Response of LTI Systems to Complex Exponentials

Complex Exponentials

Continuous-time:

\[ x(t) = e^{st}, \quad s \in \mathbb{C} \quad \xrightarrow{s = \sigma + j\omega} \quad x(t) = e^{\sigma t} e^{j\omega t} \]  

Envelope periodic

Discrete-time:

\[ x[n] = z^n, \quad z \in \mathbb{C} \quad \xrightarrow{z = re^{j\omega}} \quad x[n] = r^n e^{j\omega n} \]  

Figures 1 and 2 on page 5 plot \( e^{st} \) and \( z^n \) for various values of \( s \) and \( z \) in the complex plane.

The response of a LTI system to a complex exponential is the same complex exponential scaled by a constant.

\[
\begin{align*}
e^{st} & \rightarrow h(t) \rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)e^{st} \quad \triangleq H(s) \\
z^n & \rightarrow h[n] \rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)z^n \quad \triangleq H(z)
\end{align*}
\]

\( H(s) \) and \( H(z) \) are called "transfer functions" or "system functions."

Example: Find the transfer function \( H(s) \) for \( y(t) = x(t-3) \).

If \( x(t) = e^{st} \) then

\[ y(t) = x(t-3) = e^{s(t-3)} = e^{-3s}e^{st} = H(s)e^{st}. \]

Alternatively, use the impulse response \( h(t) = \delta(t-3) \):

\[ H(s) = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}. \]
Frequency Response of a LTI System

\[ x(t) = e^{j\omega t} \quad \rightarrow \quad y(t) = H(j\omega)e^{j\omega t} \]
\[ x[n] = e^{j\omega n} \quad \rightarrow \quad y[n] = H(e^{j\omega})e^{j\omega n} \]  

(7)

\[
H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \\
H(e^{j\omega}) = \sum h[k]e^{-j\omega k}
\]  

(8)

Filtering

LTI system designed such that \( H(j\omega) \) (\( H(e^{j\omega}) \) in DT) is zero or close to zero for frequencies to be eliminated.

Example: Why is the moving average system a low-pass filter?

\[
y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]
\]  

(9)

\[
H(e^{j\omega}) = \sum_{k=-M}^{M} \frac{1}{2M+1} e^{-j\omega k} = \frac{e^{j\omega M}}{2M+1} \left( \frac{1+e^{-j\omega} + \cdots + e^{-j\omega2M}}{1-e^{-j\omega}} \right)
\]

\[
= \frac{1}{2M+1} e^{j\omega M} \left( \frac{e^{-j\omega(M+\frac{1}{2})} - e^{-j\omega(M+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}} \right)
\]

\[
= \frac{1}{2M+1} e^{j\omega M} \left( \frac{\sin(\omega(M+1/2))}{\sin(\omega/2)} \right)
\]

\[
H(e^{j\omega}) = \begin{cases} 
\frac{1}{2M+1} \frac{\sin(\omega(M+1/2))}{\sin(\omega/2)} & \text{if } \omega = 0 \\
\frac{1}{\sin(\omega/2)} & \text{if } \omega \neq 0
\end{cases}
\]  

(10)

*since this is a geometric series of the form \(1 + x + \cdots + x^M\), which equals \(\frac{1-x^{M+1}}{1-x}\) when \(x \neq 1\). Substitute \(x = e^{-j\omega}\) and note that \(x \neq 1\) means \(\omega \neq 0\).
Low frequencies pass through:
\[ \omega = 0 \quad x[n] \rightarrow y[n] \]

High frequencies are attenuated:
\[ \omega = \pi \quad x[n] = e^{j\pi n} = (-1)^n \quad y[n] = (-1)^{n+M} \]

Example: Is \( y[n] = \frac{1}{2}(x[n] - x[n-1]) \) low-pass or high-pass?
\[ \omega = \pi \quad x[n] = (-1)^n \quad y[n] = (-1)^n \]

\[ \omega = 0 \quad x[n] \equiv 1 \quad y[n] \equiv 0 \]

To find \( H(e^{j\omega}) \), note that the impulse response is:
\[
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega} = \frac{1}{2} - \frac{1}{2}e^{-j\omega} = \frac{1}{2}(1 - e^{-j\omega}) = \frac{1}{2}e^{-j\omega/2}2\sin(\omega/2) \]

\( |H(e^{j\omega})| = \sin(\omega/2) \)

**FIR vs. IIR Systems**

Note that the examples above have impulse responses of finite duration. Such systems are called Finite Impulse Response (FIR) systems.
A causal finite impulse response (FIR) system has the form:

\[ y[n] = b_0x[n] + \ldots + b_Mx[n-M] \tag{12} \]

and its impulse response is:

\[ h[n] = b_0\delta[n] + b_1\delta[n-1] + \ldots + b_M\delta[n-M]. \tag{13} \]

Note that a FIR system is always stable, because the sum \(\sum_n |h[n]|\) is over a finite duration and, thus, finite.

An infinite impulse response (IIR) example:

\[ y[n] - y[n-1] = x[n], \quad y[-1] = 0, \quad x[n] = 0 \text{ for } n < 0 \text{ (accumulator)} \]

Impulse response: \( h[n] = u[n] \)

Constant-coefficient linear difference equations like those above (and differential equations in continuous time) are a rich source of LTI systems. The general form of a constant-coefficient linear difference equation is:

\[ a_0y[n] + a_1y[n-1] + \ldots + a_Ny[n-N] = b_0x[n] + \ldots b_Mx[n-M] \tag{14} \]

which is causal and LTI if \(a_0 \neq 0\) and the system is "initially at rest" (that is, \(y[n] = 0\) for \(n < n_0\), where \(n = n_0\) is the first instant when \(x[n] \neq 0\)). Recall that a linear system must return a zero output in response to a zero input, and this property is destroyed when the initial conditions are non-zero.

We usually take \(a_0 = 1\), since otherwise we can divide all coefficients by \(a_0\) and normalize the coefficient of \(y[n]\) to one. Thus, we can implement the system \(\text{(14)}\) with the recurrence relation:

\[ y[n] = -a_1y[n-1] - \ldots - a_Ny[n-N] + b_0x[n] + \ldots b_Mx[n-M]. \]

When \(a_1 = \ldots = a_N = 0\) the difference equation \(\text{(14)}\) reduces to the FIR system \(\text{(12)}\). Therefore, the source of IIR behavior is the presence of feedback terms \(-a_1y[n-1] - \ldots - a_Ny[n-N]\).
Figure 1: The real part of $e^{st}$ for various values of $s$ in the complex plane. Note that $e^{st}$ is oscillatory when $s$ has an imaginary component. It grows unbounded when $\text{Re}(s) > 0$, decays to zero when $\text{Re}(s) < 0$, and has constant amplitude when $\text{Re}(s) = 0$.

Figure 2: The real part of $z^n$ for various values of $z$ in the complex plane. It grows unbounded when $|z| > 1$, decays to zero when $|z| < 1$, and has constant amplitude when $z$ is on the unit circle ($|z| = 1$).