**Feedback Control**

\[ r(t) : \text{reference signal to be tracked by } y(t) \]

\( H_c(s) : \text{controller; } H_p(s) : \text{system to be controlled ("plant")} \)

Closed-loop transfer function:

\[
H(s) = \frac{Y(s)}{R(s)} = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}
\]

**Constant-gain control: \( H_c(s) = K \)**

\[
H(s) = \frac{KH_p(s)}{1 + KH_p(s)}
\]

Closed-loop poles: roots of \( 1 + KH_p(s) = 0 \)

**Example 1 (Speed Control)**

\[ H_p(s) = \frac{1}{Ms} \rightarrow \text{open-loop pole: } s = 0 \]

Closed-loop pole: \( 1 + K \frac{1}{Ms} = 0 \Rightarrow s = -\frac{K}{M} \)

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Example 2 (Position Control) \( y(t) \) : position

\[
M \frac{d^2 y}{dt^2} + b \frac{dy}{dt} = x(t) \quad H_p(s) = \frac{1}{Ms^2 + bs} = \frac{1}{s(Ms + b)}
\]

Open-loop poles: \( s = 0, -\frac{b}{M} \)

Closed-loop poles:

\[
1 + \frac{K}{s(Ms + b)} = 0 \quad \Rightarrow \quad Ms^2 + bs + K = 0
\]

\[
s = -\frac{b \pm \sqrt{b^2 - 4KM}}{2M}
\]

Root-Locus Analysis

How do the roots of

\[
1 + KH(s) = 0
\]

move as \( K \) is increased from \( K = 0 \) to \( K = +\infty \)?

If a point \( s_0 \in \mathbb{C} \) is on the root locus, then \( H(s_0) = \frac{1}{K} \) for some \( K > 0 \), therefore \( \angle H(s_0) = \pi \). The rules for sketching the root locus below are derived from this property.

Rules for sketching the root locus:

Let

\[
H(s) = \frac{s^m + b_{m-1}s^{m-1} + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0} \quad m \leq n
\]

\[
= \frac{\prod_{k=1}^{m}(s - \beta_k)}{\prod_{k=1}^{n}(s - \alpha_k)} \quad \beta_k : \text{zeros } k = 1, ..., m \quad \alpha_k : \text{poles } k = 1, ..., n
\]

1) As \( K \to 0 \), the roots converge to the poles of \( H(s) \):

\[
H(s) = -\frac{1}{K} \to \infty
\]

Since there are \( n \) poles, the root locus has \( n \) branches, each starting at a pole of \( H(s) \).
2) As $K \to \infty$, $m$ branches approach the zeros of $H(s)$. If $m < n$, then $n - m$ branches approach infinity following asymptotes centered at:

$$\frac{\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k}{n - m}$$

with angles:

$$\frac{180^\circ + (i-1)360^\circ}{n - m} \quad i = 1, 2, ..., n - m.$$

Example 2 above: $n - m = 2$, poles: $0, -b/M$ with center $= \frac{-b}{2M}$, and angles $= 90^\circ, -90^\circ$

3) Parts of the real line that lie to the left of an odd number of real poles and zeros of $H(s)$ are on the root locus.

Example 1 above: Example 2:

Proof of Property 3:

$$\angle H(s_0) = \sum_{k=1}^{m} \angle (s_0 - \beta_k) - \sum_{k=1}^{n} \angle (s_0 - \alpha_k)$$

If $s_0$ is on the real line:

$$\angle (s_0 - a) = \begin{cases} 
\pi & \text{if } s_0 < a \\
0 & \text{if } s_0 > a
\end{cases}$$

Therefore,

$$\angle H(s_0) = r \pi \quad r : \text{total # of poles and zeros to the right of } s_0$$

$$\quad = \pi \quad \text{if } r \text{ is odd.}$$

4) Branches between two real poles must break away into the complex plane for some $K > 0$. The break-away and break-in points can be determined by solving for the roots of

$$\frac{dH(s)}{ds} = 0$$

that lie on the real line.

Example 2 above:

$$H(s) = \frac{1}{Ms^2 + bs}$$
\[ \frac{dH}{ds} = \frac{-2Ms - b}{(Ms^2 + bs)^2} = 0 \Rightarrow s = \frac{-b}{2M} \]

**Example 3:**

\[ H(s) = \frac{s - 1}{(s + 1)(s + 2)} \]

For \( n = 2 \), \( m = 1 \), zeros: \( s = 1 \), poles: \( s = 1, -2 \).

One asymptote with angle \( 180^\circ \).

**Example 4:**

\[ H(s) = \frac{s + 2}{s(s + 1)} \]

\( n - m = 1 \) asymptote with angle \( 180^\circ \)

**Break-away/ break-in points:**

\[ \frac{dH}{ds} = \frac{s^2 + s - (2s + 1)(s + 2)}{s^2(s + 1)^2} = 0 \]

\[ s^2 + s - (2s^2 + 5s + 2) = 0 \]

\[ s^2 + 4s + 2 = 0 \Rightarrow s = \frac{-4 \pm \sqrt{8}}{2} = -2 \pm \sqrt{2} \]

**Example 5:**

\[ H(s) = \frac{s + 2}{s(s + 1)(s + a)} \]

For \( a > 2 \)

Pole at \( -a \) added to the previous example.

\( n - m = 2 \), therefore two asymptotes with angles \( \mp 90^\circ \)

Center of the asymptotes: \( \frac{(0-1-a)-(-2)}{2} = \frac{1-a}{2} \)
For large enough \( a \), \( \frac{dH(s)}{ds} = 0 \) has three real, negative roots:

**High-Gain Instability:**
Large feedback gain causes instability if:
1) \( H(s) \) has zeros in the right-half plane
2) \( n - m \geq 3 \)

\( n - m = 2 \)
stable but poorly damped as \( K \)

\( n - m = 3 \)

\( n - m = 4 \)

\( n - m = 5 \)
Control Design by Root Locus

Example:

\[ M \frac{d^2y}{dt^2} + b \frac{dy}{dt} = x(t) \rightarrow H_p(s) = \frac{1}{Ms^2 + bs} \]

Suppose a damping ratio of \( \zeta = 0.7 \) is desired:

\[ \text{select the gain } K \text{ that corresponds to this point on the root locus} \]

Suppose, in addition to \( \zeta \), a lower bound on \( \omega_n \) is specified:

The root locus doesn’t go through the desired region, therefore constant gain control won’t work. Try the controller:

\[ H_c(s) = K \frac{s - \beta}{s - \alpha} \quad \alpha < \beta < 0 \quad \text{(pole to the left of zero)} \]

Closed-loop poles:

\[ 1 + K \frac{s - \beta}{s - \alpha \ s(Ms + b)} \frac{1}{H(s)} = 0 \]

Select \( \alpha, \beta \) such that the root locus passes through the desired region.
A controller of the form

\[ H_c(s) = K \frac{s - \beta}{s - \alpha} \quad \alpha < \beta < 0 \]

is called a "lead controller".

**Example:**

\[ H_c(s) = \frac{s + 1}{s + 10} \]

\[ H_c(j\omega) = \frac{1}{10} \frac{1 + j\omega}{1 + j\omega/10} \]

\[ 20 \log_{10}|H_c(j\omega)| = -20 - 20 \log_{10}|1 + j\omega/10| + 20 \log_{10}|1 + j\omega| \]