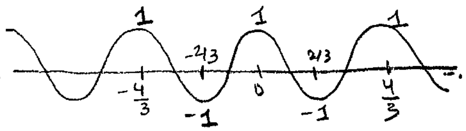
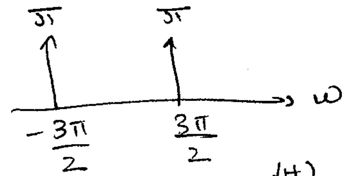
(1)

$x(t) = \cos\left(\frac{3\pi}{2} t\right)$ . . .

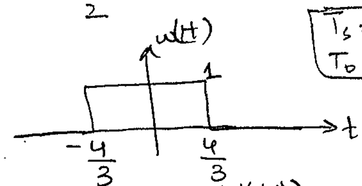


$X(j\omega) =$

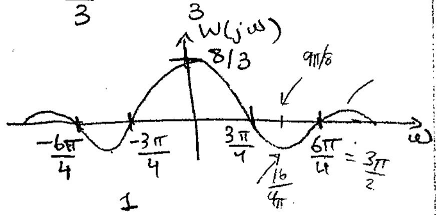


$\boxed{\begin{matrix} T_s = 1/3 \\ T_0 = 8/3 \end{matrix}}$

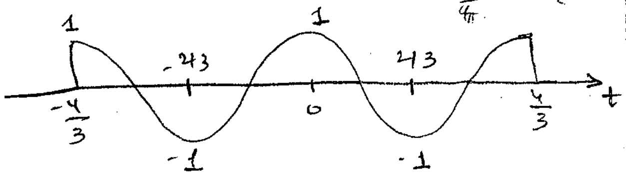
$w(t) = \Pi\left(\frac{t}{T_0}\right)$



$W(j\omega) = \dfrac{2\sin\left(\frac{4}{3}\omega\right)}{\omega}$

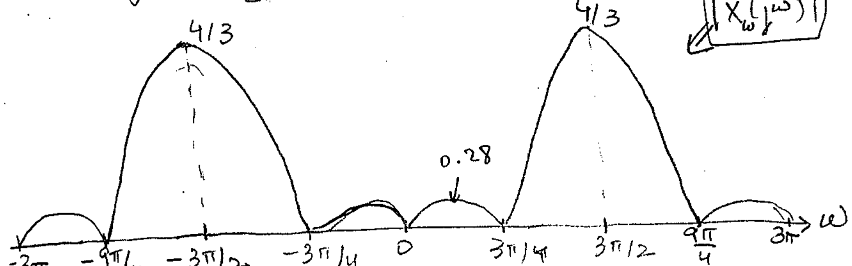


$x_\omega(t) =$



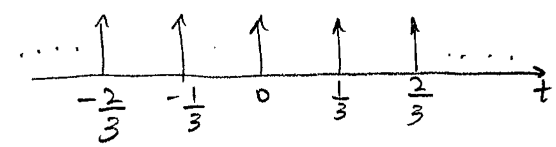
$X_\omega(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$

$\boxed{|X_\omega(j\omega)|}$

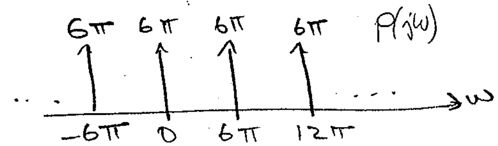


(i)

$p(t) = \sum \delta(t - nT_s) =$

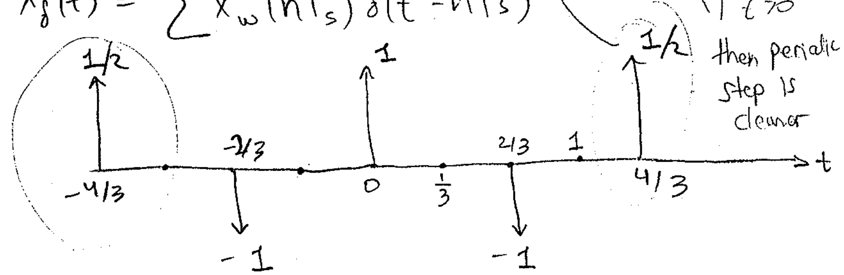


$\frac{2\pi}{1/3} \sum \delta\left(\omega - k\frac{2\pi}{1/3}\right)$

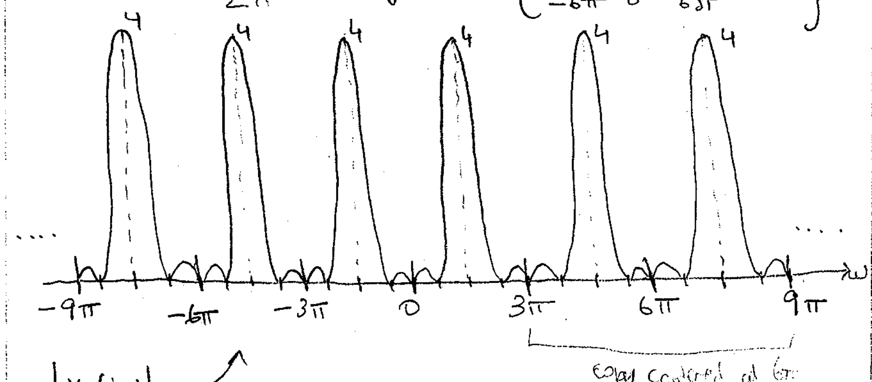


$x_\delta(t) = \sum x_\omega(nT_s)\, \delta(t - nT_s)$

if $u(t) = \begin{cases} 0 & t<0 \\ 1/2 & t=0 \\ 1 & t>0 \end{cases}$ then periodic step is cleaner



$X_\delta(j\omega) = \frac{1}{2\pi} X_w(j\omega) * \{\ldots\}$



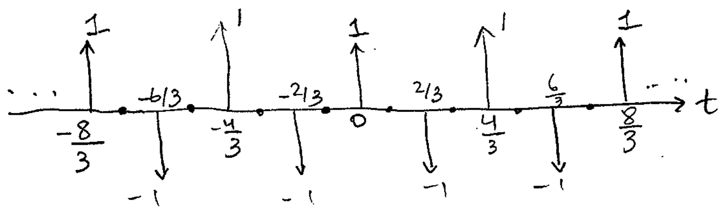
$|X_\delta(j\omega)|$

$\boxed{i}$

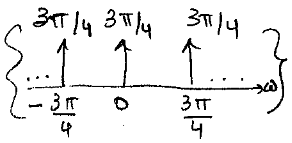
$$x'(t) = x_\delta(t) * \sum_{n=-\infty}^{n=\infty} \delta(t - n8/3)$$

$x'(nT) \Longrightarrow x[n]$ for $0 \le n < N$

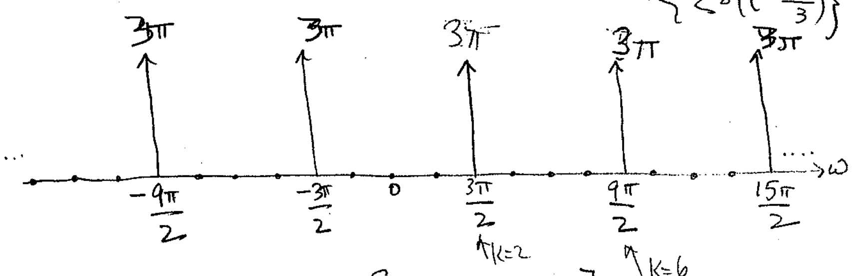
So should have area $\{x'(nT)\}$ in range $-1 \to 1$



$X'(j\omega) = X_\delta(j\omega) \cdot \{\ldots\}$; $\frac{2\pi}{8/3} = \frac{3\pi}{4}$



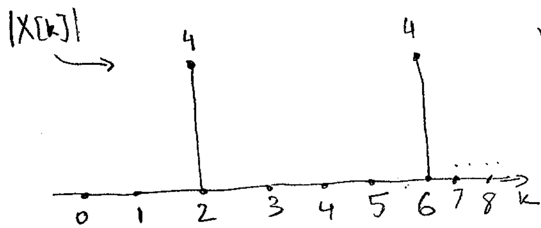
$\leftarrow F\{\sum \delta(t - n\frac{8}{3})\}$



$$X[k] = \frac{T_0}{2\pi} \text{Area}\left[X'\left(jk\frac{2\pi}{T_0}\right)\right]$$

$$X[k] = \frac{4}{3\pi} \text{Area}\left[X'\left(jk\frac{3\pi}{4}\right)\right]$$

$|X[k]|$



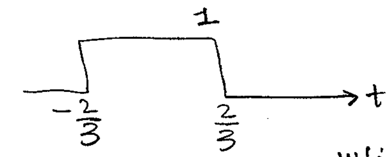
$\boxed{\measuredangle X[k] = 0}$

optional check:

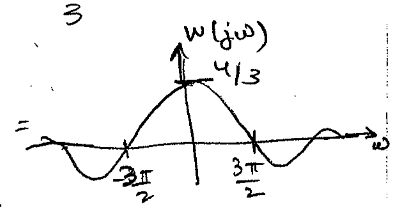
$\frac{1}{2}\sum_{n=0}^{7} e^{j\pi n(\frac{1}{2} - \frac{2k}{8})} + e^{-j\pi n(\frac{1}{2} + \frac{k}{8})}$

$= 4\delta[k-2] + 4\delta[2+k]$

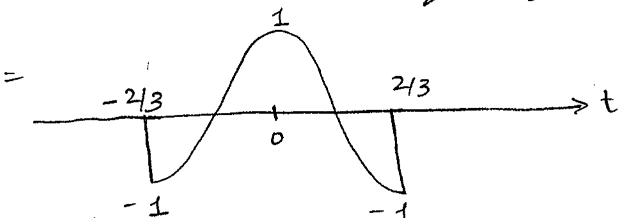
ii) $w(t) =$



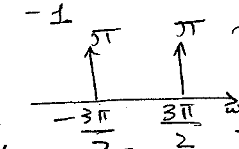
$$W(j\omega) = \frac{2\sin(\frac{2}{3}\omega)}{\omega} =$$

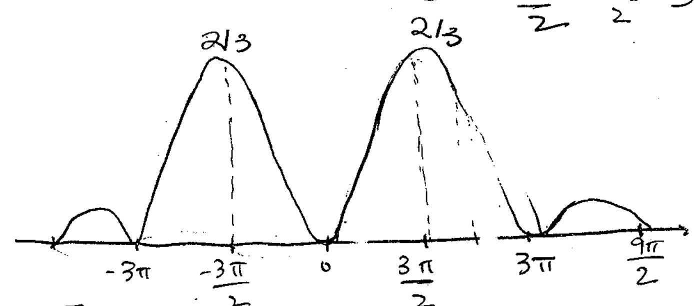


$x_w(t) =$



$X_w(j\omega) = \frac{1}{2\pi} W(j\omega) * \{\ldots\}$

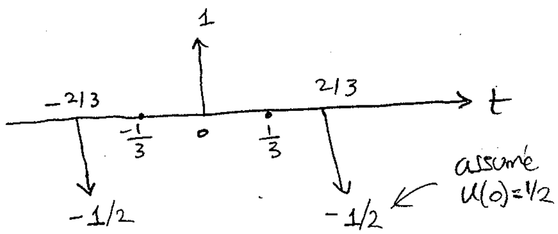


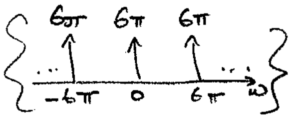


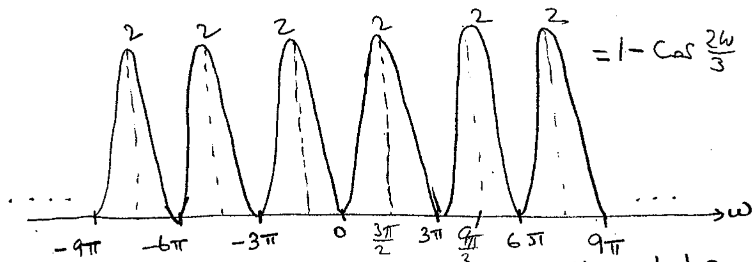
$\boxed{|X_w(j\omega)|}$

note that peak shifts a tiny bit from $\pm\frac{3\pi}{2}$, because the second lobe from each sinc overlaps the first lobe, and is subtracted appropriately

$X_\delta(t) =$



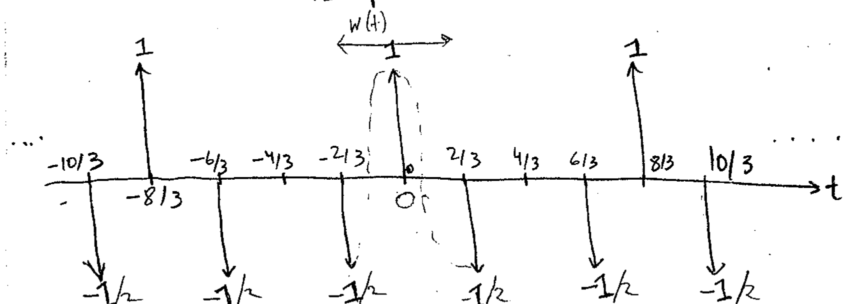
$X_\delta(j\omega) = \frac{1}{2\pi} X_w(j\omega) * \Big\{$  $\Big\}$

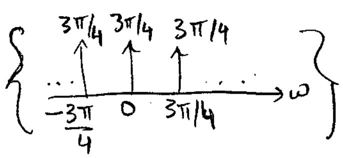


$= 1 - \cos\frac{2\omega}{3}$

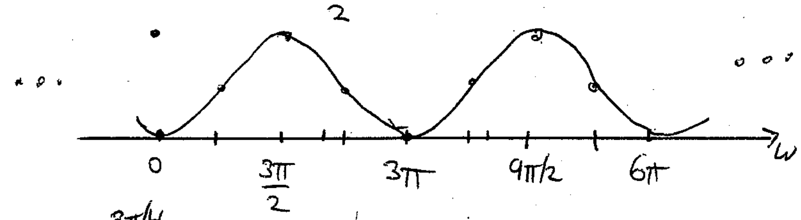
↳ note that the negative lobe contribution from neighboring sincs are smaller than the main lobe of each sinc; thus, net result is positive.

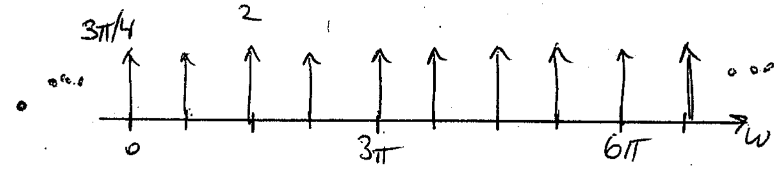
$X'(t) =$ ⇒



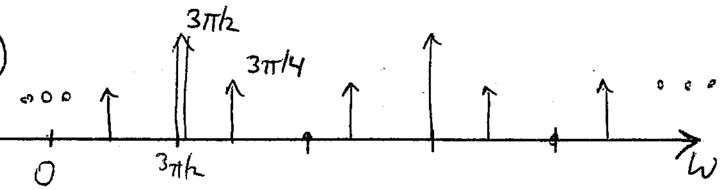
$X'(j\omega) = X_\delta(j\omega) \cdot \Big\{$  $\Big\}$

$X_\delta(j\omega)$ in range $0 < \omega < N\cdot\frac{2\pi}{T_0} = 8\cdot\frac{3\pi}{4} = 6\pi$

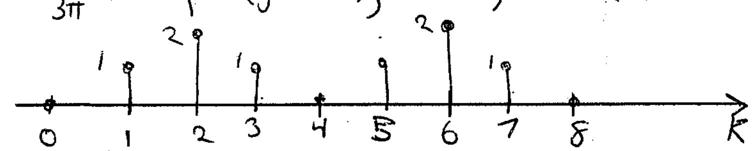


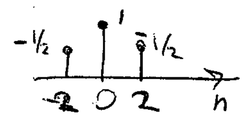


$= X'(j\omega)$



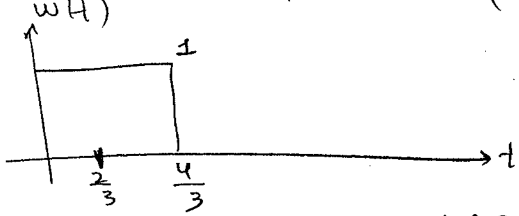
$X[k] = \frac{4}{3\pi}\,\text{Area}\{X'(jk\,3\pi/4)\}$, $\angle X[k] = 0$



Check: $x[n] =$  , $X[k] = \sum_{n=0}^{7} x[n] e^{-j2\pi nk/8}$

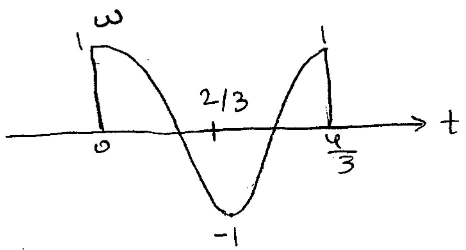
$X[k] = -\frac{1}{2}\left(e^{+j\pi k/2} + e^{-j\pi k/2}\right) + 1 = 1 - \cos k\pi/2$ ✓

iii) $w(t) = \Pi\left(\dfrac{2(t - T_0/4)}{T_0}\right) = \Pi\left(\dfrac{2(t-2/3)}{8/3}\right)$



$$W(j\omega) = \frac{2\sin\left(\frac{2}{3}\omega\right)}{\omega} e^{-j\frac{2}{3}\omega}$$

$X_w(t)$:



$$X_w(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$$

$$= \frac{1}{2\pi}\left[\pi\delta\left(\omega - \tfrac{3\pi}{2}\right) + \pi\delta\left(\omega + \tfrac{3\pi}{2}\right)\right] * W(j\omega)$$

$$= \frac{1}{2}\left[W\left(j\left(\omega - \tfrac{3\pi}{2}\right)\right) + W\left(j\left(\omega + \tfrac{3\pi}{2}\right)\right)\right]$$

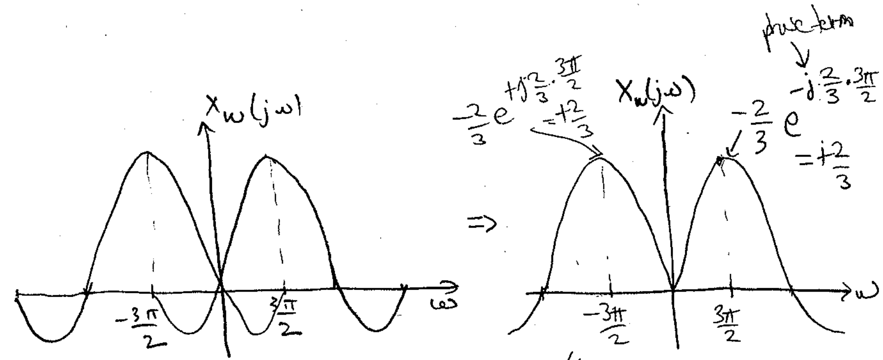
$$e^{-j\frac{2}{3}\left(\omega - \frac{3\pi}{2}\right)} = e^{-j\frac{2}{3}\omega} e^{j\pi} = -e^{-j\frac{2}{3}\omega}$$

$$\therefore X_w(j\omega) = -\frac{e^{-j\frac{2}{3}\omega}}{2}\left[\frac{2\sin\left(\frac{2}{3}\left(\omega - \frac{3\pi}{2}\right)\right)}{\omega - \frac{3\pi}{2}} + \frac{2\sin\left(\frac{2}{3}\left(\omega + \frac{3\pi}{2}\right)\right)}{\omega + \frac{3\pi}{2}}\right]$$

$\measuredangle X_w(j\omega) = -\frac{2\omega}{3} + \{-\pi, 0, \text{or } \pi\}$

both real, so phase will be 0, or $\pm\pi$

magnitude, $|X_w(j\omega)|$ is same as in b), so this next plot shows $X_w(j\omega)$ with phase annotated on the plot.



note that subtracting 1st lobe from main lobe produces net (+) regions. Also, $X_w(j\omega)$ is scaled by $\measuredangle X_w(j\omega)$.

$|X_\delta(j\omega)|$ is same as in b), but the phase from $X_w(j\omega)$ carries over.

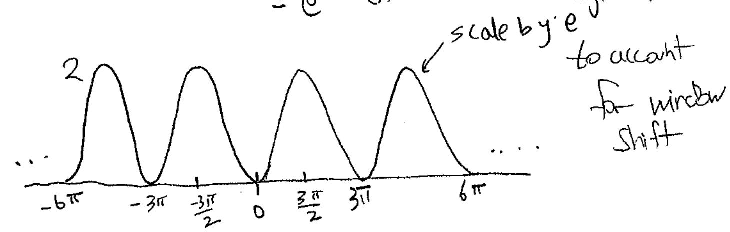
$$X_\delta(j\omega) = X_w(j\omega) * \frac{1}{2\pi}\cdot\frac{2\pi}{T_s}\sum_k \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \frac{1}{T_s}\sum_k \underbrace{X_w\left(j\left(\omega - k\,6\pi\right)\right)}$$

$\frac{1}{T_s} = 3$

note that $e^{-j\left(\frac{2}{3}(\omega - 6k\pi)\right)} = e^{-j\left(\frac{2\omega}{3}\right)}$

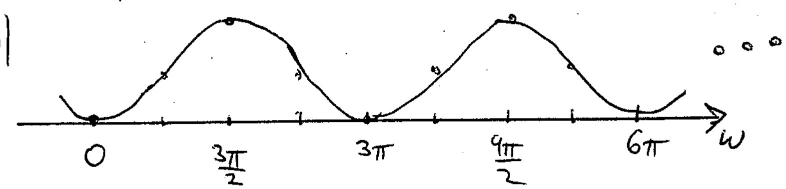
$\therefore X_\delta(j\omega) \Longrightarrow$



1(c ii) cont

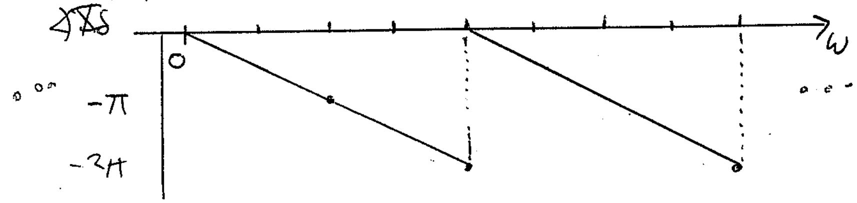
magnitude and phase of $X_s(j\omega)$.

$|X_s(j\omega)|$

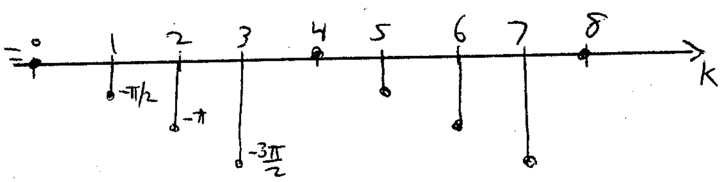


$$\angle X_s(j\omega) = \angle e^{-j(2\omega/3)} = -2\omega/3$$

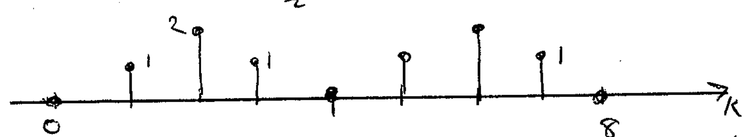
Wrapping phase at 2π:



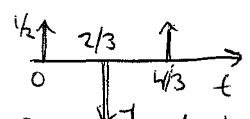
$\angle X[k] =$



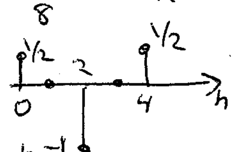
$|X[k]| =$



Check: $x_s(t) =$



$\Longrightarrow x[n] =$



$$X[k] = x[0]e^{-j\frac{2\pi k\cdot 0}{8}} + x[2]e^{-j\frac{2\pi k\cdot 2}{8}} + x[4]e^{-j\frac{2\pi k 4}{8}}$$

$$= e^{-j\frac{2\pi k\cdot 2}{8}}\left[\frac{1}{2}e^{+j2\pi k2/8} - 1\cdot e^{-j2\pi k\cdot 0/8} + \frac{1}{2}e^{-j2\pi k2/8}\right]$$

$$= e^{-j\pi k/2}\cdot\left(\cos\frac{k\pi}{2} - 1\right) \checkmark$$

② To make our plots in (a) more concrete, let's solve part (b) first.

b) Note that $x[n]$ real & even $\Rightarrow X[k]$ real (& even)

proof:

$$X^*[k] = \sum_{n=0}^{N-1} x^*[n]\, e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{j\frac{2\pi}{N}kn} \quad \leftarrow x[n] \text{ real}$$

$$= \sum_{m=-N+1}^{0} x[-m]\, e^{-j\frac{2\pi}{N}km} \quad \leftarrow m=-n$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \leftarrow x[n] \text{ even, periodic}$$

$$= X[k].$$

So, in order for $X[k]$ to be real, we select τ to make $x[n]$ even.

Let $\tilde{\omega}_0 \triangleq \frac{\omega_0 T_0}{N}$ and $\Delta = \frac{\tau N}{T_0}$.

$$x[n] = \cos(\tilde{\omega}_0(n-\Delta)) = \cos(\tilde{\omega}_0 n - \tilde{\omega}_0\Delta)$$

$$x[-n] = x[N-n] \quad \leftarrow \text{periodicity of DFT}$$

$$= \cos(\tilde{\omega}_0(N-n-\Delta))$$

$$= \cos(-\tilde{\omega}_0 n - \tilde{\omega}_0\Delta + \tilde{\omega}_0 N)$$

$$= \cos(\tilde{\omega}_0 n + \tilde{\omega}_0\Delta - \tilde{\omega}_0 N)$$

So $x[n] = x[-n] \Rightarrow \tilde{\omega}_0\Delta - \tilde{\omega}_0 N = -\tilde{\omega}_0\Delta$

$$2\tilde{\omega}_0\Delta = \tilde{\omega}_0 N$$

$$\Delta = \frac{N}{2}$$

$$\tau = \frac{T_0\Delta}{N} = \frac{T_0}{2} = \frac{1}{2}$$

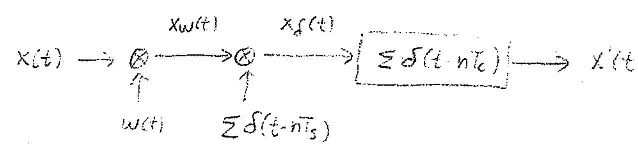
alternative:

Choose to shift by N/2

$$x[n-\tfrac{N}{2}] \xrightarrow{DFT} e^{-j\pi k} X[k] = (-1)^k X[k].$$

Shifting by $\frac{T_0}{2} = 180°$ = sign reversal, but real.

a) Now let's consider the DFT of $x[n]$ by considering figure 1:



② a) (cont'd)

$x(t) = \cos(\omega_0 (t - \tfrac{1}{2}))$ $\longleftrightarrow$ $X(j\omega)$

$w(t) = \Pi(t - \tfrac{1}{2})$ $\longleftrightarrow$ $W(j\omega) = \frac{\sin \omega/2}{\omega/2} \cdot e^{-j\omega/2}$

$x_w(t) = x(t)\, w(t)$ $\longleftrightarrow$ $X_w(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$

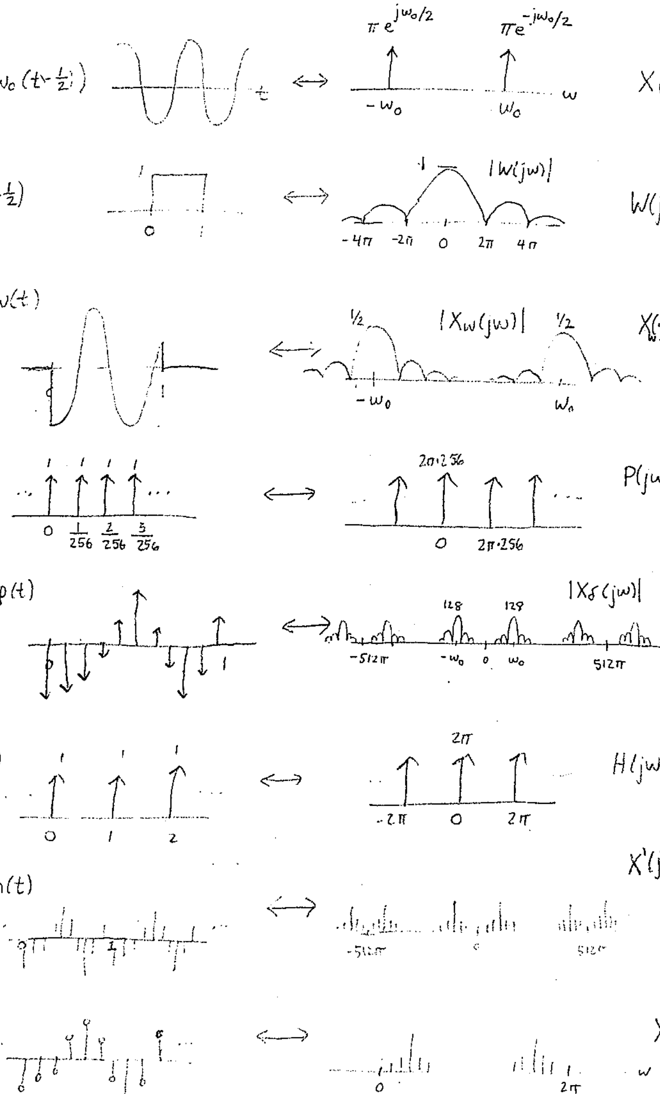
$p(t) = \sum \delta(t - nT_s)$ $\longleftrightarrow$ $P(j\omega)$

$x_\delta(t) = x_w(t) \cdot p(t)$ $\longleftrightarrow$ $X_\delta(j\omega) = \frac{1}{2\pi} X_w(j\omega) * P(j\omega)$

$h(t) = \sum \delta(t - nT_0)$ $\longleftrightarrow$ $H(j\omega)$

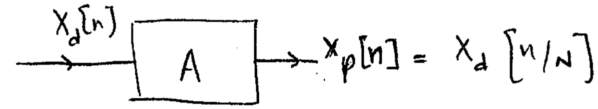
$x'(t) = x_\delta(t) * h(t)$ $\longleftrightarrow$ $X'(j\omega) = X_\delta(j\omega)\, H(j\omega)$

$x[n]$ $\longleftrightarrow$ $X[K]$



Comparing to $X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$, we see that $X[K]$ has residual energy around $\pm\tilde{\omega}_0$ (not just at a single frequency). This comes about because our window does not capture an integer number of cycles. Also, $x[n]$ is sampled from a delayed version of $x(t)$, which contributes a phase shift

③

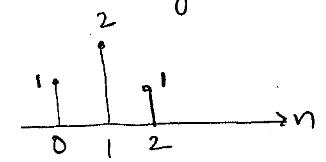


a) Through inspection we note that:

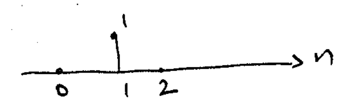
$$A\{a\, x_d[n]\} = a\, A\{x_d[n]\} = a\, x_d[n/N]$$

i.e., merely scaling the input causes the $x_p[n]$ to scale by that same amount.

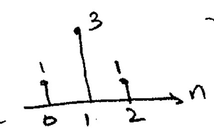
Let, $x_{d1}[n] =$



Let, $x_{d2}[n] =$

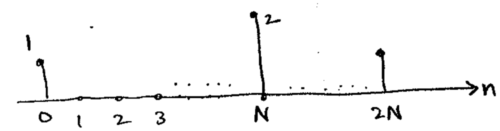


then, $A\{x_{d1}[n] + x_{d2}[n]\} = A\{$

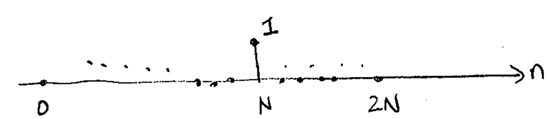


$\}$

$A\{x_{d1}[n]\} =$



$A\{x_{d2}[n]\} =$



Summarizing all of the above,

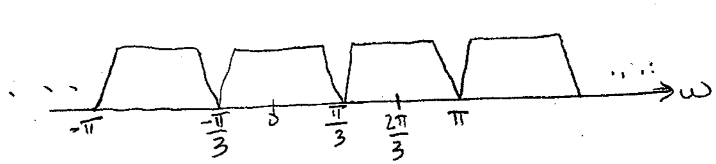
$$A\{a\, x_{d1}[n] + b\, x_{d2}[n]\} = a\, A\{x_{d1}[n]\} + b\, A\{x_{d2}[n]\}$$

Hence, A is a linear operator.

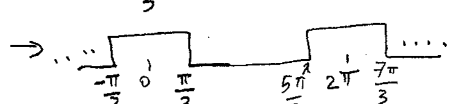
b) A is not time-invariant. Assume input is shifted by 1 unit of time. Through inspection we note that this shift of input results in a shift by N indices of time on the output. A shift of n_0 time units in $x_p[n]$ produces a shift of Nn_0 in $x_d[n]$

c)
$$X_p(e^{j\omega}) = \sum_{n=-\infty}^{n=\infty} x_p[n]\, e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{n=\infty} X_d[n/N]\, e^{-j\omega n}$$
$$= \sum_{\tilde{n}=-\infty}^{\tilde{n}=\infty} X_d[\tilde{n}]\, e^{-j(\omega N)\tilde{n}}$$
$$= X_d(e^{j3\omega})$$

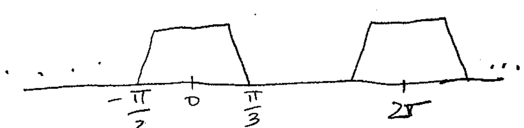
$X_p(e^{j\omega})$



d) Use $H(e^{j\omega}) \rightarrow$



$\therefore\ X(e^{j\omega}) =$



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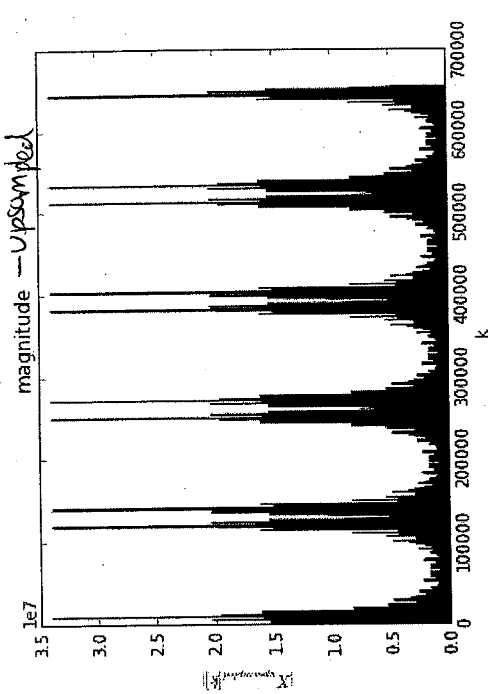
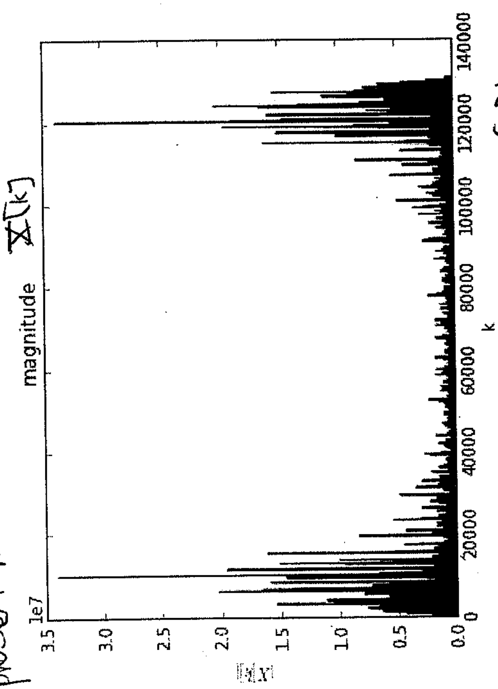
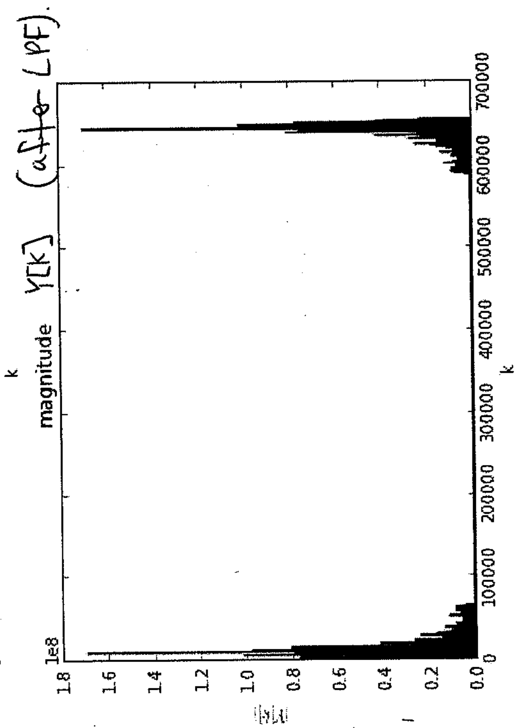
Problem 4.



Problem 4.





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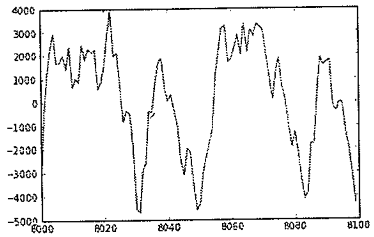
```
In [35]: # plot data
         sound = np.zeros(length)
         for i in range(0,length):
             sound[i] = data1[i][0] # copy left channel
         # Choose N
         sound = np.concatenate( (sound, np.zeros(2**17 - length)))
         _ = plt.plot(range(8000,8100), sound[8000:8100])
```

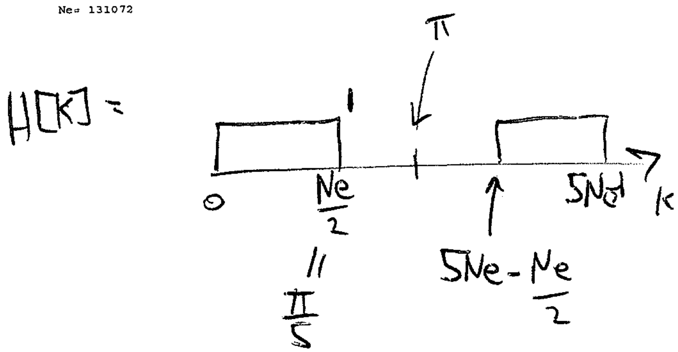


Add necessary upsampling steps to cells below

```
In [36]: # calculate FFT for a segment to see filter before downsample
         Ne = np.size(sound)
         print 'Ne=', Ne
         X = np.fft.fft(sound)  # calculate X[k]
         n1 = np.linspace(0,Ne-1, Ne)

         x_upsampled = np.zeros((Ne*5))
         x_upsampled[np.arange(0,Ne*5,5)]=sound
         X_upsampled = np.fft.fft(x_upsampled)
         H = np.zeros((Ne*5))
         H[np.arange(0,Ne/2+1)] = 5 # passes frequencies below pi/5, also compensates for energy loss
         H[np.arange(-Ne/2+5*Ne,5*Ne)]=5 # don't forget the negative frequencies
         Y = np.multiply(X_upsampled,H)

         Ne= 131072
```



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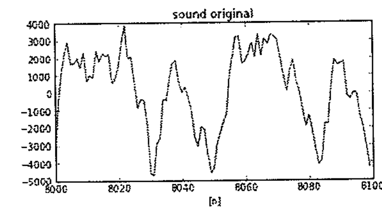
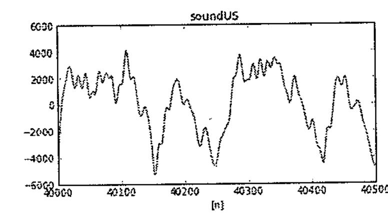
```
In [40]: # upsample
         soundUS = np.zeros((Ne*5,2))# setup array for stereo
         print 'size soundUS =', np.size(soundUS)

         y = np.fft.ifft(Y)
         for i in range(0,Ne*5):
             soundUS[i,0] = np.real(y[i])
             soundUS[i,1] = np.real(y[i])

         setup_graph(title='sound original', x_label='[n]', fig_size=(6,3))
         _ = plt.plot(range(8000,8100), sound[8000:8100])

         setup_graph(title='soundUS', x_label='[n]', fig_size=(6,3))
         _ = plt.plot(range(40000,40500), soundUS.real[40000:40500])

         # now write  data file
         wavfile.write('music-up.wav', rate1, soundUS.astype(int16)) # 16 bit integer

         size soundUS = 1310720
```





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