Homework 1 Solutions

Email your scores to ee120.gsi@gmail.com
Check the course website for more details on the homework self-grading policy.

Problem 1 *(Complex numbers.)*

(a)

Cartesian: $e^{j5\pi} = \cos(5\pi) + j\sin(5\pi) = -1$

Polar: $e^{j5\pi} = e^{j\pi}$

Cartesian: $\left(\frac{1}{3}e^{j\pi/4}\right)^* = \frac{\sqrt{2}}{6} - j\frac{\sqrt{2}}{6}$

Polar: $\left(\frac{1}{3}e^{j\pi/4}\right)^* = \frac{1}{3}e^{j(-\pi/4)}$

Cartesian: $j^{-j} = (e^{j\pi/2})^{-j} = e^{\pi/2}$

Polar: $j^{-j} = e^{\pi/2} = e^{\pi/2} \cdot e^{j0}$

(b)

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{ac+bd+j(bc-ad)}{c^2+d^2}$$

We must examine the real and imaginary parts to determine the quadrant in the complex plane in which the complex number lies.

$$\frac{a+jb}{c+jd} = \begin{cases} \sqrt{\frac{a^2+b^2}{c^2+d^2}}e^{j\left(\arctan\left(\frac{bc-ad}{ac+bd}\right)+\pi\right)}, & ac+bd<0 \text{ and } bc-ad>0 \\ \sqrt{\frac{a^2+b^2}{c^2+d^2}}e^{j\left(\arctan\left(\frac{bc-ad}{ac+bd}\right)-\pi\right)}, & ac+bd<0 \text{ and } bc-ad<0, \\ \sqrt{\frac{a^2+b^2}{c^2+d^2}}e^{j\left(\arctan\left(\frac{bc-ad}{ac+bd}\right)\right)}, & \text{otherwise} \end{cases}$$

(c) See plots in Figure 1.

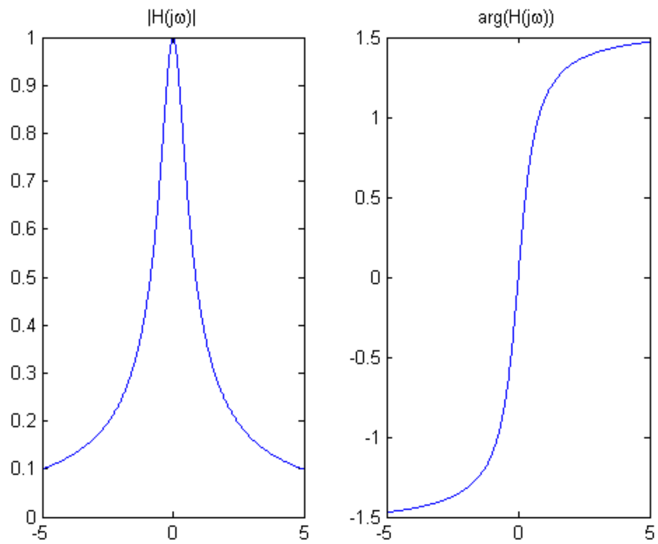

Figure 1: Problem 1(c).

Problem 2 *(Implementation of Complex Systems.)*

$$
\begin{aligned}
y[n] &= (2-j3)\,x[n] + (3+j)\,x[n-1] \\
y_R[n] + jy_I[n] &= (2-j3)(x_R[n] + jx_I[n]) + (3+j)(x_R[n-1] + jx_I[n-1]) \\
y_R[n] + jy_I[n] &= 2x_R[n] + j2x_I[n] - j3x_R[n] + 3x_I[n] + 3x_R[n-1] + j3x_I[n-1] + jx_R[n-1] - x_I[n-1] \\
y_R[n] + jy_I[n] &= (2x_R[n] + 3x_I[n] + 3x_R[n-1] - x_I[n-1]) + j(2x_I[n] - 3x_R[n] + 3x_I[n-1] + x_R[n-1]) \\
y_R[n] &= 2x_R[n] + 3x_I[n] + 3x_R[n-1] - x_I[n-1] \\
y_I[n] &= 2x_I[n] - 3x_R[n] + 3x_I[n-1] + x_R[n-1]
\end{aligned}
$$

The block diagram is shown in Figure 2

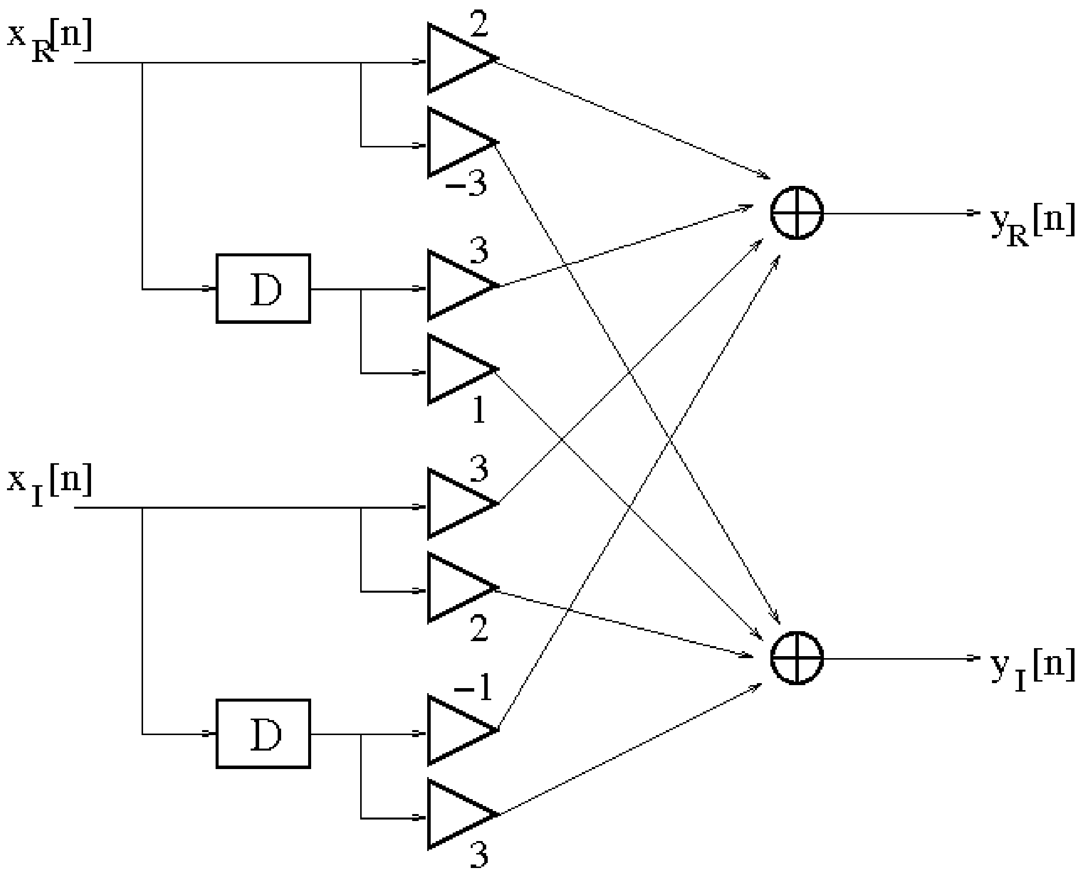

Figure 2: Problem 2.

Problem 3 *(Elementary functions and their graphs.)*

(a)

$$
\begin{aligned}
Re\{y(t)\} &= \cos(2\pi t) \\
Im\{y(t)\} &= \sin(2\pi t) \\
|y(t)| &= 1 \\
\arg(y(t)) &= 2\pi(t - \lfloor t \rfloor)
\end{aligned}
$$

See plots in Figure 3.

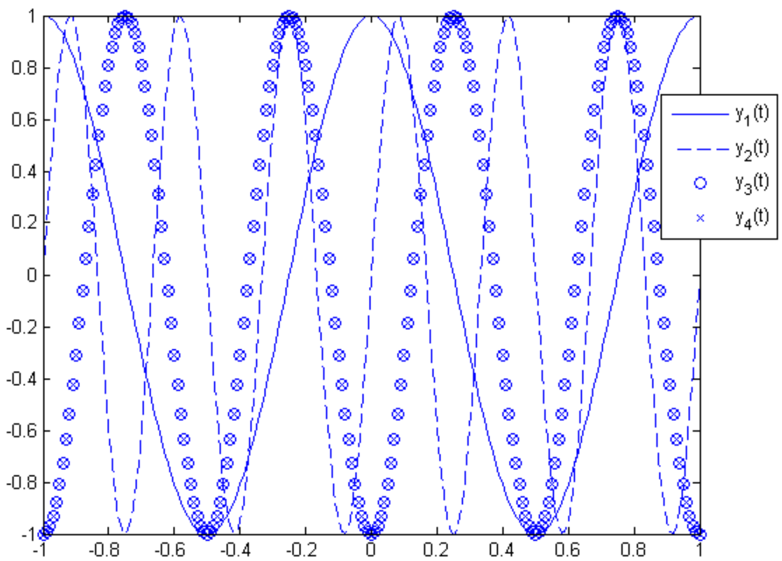

Figure 3: Problem 3(a).

Problem 4 *(Periodic continuous-time signals.)*

(a)

$\cos(\frac{\pi}{3}t)$ has a period of $\frac{2\pi}{\pi/3} = 6$.

$\cos(\frac{2\pi}{5}t)$ has a period of $\frac{2\pi}{2\pi/5} = 5$.

The fundamental period is 30, the least common multiple of 6 and 5.

(b)

$w(t) = \cos(\frac{\pi}{3}t) + \cos(t)$ is aperiodic.

$\cos(\frac{\pi}{3}t)$ has a period of $\frac{2\pi}{\pi/3} = 6$ and $\cos(t)$ has a period of 2π. Since one is rational and one is irrational, they have no common multiples.

Problem 5 *(Transformations of functions.)*

See Figure 4.

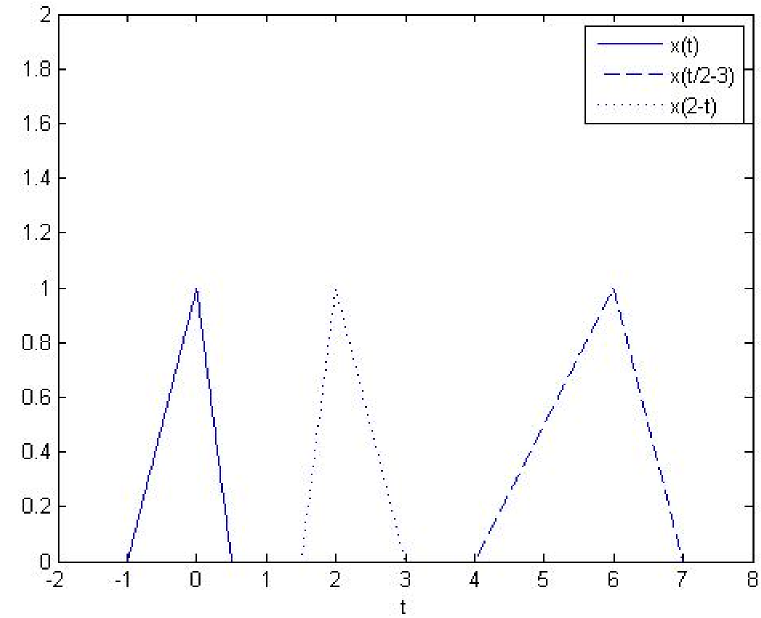

Figure 4: Problem 5.

Problem 6 *(Periodic discrete-time signals.)*

(a)

$y_1[n] = \cos(\frac{\pi}{3}n)$ has a period of 6.

$y_2[n] = \cos(\frac{3\pi}{5}n)$ has a period of 10.

See plots in Figure 5.

(b)

The fundamental period is 30, the least common multiple of 6 and 10.

See plots in Figure 5.

(c)

$z[n] = \cos(2n)$ is aperiodic. There is no integer n such that $2n$ is divisible by 2π.

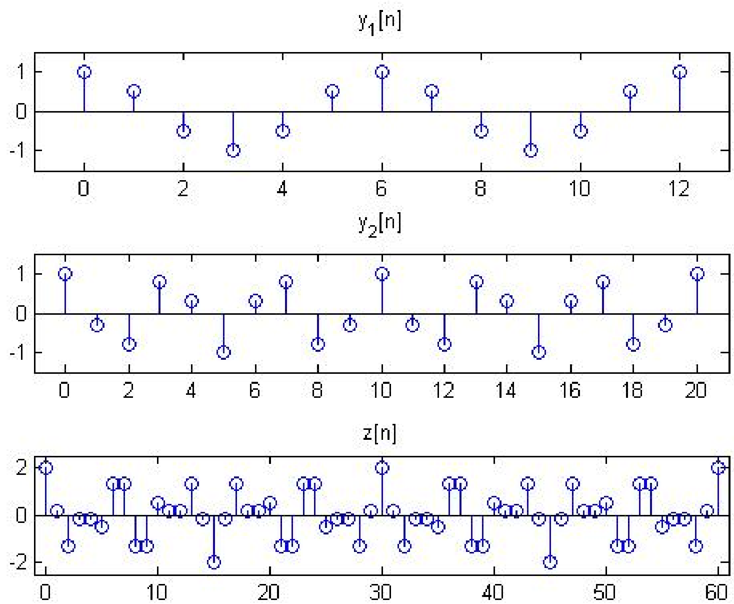

Figure 5: Problem 6.

Problem 7 *(Properties of systems.)*

Let $f(t) \triangleq k_1 x_1(t) + k_2 x_2(t)$ where k_1, k_2 are constants.

Let $g(t) \triangleq x(t-\tau)$ where τ is a constant.

(a) $\mathcal{H}[x(t)] = x(at+b)$ is linear and not time-invariant.

$$\begin{aligned}
\mathcal{H}[f(t)] &= f(at+b) = k_1 x_1(at+b) + k_2 x_2(at+b) \\
&= k_1 \mathcal{H}[x_1(t)] + k_2 \mathcal{H}[x_2(t)] \\
\mathcal{H}[g(t)] &= g(at+b) = x(at+b-\tau) \\
&\neq x(a(t-\tau)+b)
\end{aligned}$$

The last line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $x(at+b)$, by τ.

(b) $\mathcal{H}[x(t)] = x(at^2+b)$ is linear and not time-invariant.

$$\begin{aligned}
\mathcal{H}[f(t)] &= f(at^2+b) = k_1 x_1(at^2+b) + k_2 x_2(at^2+b) \\
&= k_1 \mathcal{H}[x_1(t)] + k_2 \mathcal{H}[x_2(t)] \\
\mathcal{H}[g(t)] &= g(at^2+b) = x(at^2+b-\tau) \\
&\neq x(a(t-\tau)^2+b)
\end{aligned}$$

The last line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $x(at^2+b)$, by τ.

Problem 8 *(Properties of systems.)*

(a) $\mathcal{H}[x(t)] = x(at)+b$ is not linear and not time-invariant.

$$\begin{aligned}
\mathcal{H}[f(t)] &= f(at)+b = k_1 x_1(at) + k_2 x_2(at) + b \\
&\neq k_1 \mathcal{H}[x_1(t)] + k_2 \mathcal{H}[x_2(t)] = k_1 x_1(at) + k_1 b + k_2 x_2(at) + k_2 b \\
\mathcal{H}[g(t)] &= g(at)+b = x(at-\tau)+b \\
&\neq x(a(t-\tau))+b
\end{aligned}$$

The last line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $x(at)+b$, by τ.

(b) $\mathcal{H}[x(t)] = \frac{d}{dt}x(t)$ is linear and time-invariant.

$$\begin{aligned}
\mathcal{H}[f(t)] &= \frac{d}{dt}f(a) = k_1 \frac{d}{dt}x_1(t) + k_2 \frac{d}{dt}x_2(t) \\
&= k_1 \mathcal{H}[x_1(t)] + k_2 \mathcal{H}[x_2(t)] \\
\mathcal{H}[g(t)] &= \frac{d}{dt}g(t) \\
&= \frac{d}{dt}x(t-\tau)
\end{aligned}$$

The above line is the result of shifting the output of $\mathcal{H}[x(t)]$, i.e. $\frac{d}{dt}x(t)$, by τ.

Problem 9 *(Properties of continuous-time systems.)*

(a) $y(t) = x(t)\frac{e^t+e^{-t}}{2}$

(i) memoryless (OWN definition): The current output value $y(t)$ is determined by the value of the current input $x(t)$.

not memoryless (LV/Handout 1 definition): The output at time t requires knowledge of the value of t.

(ii) unstable: For the trivially bounded input $x(t) = 1 \ \forall t$, the output $y(t)$ is unbounded as $t \to \pm\infty$.

(iii) causal: The current output value $y(t)$ does not depend on future inputs $x(\tau), \tau > t$.

(iv) linear:

Let $y_1(t) = x_1(t)\frac{e^t+e^{-t}}{2}$ and $y_2(t) = x_2(t)\frac{e^t+e^{-t}}{2}$.

If the input to the system is $f(t) = k_1 x_1(t) + k_2 x_2(t)$, where k_1, k_2 are constants,

then the output is $g(t) = f(t)\frac{e^t+e^{-t}}{2} = (k_1 x_1(t) + k_2 x_2(t))\frac{e^t+e^{-t}}{2} = k_1 y_1(t) + k_2 y_2(t)$.

(v) not time-invariant:

If the input is $f(t) = x(t-\tau)$, where τ is a constant,

then the output is $g(t) = f(t)\frac{e^t+e^{-t}}{2} = x(t-\tau)\frac{e^t+e^{-t}}{2}$.

Since $y(t-\tau) = x(t-\tau)\frac{e^{t-\tau}+e^{-(t-\tau)}}{2}$, $g(t) \neq y(t-\tau)$.

(b) $y(t) = Im\{x(t)\}$

(i) memoryless: The output at time t_0 depends only on the input at the same time t_0.

(ii) stable: If $|x(t)| < \infty$, then $|Im\{x(t)\}| < \infty$.

(iii) causal: At each time, the output depends only on the current input.

(iv) non-linear:

Consider an input $x_1(t) = jt$ which produces the output $y_1(t) = t$

The input $x_2(t) = jx_1(t) = -t$ produces the output $y_2(t) = 0$, which is not equal to $jy_1(t)$

(v) time-invariant:

If the input is $f(t) = x(t-\tau)$, where τ is a constant,

then the output is $g(t) = Im\{f(t)\} = Im\{x(t-\tau)\}$.

Since $y(t-\tau) = Im\{x(t-\tau)\}$, the system is time-invariant.

Problem 10 *(Properties of continuous-time systems.)*

(a) $y(t) = \int_{-\infty}^{t/2} x(3\alpha)d\alpha = \frac{1}{3}\int_{-\infty}^{3t/2} x(\beta)d\beta$

(i) not memoryless: The current output $y(t)$ uses many past values of the input $x(\tau), \tau < t$.

(ii) unstable: $y(t) = \infty \ \forall t$ if $x(t) = u(-t)$.

(iii) not causal: For $t > 0$, the current output value $y(t)$ depends on future inputs $x(\tau), \tau > t$.

(iv) linear: Integration is linear.

(v) not time-invariant:

If the input is $f(t) = x(t-\tau)$, where τ is a constant,

then the output is $g(t) = \frac{1}{3}\int_{-\infty}^{3t/2} f(\beta)d\beta = \frac{1}{3}\int_{-\infty}^{3t/2} x(\beta-\tau)d\beta = \frac{1}{3}\int_{-\infty}^{3t/2-\tau} x(\sigma)d\sigma$.

Since $y(t-\tau) = \frac{1}{3}\int_{-\infty}^{3(t-\tau)/2} x(\beta)d\beta$, $g(t) \neq y(t-\tau)$.

(b) $y(t) = x(-t/2)$

(i) not memoryless: The output at time t cannot be determined from the input at time t.

(ii) stable: If $x(t)$ is bounded, then $y(t)$ is bounded.

(iii) noncausal: For $t < 0$, the current output $y(t)$ depends on a future value of $x(t)$.

(iv) linear:
Let $y_1(t) = x_1(-t/2)$ and $y_2(t) = x_2(-t/2)$.
If the input is $f(t) = k_1x_1(t) + k_2x_2(t)$, where k_1, k_2 are constants,
then the output is $g(t) = f(-t/2) = k_1x_1(-t/2) + k_2x_2(-t/2) = k_1y_1(t) + k_2y_2(t)$.

(v) not time-invariant:
If the input is $f(t) = x(t - \tau)$, where τ is a constant,
then the output is $g(t) = f(-t/2) = x(-t/2 - \tau)$.
Since $y(t - \tau) = x(-(t - \tau)/2)$, $g(t) \neq y(t - \tau)$.

Problem 11 *(Properties of discrete-time systems.)*

(a) $y[n] = \sum_{k=-\infty}^{n} x[k+1] = \sum_{l=-\infty}^{n+1} x[l]$

(i) not memoryless: The current output $y[n]$ depends on many past input values $x[\tau], \tau < n$.

(ii) not stable: $y[n] = \infty \;\; \forall n$ if $x[n] = u[-n]$.

(iii) not causal: The current output $y[n]$ depends on future input $x[n+1]$.

(iv) linear:
Let $y_1[n] = \sum_{l=-\infty}^{n+1} x_1[l]$ and $y_2[n] = \sum_{l=-\infty}^{n+1} x_2[l]$.
If the input is $f[n] = k_1x_1[n] + k_2x_2[n]$, where k_1, k_2 are constants,
then the output is $g[n] = \sum_{l=-\infty}^{n+1} f[l] = \sum_{l=-\infty}^{n+1}(k_1x_1[l] + k_2x_2[l]) = k_1y_1[n] + k_2y_2[n]$.

(v) time-invariant:
If the input is $f[n] = x[n - \tau]$, where τ is a constant,
then the output is $g[n] = \sum_{l=-\infty}^{n+1} f[l] = \sum_{l=-\infty}^{n+1} x[l - \tau] = \sum_{m=-\infty}^{n+1-\tau} x[m]$.
Therefore $y[n - \tau] = g[n]$.

(b) $y[n] = x[n] \cdot \sum_{k=-\infty}^{\infty} \delta[n - 3k]$

Observe that $y[n] = x[n]$ if n is a multiple of 3, and $y[n] = 0$ otherwise.

(i) memoryless (OWN definition): The current output $y[n]$ does not depend on past or future inputs $x[n]$.
not memoryless (LV/Handout 1 definition): The output at time n requires knowledge of the value of n.

(ii) stable: $y[n]$ is bounded if $x[n]$ is bounded.

(iii) causal: The current output does not depend on future inputs.

(iv) linear:
Let $y_1[n] = x_1[n] \sum_{k=-\infty}^{\infty} \delta[n - 3k]$ and $y_2[n] = x_2[n] \sum_{k=-\infty}^{\infty} \delta[n - 3k]$.
If the input is $f[n] = k_1x_1[n] + k_2x_2[n]$, where k_1, k_2 are constants,
then the output is $g[n] = f[n] \sum_{k=-\infty}^{\infty} \delta[n - 3k] = k_1y_1[n] + k_2y_2[n]$.

(v) not time-invariant:
If the input is $f[n] = x[n - \tau]$, where τ is a constant,
then the output is $g[n] = f[n] \sum_{k=-\infty}^{\infty} \delta[n - 3k] = x[n - \tau] \sum_{k=-\infty}^{\infty} \delta[n - 3k]$.
Since $y[n - \tau] = x[n - \tau] \sum_{k=-\infty}^{\infty} \delta[n - \tau - 3k]$, $g[n] \neq y[n - \tau]$.

Problem 12 *(Convolution of step functions.)*

(a) $h(t) = u(t) - u(t-1)$, $x(t) = u(t)$

$$\begin{aligned} h(t) * x(t) &= \int_{-\infty}^{\infty} (u(\tau) - u(\tau - 1)) \cdot u(t - \tau) d\tau \\ &= \int_{-\infty}^{t} u(\tau) - u(\tau - 1) d\tau \\ &= u(t) \cdot \int_{0}^{\min(t,1)} d\tau \\ &= \begin{cases} 0, & t < 0 \\ t, & 0 \leq t \leq 1 \\ 1, & t > 1 \end{cases} \end{aligned}$$

See Figure 6.

(b) $h(t) = u(t) - u(t-1)$, $x(t) = u(t) - u(t-1)$

We will use the linearity and time invariance properties of the system $h(t)$ to make use of part (a) of this problem. Suppose $u(t)$ is the input to the system as in part (a). Call the output $z(t)$. Now, the input is $x(t) = u(t) - u(t-1)$, the difference of $u(t)$ and a shifted copy of $u(t)$. Using the fact that the system is linear and time invariant, the output should now be $z(t) - z(t-1)$.

$$\begin{aligned} h(t) * x(t) &= [u(t) * (u(t) - u(t-1))] - [u(t-1) * (u(t) - u(t-1))] \\ &= z(t) - z(t-1) \\ &= \begin{cases} t, & 0 \leq t \leq 1 \\ 2 - t, & 1 \leq t \leq 2 \\ 0, & t < 0 \text{ or } t > 2 \end{cases} \end{aligned}$$

See Figure 7.

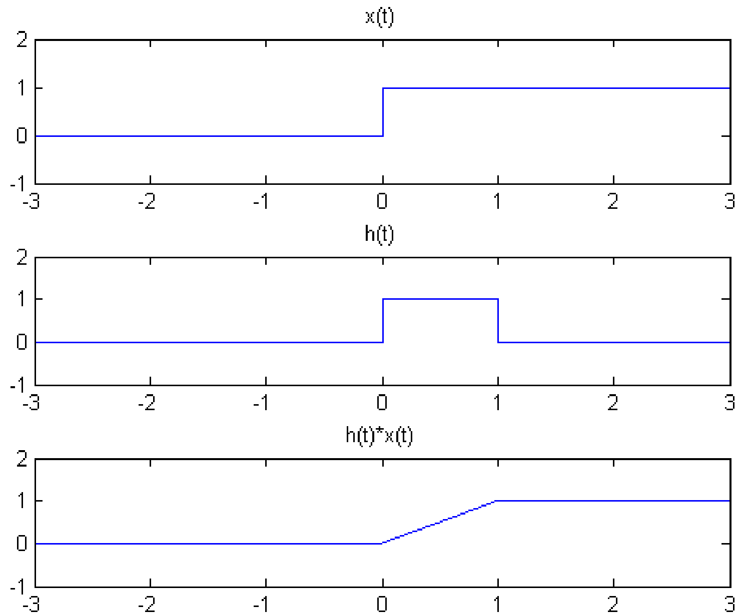

Figure 6: Problem 12(a).

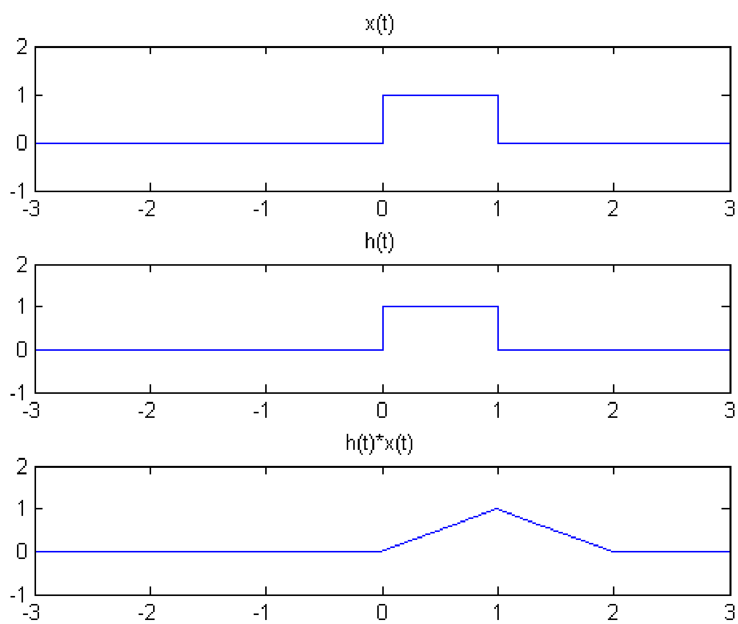

Figure 7: Problem 12(b).

Problem 13 *(LTI systems.)*

We observe that $z(t) = -x(t-1) + 2x(t-2)$. Because the system is LTI, the new output is equal to $-y(t-1) + 2y(t-2)$. The output is plotted in Figure 8.

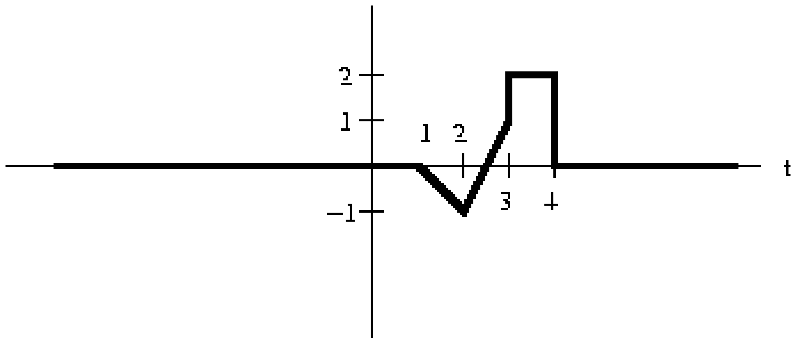

Figure 8: Problem 13.

Problem 14 *(Convolution via matlab.)*

The following Matlab code generates Figure 9.

```
t = -2:.01:2;
xa = (t>=0);
ha = zeros(size(t));
ha(t>=0) = exp(-(0:.01:2));
za = conv(ha,xa);

xb = ha;
hb = zeros(size(t));
hb(t>=0) = exp(-.5.*(0:.01:2));
zb = conv(hb,xb);
```

The Matlab command `z = conv(h,x)` convolves vectors $\mathbf{h}$ (of length a) and $\mathbf{x}$ (of length b), and produces a vector $\mathbf{z}$ of length $a+b-1$. Matlab effectively computes the convolution sum of the discrete-time signals $g[n]$ and $y[n]$, whose nonzero values are contained in $\mathbf{h}$ and $\mathbf{x}$, and assumes that the signals are zero otherwise. Thus $\mathbf{z}$ contains the nonzero values of $g[n] * y[n]$. Since our functions $h(t)$ and $x(t)$ are not zero outside of $t \in [-2, 2]$, we need to discard the extraneous (and incorrect) outputs.

Furthermore, because we are approximating $\int h(\tau)x(t-\tau)d\tau$ with $\sum_\tau h(\tau)x(t-\tau)\Delta\tau$, we need to multiply the output of `conv`, $\sum_\tau h(\tau)x(t-\tau)$, by $\Delta\tau = 0.01$.

```
za = za(201:601)./100;
zb = zb(201:601)./100;
```

Now we are ready to plot the figures.

```
subplot(2,3,1); plot(t,ha); axis([-2 2 -.5 1.5]); title('8a:  h(t)');
subplot(2,3,2); stairs(t,xa); axis([-2 2 -.5 1.5]); title('8a:  x(t)');
subplot(2,3,3); plot(t,za); axis([-2 2 -.5 1.5]); title('8a:  conv(h,x)');

subplot(2,3,4); plot(t,hb); axis([-2 2 -.5 1.5]); title('8b:  h(t)');
subplot(2,3,5); plot(t,xb); axis([-2 2 -.5 1.5]); title('8b:  x(t)');
subplot(2,3,6); plot(t,zb); axis([-2 2 -.5 1.5]); title('8b:  conv(h,x)');
```

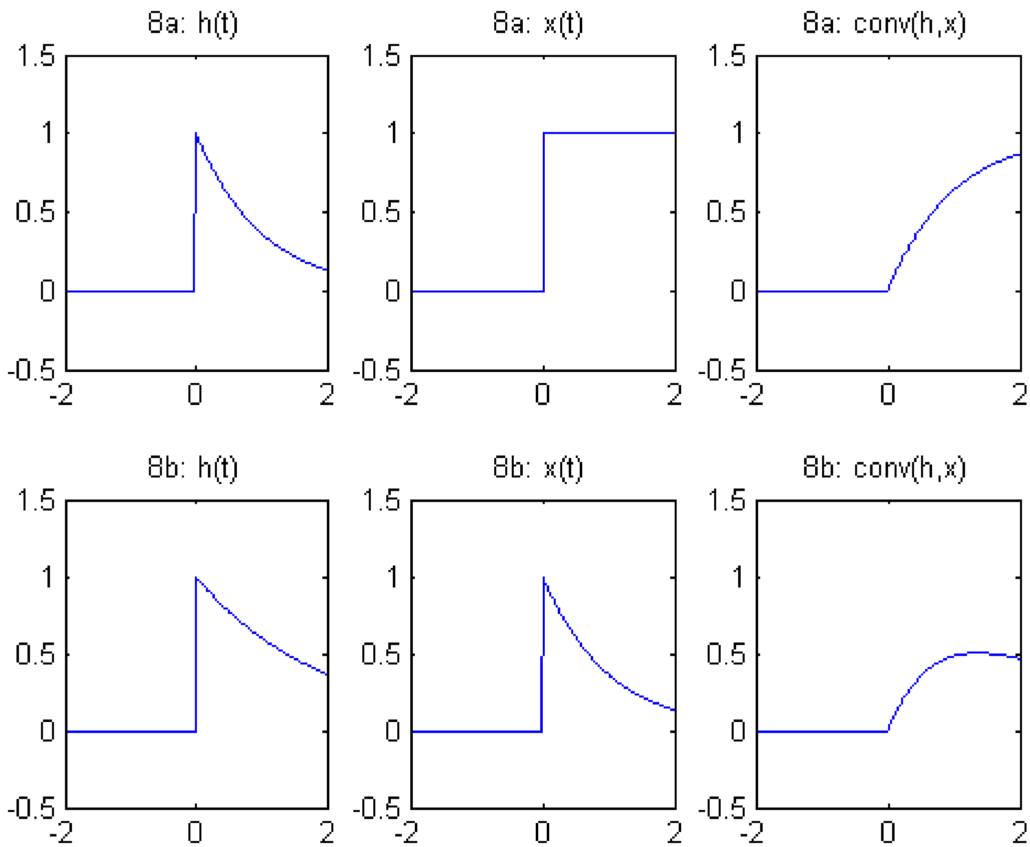

Figure 9: Problem 14.