Continuous Time Fourier Transform

Unlike Fourier Series, the Fourier Transform is applicable to aperiodic signals. It has the form

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad (1)$$

where $\omega$ is a continuous frequency variable. To motivate this definition we treat the aperiodic signal $x$ as the limit of a periodic signal $\tilde{x}$ as period $T \to \infty$ (see example below). As $T$ increases, the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases and the harmonic components become closer in frequency, forming a continuum in the limit $T \to \infty$.

Example 1:

The definition (1) applied to the aperiodic signal $x$ gives

$$X(\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \begin{cases} 2T_1 & \omega = 0 \\ \frac{1}{j\omega}e^{-j\omega T_1}\bigg|_{-T_1}^{T_1} = \frac{2\sin(\omega T_1)}{\omega} & \omega \neq 0 \end{cases} \quad (2)$$

Now recall from Lecture 4 that $\tilde{x}$ has Fourier Series coefficients:

$$a_k = \begin{cases} \frac{2T_1}{k\omega_0 T} & k = 0 \\ \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} & k \neq 0 \end{cases} \quad (3)$$

where $\omega_0 = \frac{2\pi}{T}$. Comparing (2) and (3), we see that

$$T a_k = X(\omega)|_{\omega=k\omega_0} \quad (4)$$

which means that $X(\omega)$ is an envelope for the coefficients $T a_k$:
Thus, the Fourier Transform of $x$ emerges from the Fourier Series coefficients of $\tilde{x}$, which get densely packed as $T \to \infty$ and form the silhouette of a function, $X(\omega)$, of a continuous frequency variable $\omega$.

The square pulse example above is easy to generalize to any function $x$ of finite duration. Create periodic signal $\tilde{x}$ as above, with $T$ large enough to avoid overlaps. Then,

$$a_k = \frac{1}{T} \int x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

if we integrate over an interval encompassing the full duration of $x$. It follows that

$$T a_k = \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt = X(\omega)|_{\omega=\omega_0}$$

where the envelope $X(\omega)$ is as defined in (1).

To reconstruct $x(t)$ from its Fourier Transform $X(\omega)$, recall from the synthesis equation for Fourier Series that

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

and substitute $a_k$ from (5):

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{T} X(\omega) \right)|_{\omega=\omega_0} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} \left( X(\omega) e^{j\omega_0 t} \right)|_{\omega=\omega_0} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 \left( X(\omega) e^{j\omega_0 t} \right)|_{\omega=\omega_0}$$

The $k$th term in this summation can be pictured as the shaded bar in the figure on the right. Thus, as $T \to \infty (\omega_0 \to 0)$, the summation converges to the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega_0 t} d\omega.$$

Since $\tilde{x}$ recovers $x$ in the limit as $T \to \infty$, this expression serves as the synthesis equation to reconstruct $x(t)$. To summarize:

- $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ (Analysis Equation)
- $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ (Synthesis Equation)

**Example 2:** For $x(t) = e^{-at} u(t)$, $a > 0$,

$$X(\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = \left. \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \right|_{0}^{\infty} = \frac{1}{a+j\omega}, \quad |a+j\omega| = \sqrt{a^2 + \omega^2}, \quad \angle(a+j\omega) = \tan^{-1}(\omega/a)$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad \angle X(\omega) = -\tan^{-1}(\omega/a)$$
Example 3: Given the Fourier Transform

\[ X(\omega) = \begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| \geq \pi \end{cases} \]  

find

\[ x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega t} d\omega. \]

When \( t = 0 \) the integral gives \( x(0) = 1 \). When \( t \neq 0 \),

\[ x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega t} d\omega = \frac{1}{2\pi} \left[ \frac{e^{j\pi t} - e^{-j\pi t}}{2jt} \right]_{-\pi}^{\pi} = \frac{\sin \pi t}{\pi t}. \]

Thus,

\[ x(t) = \text{sinc}(t) := \begin{cases} 1 & t = 0 \\ \frac{\sin \pi t}{\pi t} & t \neq 0. \end{cases} \]

The Fourier Transform (2) in Example 1 can be expressed as a (scaled) sinc function as well:

\[ X(\omega) = 2T_1 \text{sinc} \left( \frac{T_1}{\pi} \omega \right). \]

Note the duality in Examples 1 and 3:

<table>
<thead>
<tr>
<th>rectangular pulse</th>
<th>( \leftrightarrow )</th>
<th>sinc</th>
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<tr>
<td>sinc</td>
<td>( \leftrightarrow )</td>
<td>rectangular pulse</td>
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**Properties of the Fourier Transform**

Consider two signals \( x(t) \xrightarrow{FT} X(\omega) \) and \( y(t) \xrightarrow{FT} Y(\omega) \).

**Linearity:** For any constants \( a, b \),

\[ ax(t) + by(t) \xrightarrow{FT} aX(\omega) + bY(\omega) \]
Time-Shift:

\[ x(t-t_0) \overset{FT}{\longleftrightarrow} e^{-j\omega t_0}X(\omega) \]  

(10)

Proof: \[
\int_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(t-t_0)}d\tau \\
= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega \tau}d\tau = X(\omega)
\]

Conjugation and Conjugate Symmetry:

\[ x^*(t) \overset{FT}{\longleftrightarrow} X^*(-\omega) \]  

(11)

If \( x(t) \) is real: \( X(\omega) = X^*(-\omega) \) (because \( x(t) = x^*(t) \))

\[ \Rightarrow |X(\omega)| = |X(-\omega)| \quad \text{even symmetry} \]  

(12)

\[ \angle X(\omega) = -\angle X(-\omega) \quad \text{odd symmetry} \]  

(13)

You can see such symmetry in the plots of Example 2 above.

Differentiation:

\[ \frac{dx(t)}{dt} \overset{FT}{\longleftrightarrow} j\omega X(\omega) \]  

(14)

Proof: Take the derivative of both sides of the synthesis equation.

Time and Frequency Scaling:

\[ x(at) \overset{FT}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{\omega}{a}\right), \quad a \neq 0 \]  

(15)

Proof: \[
\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega \tau/\omega}d\tau/|a|, \quad \text{if } a > 0 \\
= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega t/|a|}d\tau/|a|, \quad \text{if } a < 0 \\
= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\tau)e^{-j\omega t/|a|}d\tau = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)
\]

Example 3 revisited: Applying (15) with \( a = \frac{W}{\pi} \) to \( x(t) = \text{sinc}(t) \),

\[ x(at) = \text{sinc}\left(\frac{W}{\pi}t\right) \overset{FT}{\longleftrightarrow} \frac{\pi}{W}X\left(\frac{\pi}{W}\omega\right) \]

where \( X(\cdot) \) is as in (7). Thus,

\[ \frac{W}{\pi} \text{sinc}\left(\frac{W}{\pi}t\right) \overset{FT}{\longleftrightarrow} X\left(\frac{\pi}{W}\omega\right) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| \geq W \end{cases} \]

which generalizes Example 3 to an arbitrary bandwidth \( W \).
Special case of (15) with \( a = -1 \):

\[ x(-t) \leftrightarrow X(-\omega) \quad (16) \]

If \( x(-t) = x(t) \) then \( X(-\omega) = X(\omega) \)
If \( x(t) \) is also real: \( X(-\omega) = X^*(\omega) \) i.e., \( X(\omega) \) is real.

Note that \( X(\omega) \) is real in Examples 1 and 3 where \( x \) is real and even-symmetric.

Parseval's Relation:

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad (17)
\]

Example: \( x(t) = e^{-at}u(t), \ a > 0 \leftrightarrow X(\omega) = \frac{1}{a+j\omega} \)

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} e^{-2at} dt = \frac{1}{2a} e^{-2at}\bigg|_{0}^{\infty} = \frac{1}{2a}
\]

\[
\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \frac{1}{a^2 + \omega^2} d\omega = \frac{1}{a} \tan^{-1}\left(\frac{\omega}{a}\right)\bigg|_{-\infty}^{\infty} = \frac{\pi}{a} = 2\pi \frac{1}{2a}
\]

Initial Value:

\[ x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega \quad \text{(synthesis eq'n with } t = 0) \quad (18) \]

DC Component:

\[ X(0) = \int_{-\infty}^{\infty} x(t) dt \quad \text{(analysis equation with } \omega = 0) \quad (19) \]

Convolution Property:

\[ (x_1 * x_2)(t) \quad \text{FT} \leftrightarrow X_1(\omega)X_2(\omega) \quad (20) \]

Example: The triangular pulse shown on the right is the convolution of the rectangular pulse in Example 1 \((T_1 = 0.5)\) with itself.

Thus, squaring the transform (8) and substituting \( T_1 = 0.5 \), we conclude that the Fourier Transform of the triangular pulse is:

\[
\left( \text{sinc}\left(\frac{\omega}{2\pi}\right) \right)^2.
\]