Step Response of Second Order Systems

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2} \]

\( \zeta: \) damping ratio, \( \omega_n: \) natural frequency

Poles: \( s_{1,2} = -\omega_n\cos\theta \pm j\omega_n\sin\theta \) where \( \cos\theta = \zeta \)

Below are the step responses for various values of \( \zeta \). Note that \( \omega_n \) changes only the time scale, not the shape of the response.

Important Features of the Step Response:

1) Rise time (\( tr \)): time to go from 10% to 90% of steady-state value
2) Peak overshoot (\( Mp \)): (peak value - steady state)/steady state
3) Peaking time (\( tp \)): time to peak overshoot
4) Settling time (\( ts \)): time after which the step response stays within 1% of the steady-state value
How do these parameters depend on $\zeta$ and $\omega_n$?

$$u(t) : \text{unit step} \xrightarrow{\ell} \frac{1}{s}$$

Step response:

$$Y(s) = \frac{1}{s} H(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

$$= \frac{A}{s} + \frac{B}{s + \sigma + j\omega_d} + \frac{B^*}{s + \sigma - j\omega_d}$$

$$A = 1 \quad B = -\frac{1}{2} \left(1 + j\frac{\sigma}{\omega_d}\right)$$

$$y(t) = \left(1 + Be^{-\sigma t}e^{-j\omega_d t} + B^*e^{-\sigma t}e^{j\omega_d t}\right) u(t)$$

$$= \left(1 + \left(Be^{-j\omega_d t} + B^*e^{j\omega_d t}\right)e^{-\sigma t}\right) u(t)$$

$$= 2\Re\{Be^{-j\omega_d t}\}$$

$$= -\frac{1}{2}\left(1 + j\frac{\sigma}{\omega_d}\right)(\cos\omega_dt - j\sin\omega_dt)$$

$$= -\left(\cos\omega_dt + \frac{\sigma}{\omega_d}\sin\omega_dt\right)$$

$$y(t) = \left[1 - \left(\cos\omega_dt + \frac{\sigma}{\omega_d}\sin\omega_dt\right)e^{-\sigma t}\right] u(t)$$

Peaking time:

$$\frac{dy(t)}{dt} = \sigma e^{-\sigma t}\left(\cos\omega_dt + \frac{\sigma}{\omega_d}\sin\omega_dt\right) - e^{-\sigma t}\left(-\omega_d\sin\omega_dt + \sigma\cos\omega_dt\right)$$

$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d^2} + \omega_d\right)\sin\omega_dt$$

$$\frac{dy(t)}{dt} = 0 \implies \sin\omega_dt = 0 \quad t_p = \frac{\pi}{\omega_d}$$

Peak overshoot: $M_p = y(t_p) - 1$

$$y(t_p) = \left(1 - \cos\omega_dt_p e^{-\sigma t_p}\right) = 1 + e^{-\sigma t_p} = 1 + e^{-\sigma \frac{\pi}{\omega_d}}$$

$$M_p = e^{-\pi \frac{\sigma}{\omega_d}} = e^{-\pi \frac{\zeta}{\sqrt{1-\zeta}}}$$

$$\zeta \nearrow \implies M_p \searrow \quad M_p \to 0 \text{ as } \zeta \to 1 \quad M_p \approx \begin{cases} 0.05 & \zeta = 0.7 \\ 0.16 & \zeta = 0.5 \end{cases}$$

Approximate expressions for rise time and settling time:

$$t_s \approx \frac{4.6}{\sigma}$$

(obtained from $e^{-\sigma t_s} = 0.01$)

$$t_r \approx \frac{1.8}{\omega_n}$$

for $\zeta = 0.5$ (changes little with $\zeta$)
Note that $t_p$, $t_s$, $t_r$ are inversely proportional to $\omega_n$:

$$t_p = \frac{\pi}{\omega_n} = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}} \quad t_s \approx \frac{4.6}{\sigma} \frac{4.6}{\omega_n \xi} \quad t_r \approx \frac{1.8}{\omega_n}.$$  

This is consistent with our observation on page 1 that $\omega_n$ changes only the time scale, not the shape of the response. We make this property explicit in the following statement:

If $\zeta$ is kept constant and $\omega_n$ is scaled by a factor of $\alpha > 0$ ($\omega_n \rightarrow \alpha \omega_n$) then the step response is scaled in time by $\alpha$: $y(t) \rightarrow y(\alpha t)$.

**Proof:** If we replace $\omega_n$ with $\alpha \omega_n$ in (1), we get

$$\frac{(\alpha \omega_n)^2}{s (s^2 + 2 \zeta (\alpha \omega_n) s + (\alpha \omega_n)^2)} = \frac{\omega_n^2}{s \left( \left( \frac{s}{\alpha} \right)^2 + 2 \zeta \omega_n \left( \frac{s}{\alpha} \right) + \omega_n^2 \right)} = \frac{1}{\alpha} Y \left( \frac{s}{\alpha} \right).$$

The statement above then follows from the scaling property of Laplace transform:

$$y(\alpha t) \leftrightarrow \frac{1}{\alpha} Y \left( \frac{s}{\alpha} \right).$$

**Summary:** $\omega_n \nearrow$ increases speed of the response  
$\zeta \nearrow$ reduces overshoot

Although the formulas above are for second order systems, they can be applied as approximate expressions to higher order systems with two dominant poles:
The Unilateral Laplace Transform

\[ \mathcal{X}(s) = \int_{0}^{\infty} x(t)e^{-st}dt \]  

(2)

Identical to the bilateral Laplace transform if \( x(t) = 0 \) for \( t < 0 \).

Example: \( x(t) = e^{-a(t+1)}u(t+1) \)

\[ X(s) = \frac{e^s}{s+a}, \quad \text{Re}\{s\} > -a \]

\[ \mathcal{X}(s) = \frac{e^{-a}}{s+a}, \quad \text{Re}\{s\} > -a \]

Properties of the Unilateral Laplace Transform

Most properties of the bilateral Laplace transform also hold for the unilateral Laplace transform.

Exceptions:

Convolution:

\( (x_1 \ast x_2)(t) \leftrightarrow \mathcal{X}_1(s)\mathcal{X}_2(s) \quad \text{if} \quad x_1(t) = x_2(t) = 0 \quad \text{for all} \quad t < 0 \)

This follows from the convolution property of the bilateral Laplace transform which coincides with the unilateral transform because \( x_1(t) = x_2(t) = 0, \quad t < 0 \).

Differentiation in Time:

\[ \frac{dx(t)}{dt} \leftrightarrow s\mathcal{X}(s) - x(0^-) \]

Repeated application gives:

\[ \frac{d^2x(t)}{dt^2} = \frac{d}{dt} \left\{ \frac{dx(t)}{dt} \right\} \leftrightarrow s\mathcal{X}(s) - x(0^-) - \frac{dx}{dt}(0^-) \]

\[ = s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-) \]

\[ \frac{d^3x(t)}{dt^3} = \frac{d}{dt} \left\{ \frac{d^2x(t)}{dt^2} \right\} \leftrightarrow s \left( s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-) \right) - \frac{d^2x}{dt^2}(0^-) \]

\[ = s^3\mathcal{X}(s) - s^2x(0^-) - s\frac{dx}{dt}(0^-) - \frac{d^2x}{dt^2}(0^-) \]
Solving differential equations with the unilateral Laplace transform

Example:

\[ \frac{d^2y(t)}{dt^2} + 3 \frac{dy}{dt} + 2y(t) = e^t \quad t \geq 0 \]  

(3)

Initial condition \( y(0^-) = a, \frac{dy}{dt}(0^-) = b \).

\[ (s^2Y(s) - as - b) + 3(sY(s) - a) + 2Y(s) = \frac{1}{s-1} \]

\[ (s^2 + 3s + 2)Y(s) = as + b + 3a + \frac{1}{s-1} = \frac{as^2 + (b+2a)s + (1-b-3a)}{s-1} \]

\[ Y(s) = \frac{as^2 + (b+2a)s + (1-b-3a)}{(s+1)(s+2)(s-1)} \]

Partial fraction expansion:

\[ Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{B}{s-1} \]

\[ = \frac{(A_1 + A_2 + B)s^2 + (A_1 + 3B)s + (2B - 2A_1 - A_2)}{(s+1)(s+2)(s-1)} \]

Match coefficients:

\[ \begin{align*}
A_1 + A_2 + B &= a \\
A_1 + 3B &= b + 2a \\
2B - 2A_1 - A_2 &= 1 - b - 3b
\end{align*} \]

\[ \begin{align*}
B &= 1/6 \\
A_1 &= -\frac{1}{2} + 2a + b \\
A_2 &= \frac{1}{3} - a - b
\end{align*} \]

Then,

\[ y(t) = \frac{1}{6}e^t + \left(-\frac{1}{2} + 2a + b\right)e^{-t} + \left(\frac{1}{3} - a - b\right)e^{-2t} \quad t \geq 0. \]

Compare this to the standard method for solving linear constant coefficient differential equations:

The first term in \( y(t) \) above is the particular solution. If we substitute \( y_p(t) = \frac{1}{6}e^t \) in (3):

\[ \frac{d^2y_p(t)}{dt^2} + 3 \frac{dy_p}{dt} + 2y_p(t) = e^t. \]

The second and third terms constitute the homogeneous solution. If we substitute \( y_h(t) = A_1e^{-t} + A_2e^{-2t} \):

\[ \frac{d^2y_h(t)}{dt^2} + 3 \frac{dy_h}{dt} + 2y_h(t) = 0. \]

Thus, \( y(t) = y_p(t) + y_h(t) \) and \( A_1 \) and \( A_2 \) are selected to satisfy the initial conditions.