{s\} > 2$
- Top Right (TR): $\Re\{s\} < -2$ or $\Re\{s\} > -2$
- Bottom Left (BL): $\Re\{s\} < 2$ or $\Re\{s\} > 2$
- Bottom Right (BR): Entire s-plane

Also, suppose that the signal $x(t)$ has Laplace transform $X(s)$ with ROC R

- (1)

  If $x(t)e^{-3t}$ is absolutely integrable, then the ROC of $x(t)$ must contain the line $\Re\{s\} = 3$. This can be seen by looking at the definition of the ROC in equation 9.36 in OWN. The ROC R of $x(t)$ is:

  (TL) $\Re\{s\} > 2$, (TR) $\Re\{s\} > -2$, (BL) $\Re\{s\} > 2$, (BR) Entire s-plane

- (2)
We know from Table 9.2 that $e^{-t}u(t)$ has Laplace transform $\frac{1}{s+1}$ with ROC $\Re\{s\} > -1$. From Table 9.1, the Laplace transform of $x(t) \star [e^{-t}u(t)]$ is

$$\frac{X(s)}{s+1}$$

with ROC $R_2 = R \cap [\Re\{s\} > -1]$. If $x(t) \star [e^{-t}u(t)]$ is absolutely integrable, then R_2 must include the $j\omega$ axis, which means that R must include the $j\omega$ axis. The ROC R is:
(TL) $-2 < \Re\{s\} < 2$, (TR) $\Re\{s\} > -2$, (BL) $\Re\{s\} < 2$, (BR) Entire s-plane

- (3)
If $x(t) = 0$ for $t > 1$, then $x(t)$ is left-sided or finite duration. This implies that if $\Re\{s\} = \sigma_1$ is in the ROC, then all values of s for which $\Re\{s\} < \sigma_1$ will also be in the ROC. The ROC R is:
(TL) $\Re\{s\} < -2$, (TR) $\Re\{s\} < -2$, (BL) $\Re\{s\} < 2$, (BR) Entire s-plane

- (4)
If $x(t) = 0$ for $t < -1$, then $x(t)$ is right-sided or finite duration. This implies that if $\Re\{s\} = \sigma_1$ is in the ROC, then all values of s for which $\Re\{s\} > \sigma_1$ will also be in the ROC. The ROC R is:
(TL) $\Re\{s\} > 2$, (TR) $\Re\{s\} > -2$, (BL) $\Re\{s\} > 2$, (BR) Entire s-plane

Problem 6 *(Inverse Laplace.)*

(a) OWN Problem 9.22 (a).

Using the Laplace transform pairs Table 9.2 of OWN, we can immediately find the inverse Laplace transform of $X(s)$ to be $x(t) = \frac{1}{3}\sin(3t)u(t)$. Alternatively we can use partial fraction expansions to find the inverse.

$$\begin{aligned} X(s) &= \frac{1}{s^2+9} \\ &= \frac{j/6}{s+j3} + \frac{-j/6}{s-j3} \\ &\quad \text{for } Re(s) > 0 \\ x(t) &= \frac{j}{6}e^{-j3t}u(t) - \frac{j}{6}e^{j3t}u(t) \\ &= -\frac{j}{6}2j\sin(3t)u(t) \\ &= \frac{1}{3}\sin(3t)u(t) \end{aligned}$$

(b) OWN Problem 9.22 (b).

$$\begin{aligned} X(s) &= \frac{s}{s^2+9} \\ &= \frac{1/2}{s+j3} + \frac{1/2}{s-j3} \\ &\quad \text{for } Re(s) < 0 \\ x(t) &= -\frac{1}{2}e^{-j3t}u(-t) - \frac{1}{2}e^{j3t}u(-t) \\ &= -\cos(3t)u(-t) \end{aligned}$$