Linear difference and differential equations

In this section we undertake a deeper study of linear difference and differential equations than that in section **??**, using the Z transform and the Laplace transform.

A **linear time-invariant difference equation** has the form

$$y(n) + a_1 y(n-1) + \cdots + a_m y(n-m) = b_0 x(n) + \cdots + b_k x(n-k), \; n \geq 0. \tag{1}$$

This equation describes a discrete-time linear time-invariant system in which $x(n)$ is the input and $y(n)$ is the output at time n. The a_i and b_j are constant coefficients. We are given:

> the input signal $x(n), n \geq 0$, with $x(n) = 0, n < 0$,
> and the initial conditions $y(-1) = \overline{y}(-1), \cdots, y(-m) = \overline{y}(-m)$; and
> our task is to determine the output signal $y(n), n \geq 0$.

There is a procedure to calculate the output. Rewrite (1) as

$$y(n) = -a_1 y(n-1) - \cdots - a_m y(n-m) + b_0 x(n) + \cdots + b_k x(n-k), \tag{2}$$

and recursively use (2) to obtain $y(0), y(1), y(2), \cdots$. Taking $n = 0$ in (2) yields

$$\begin{aligned} y(0) &= -a_1 y(-1) - \cdots - a_m y(-m) + b_0 x(0) + \cdots + b_k x(-k) \\ &= -a_1 \overline{y}(-1) - \cdots - a_m \overline{y}(-m) + b_0 x(0). \end{aligned}$$

All the terms on the right are known from the initial conditions and the input $x(0)$, so we can calculate $y(0)$. Next, taking $n = 1$ in (2),

$$y(1) = -a_1 y(0) + \cdots + a_m y(1-m) + b_0 x(1) + \cdots + b_k x(1-k).$$

All the terms on the right are known either from the given data or from precalculated values—$y(0)$ in this case. We can proceed in this way to calculate the remaining values of the output sequence $y(2), y(3), \cdots$, one at a time.

We now use the Z transform to calculate the *entire* output sequence. To obtain the Z transform of the sequences in (1), multiply both sides by z^{-n} and sum,

$$\sum_{n=0}^{\infty} y(n) z^{-n} + a_1 \sum_{n=0}^{\infty} y(n-1) z^{-n} + \cdots + a_m \sum_{n=0}^{\infty} y(n-m) z^{-n} = b_0 \sum_{n=0}^{\infty} x(n) z^{-n} + \cdots + b_k \sum_{n=0}^{\infty} x(n-k) z^{-n}. \tag{3}$$

Define the unilateral Z transforms

$$\hat{X}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad \hat{Y}(z) = \sum_{n=0}^{\infty} y(n)z^{-n}.$$

Each sum in (3) can be expressed in terms of $\hat{Y}$ or $\hat{X}$:

$$\begin{aligned}
\sum_{n=0}^{\infty} y(n-1)z^{-n} &= \overline{y}(-1)z^0 + z^{-1}\sum_{n=1}^{\infty} y(n-1)z^{-(n-1)} = \overline{y}(-1)z^0 + z^{-1}\hat{Y}(z),\\
\sum_{n=0}^{\infty} y(n-2)z^{-n} &= \overline{y}(-2)z^0 + \overline{y}(-1)z^{-1} + z^{-2}\sum_{n=2}^{\infty} y(n-2)z^{-(n-2)}\\
&= \overline{y}(-2)z^0 + \overline{y}(-1)z^{-1} + z^{-2}\hat{Y}(z),\\
&\cdots\\
\sum_{n=0}^{\infty} y(n-m)z^{-n} &= \overline{y}(-m)z^0 + \cdots + \overline{y}(-1)z^{-(m-1)} + z^{-m}\sum_{n=m}^{\infty} y(n-m)z^{-(n-m)}\\
&= \overline{y}(-m)z^0 + \cdots + \overline{y}(-1)z^{-(m-1)} + z^{-m}\hat{Y}(z).
\end{aligned}$$

Recalling that $x(n) = 0, n < 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty} x(n-1)z^{-n} &= x(-1)z^0 + z^{-1}\hat{X}(z) = z^{-1}\hat{X}(z)\\
\sum_{n=0}^{\infty} x(n-2)z^{-n} &= x(-2)z^0 + x(-1)z^{-1} + z^{-2}\hat{X}(z) = z^{-2}\hat{X}(z)\\
&\cdots\\
\sum_{n=0}^{\infty} x(n-k)z^{-n} &= x(-k)z^0 + \cdots + x(-1)z^{-(k-1)} + z^{-k}\hat{X}(z) = z^{-k}\hat{X}(z).
\end{aligned}$$

Substituting these relations in (3) yields

$$\begin{aligned}
\hat{Y}(z) &+ a_1[z^{-1}\hat{Y}(z) + \overline{y}(-1)z^0] + \cdots + a_m[z^{-m}\hat{Y}(z) + \overline{y}(-m)z^0 + \cdots \overline{y}(-1)z^{-(m-1)}]\\
&= b_0\hat{X}(z) + b_1 z^{-1}\hat{X}(z) + \cdots b_k\hat{X}z^{-k},
\end{aligned}$$

from which, by rearranging terms, we obtain

$$[1 + a_1 z^{-1} + \cdots + a_m z^{-m}]\hat{Y}(z) = [b_0 + b_1 z^{-1} + \cdots + b_k z^{-k}]\hat{X}(z) + \hat{C}(z),$$

where $\hat{C}(z)$ is an expression involving only the initial conditions $\overline{y}(-1), \cdots, \overline{y}(-m)$. Therefore,

$$\hat{Y}(z) = \frac{b_0 + b_1 z^{-1} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}\hat{X}(z) + \frac{\hat{C}(z)}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}$$

We rewrite this relation as

$$\hat{Y}(z) = \hat{H}(z)\hat{X}(z) + \frac{\hat{C}(z)}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}. \tag{4}$$

where

$$\boxed{\hat{H}(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_k z^{-k}}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}} \tag{5}$$

Observe that if the initial conditions are all zero, $\hat{C}(z) = 0$, and we only have the first term on the right in (4); and if the input is zero—i.e., $x(n) = 0, n \geq 0$—then $\hat{X}(z) = 0$, and we only have the second term.

To determine $y(n), n \geq 0$, we take the inverse Z transform in (4). Therefore,

$$\forall n \geq 0, \quad y(n) = y_{zs}(n) + y_{zi}(n), \tag{6}$$

where $y_{zs}(n)$, the inverse Z transform of $\hat{H}\hat{X}$, is the **zero-state response**, and $y_{zi}(n)$, the inverse Z transform of $\hat{C}(z)/[1 + a_1 z^{-1} + \cdots + a_m z^{-m}]$, is the **zero-input response**. The zero-state response, also called the **forced response**, is the output when all initial conditions are zero. The zero-input response, also called the **natural response**, is the output when the input is zero.

Thus the (total) response is the sum of the zero-state and zero-input response. We first encountered this property of linearity in **??**.

By definition, the **transfer function** is the Z transform of the zero-state impulse response. Taking $\hat{C} = 0$ and $\hat{X} = 1$ in (4) shows that the transfer function is $\hat{H}(z)$. From (5) we see that $\hat{H}$ can be written down by inspection of the difference equation (1). If the system is stable—all poles of $\hat{H}$ are inside the unit circle—the frequency response is

$$\forall \omega, \quad H(\omega) = \hat{H}(e^{i\omega}) = \frac{b_0 + b_1 e^{-i\omega} + \cdots + b_k e^{-ik\omega}}{1 + a_1 e^{-i\omega} + \cdots + a_m e^{-im\omega}}.$$

We saw this relation in (**??**).

Example 0.1: Consider the difference equation

$$y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = x(n), \; n \geq 0.$$

Taking Z transforms yields

$$\hat{Y}(z) - \frac{5}{6}[z^{-1}\hat{Y}(z) + \overline{y}(-1)] + \frac{1}{6}[z^{-2}\hat{Y}(z) + \overline{y}(-2) + \overline{y}(-1)z^{-1}] = \hat{X}(z).$$

Therefore

$$\begin{aligned} \hat{Y}(z) &= \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \hat{X}(z) + \frac{\frac{5}{6}\overline{y}(-1) + \frac{1}{6}\overline{y}(-2) + \frac{1}{6}\overline{y}(-1)z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\ &= \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}} \hat{X}(z) + \frac{[\frac{5}{6}\overline{y}(-1) + \frac{1}{6}\overline{y}(-2)]z^2 + \frac{1}{6}\overline{y}(-1)z}{z^2 - \frac{5}{6}z + \frac{1}{6}}, \end{aligned}$$

from which we can obtain $\hat{Y}$ for a specified $\hat{X}$ and initial conditions $\overline{y}(-1), \overline{y}(-2)$. The transfer function is

$$\hat{H}(z) = \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{z^2}{(z - \frac{1}{3})(z - \frac{1}{2})},$$

which has poles at $z = 1/3$ and $z = 1/2$ (and two zeros at $z = 0$). The system is stable. The zero-state impulse response h is the inverse Z transform of $\hat{H}(z)$, which we obtain using partial fraction expansion,

$$\hat{H}(z) = z\left[\frac{-2}{z-\frac{1}{3}} + \frac{3}{z-\frac{1}{2}}\right]$$

so that

$$\forall n \geq 0, \quad h(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n).$$

We can recognize that the impulse response consists of two terms, each contributed by one pole of the transfer function.

Suppose the initial conditions are $\overline{y}(-1) = 1, \overline{y}(-2) = 1$ and the input x is the unit step, so $\hat{X}(z) = z/(z-1)$. Then the zero-input response, y_{zi}, has Z transform

$$\begin{aligned} \hat{Y}_{zi}(z) &= \frac{[\frac{5}{6}\overline{y}(-1) + \frac{1}{6}\overline{y}(-2)]z^2 + \frac{1}{6}\overline{y}(-1)z}{(z-\frac{1}{3})(z-\frac{1}{2})} \\ &= \frac{z^2 + \frac{1}{6}z}{(z-\frac{1}{3})(z-\frac{1}{2})} = z\left[\frac{-3}{z-\frac{1}{3}} + \frac{4}{z-\frac{1}{2}}\right], \end{aligned}$$

so

$$y_{zi}(n) = -3\left(\frac{1}{3}\right)^n u(n) + 4\left(\frac{1}{2}\right)^n u(n).$$

The zero-state response, y_{zs}, has Z transform

$$\begin{aligned} \hat{Y}_{zs}(z) &= \hat{H}(z)\hat{X}(z) = \frac{z^3}{(z-\frac{1}{3})(z-\frac{1}{2})(z-1)} \\ &= z\left[\frac{1}{z-\frac{1}{3}} + \frac{-3}{z-\frac{1}{2}} + \frac{3}{z-1}\right], \end{aligned}$$

so

$$y_{zs}(n) = \left(\frac{1}{3}\right)^n u(n) - 3\left(\frac{1}{2}\right)^n u(n) + 3u(n).$$

The (total) response

$$y(n) = y_{zs}(n) + y_{zi}(n) = 3u(n) + [-2(1/3)^n + (1/2)^n]u(n),$$

can also be expressed as the sum of the steady-state and the transient response with $y_{ss}(n) = 3u(n)$ and $y_{tr}(n) = -2(1/3)^n u(n) + (1/2)^n u(n)$. Note that the decomposition of the response into the sum of the zero-state and zero-input responses is different from its decomposition into the steady-state and transient responses.

The analogous development for continuous-time makes use of the Laplace transform. A **linear time-invariant differential equation** has the form

$$\frac{d^m y}{dt^m}(t) + a_{m-1}\frac{d^{m-1}y}{dt^{m-1}}(t) + \cdots + a_1\frac{dy}{dt}(t) + a_0 y(t) = b_k\frac{d^k x}{dt^k}(t) + \cdots + b_1\frac{dx}{dt}(t) + b_0 x(t),\ t \geq 0 \quad (7)$$

The differential equation describes a continuous-time linear time-invariant system in which $x(t)$ is the input and $y(t)$ is the output at time t. The a_i and b_j are constant coefficients. We are given:

the input signal $x(t), t \geq 0$,
and the initial conditions $y(0) = \overline{y}(0), \frac{dy}{dt}(0) = \overline{y}^{(1)}(0), \cdots, \frac{d^{m-1}y}{dt^{m-1}}(0) = \overline{y}^{(m-1)}(0)$; and
our task is to determine the output signal $y(t), t \geq 0$.

Unlike for (1) we don't need the initial conditions for the derivatives of x since those are determined from the data $x(t), t \geq 0$.

Because time is continuous, there is no recursive procedure for calculating the output from the given data as we did in (2). Instead we calculate the output signal using the Laplace transform.

Define the unilateral Laplace transforms

$$\hat{X}(s) = \int_0^\infty x(t)e^{-st}dt, \quad \hat{Y}(s) = \int_0^\infty y(t)e^{-st}dt.$$

We want to obtain the unilateral Laplace transforms of the derivatives of y and x in terms of the unilateral Laplace transform of y, x. These transforms are slightly different from each other and from those in table **??** because the interpretation of these derivatives in (7) are different.

The derivative $y^{(1)}(t) = \frac{dy}{dt}(t)$ and y are related by

$$y(t) = y(0) + \int_0^t y^{(1)}(\tau)d\tau = \overline{y}(0) + \int_0^t y^{(1)}(\tau)d\tau, \ t \geq 0.$$

Using integration by parts, and denoting by $\hat{Y}^{(1)}(s)$ the Laplace transform of $y^{(1)}$, yields

$$\begin{aligned} \hat{Y}(s) &= \int_0^\infty y(t)e^{-st}dt = \int_0^\infty \overline{y}(0)e^{-st}dt + \int_0^\infty \left(\int_0^t y^{(1)}(\tau)d\tau\right)e^{-st}dt \\ &= \frac{1}{s}\overline{y}(0) - \frac{1}{s}\int_0^t y^{(1)}(\tau)d\tau e^{-st} \mid_{t=0}^\infty + \frac{1}{s}\int_0^\infty y^{(1)}(t)e^{-st}dt \\ &= \frac{1}{s}[\hat{Y}^{(1)}(s) + \overline{y}(0)], \end{aligned}$$

Therefore,

$$\boxed{\hat{Y}^{(1)}(s) = s\hat{Y}(s) - \overline{y}(0).} \tag{8}$$

If we repeat this procedure, we get the Laplace transforms of the higher-order derivatives,

$$\begin{aligned} \hat{Y}^{(2)}(s) &= s\hat{Y}^{(1)}(s) - \overline{y}^{(1)}(0) \\ &= s^2\hat{Y}(s) - s\overline{y}(0) - \overline{y}^{(1)}(0) \\ &\cdots \\ \hat{Y}^{(m)}(s) &= s^m\hat{Y}(s) - s^{m-1}\overline{y}(0) - s^{m-2}\overline{y}^{(1)}(0) - \cdots - \overline{y}^{(m-1)}(0). \end{aligned}$$

Here $\hat{Y}^{(2)}$ is the Laplace transform of the second derivative, $y^{(2)} = \frac{d^2y}{dt^2}$, and $\hat{Y}^{(m)}$ is the Laplace transform of the mth derivative, $y^{(m)} = \frac{d^my}{dt^m}$.

On the other hand, the interpretation of the input signal x is that $x(t) = x(t)u(t)$ for all $t \in Reals$, so $x^{(1)}(t) = \frac{dx}{dt}(t)$ for all $t \in Reals$. Hence from table **??**,

$$\begin{aligned} \hat{X}^{(1)}(s) &= s\hat{X}(s) \\ &\cdots \\ \hat{X}^{(k)}(s) &= s^k\hat{X}(s), \end{aligned}$$

where $\hat{X}^{(1)}$ is the Laplace transform of $x^{(1)} = \frac{dx}{dt}$, and $\hat{X}^{(k)}$ is the Laplace transform of $x^{(k)} = \frac{d^k x}{dt^k}$.

By substituting from the relations just derived, we obtain the unilateral Laplace transforms of all the terms in (7),

$$[s^m\hat{Y}(s) - s^{m-1}\overline{y}(0) - \cdots - \overline{y}^{m-1}(0)] + a_{m-1}[s^{m-1}\hat{Y}(s) - s^{m-2}\overline{y}(0) - \cdots - \overline{y}^{m-2}(0)]$$
$$\cdots + a_1[s\hat{Y}(s) - \overline{y}(0)] + a_0\hat{Y}(s) = b_k s^k \hat{X}(s) + \cdots + b_1 s\hat{X}(s) + b_0\hat{X}(s).$$

Rearranging terms yields

$$[s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0]\hat{Y}(s) = [b_k s^k + \cdots + b_1 s + b_0]\hat{X}(s) + \hat{C}(s),$$

in which $\hat{C}$ is an expression involving only the intial conditions $\overline{y}(0), \cdots, \overline{y}^{m-1)}(0)$. Therefore,

$$\hat{Y}(s) = \frac{b_k s^k + b_{k-1}s^{k-1} + \cdots b_1 s + b_0}{s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}\,\hat{X}(s) + \frac{\hat{C}(s)}{s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}, \tag{9}$$

which we also write as

$$\hat{Y}(s) = \hat{H}(s)\hat{X}(s) + \frac{\hat{C}(s)}{s^m + a_{m-1}s^{m-1} + \cdots a_1 s + a_0}, \tag{10}$$

in which

$$\boxed{\hat{H}(s) = \frac{b_k s^k + \cdots + b_1 s + b_0}{s^m + \cdots + a_1 s + a_0}.} \tag{11}$$

If the initial conditions are all zero, $\hat{C}(s) = 0$, and we only have the first term on the right in (10); if the input is zero—i.e., $x(t) = 0$ for all t—then $\hat{X}(s) = 0$, and we only get the second term in (10).

Taking the inverse Laplace transform, we can express the output signal $y(t)$ as

$$\forall t \geq 0, \quad y(t) = y_{zs}(t) + y_{zi}(t),$$

where $y_{zs}(t)$, the inverse Laplace transform of $\hat{H}\hat{X}$, is the **zero-state** or **forced response** and $y_{zi}(t)$, the inverse Laplace transform of $\hat{C}(s)/[s^m + \cdots + a_0]$, is the **zero-input** or **natural response**. The (total) response is the sum of the zero-state and zero-input response, which is a general property of linear systems.

By definition, the **transfer function** is the Laplace transform of the zero-state impulse response. Taking $\hat{C} = 0$ and $\hat{X} = 1$—the Laplace transform of the unit impulse—in (10) shows that the transfer function is $\hat{H}(s)$ which, as we see from (11), can be written down by inspection of the differential equation (7). If the system is stable—all poles of $\hat{H}(s)$ have real parts strictly less than zero—the frequency response is

$$\forall \omega, \quad H(\omega) = \hat{H}(i\omega) = \frac{b_k(i\omega)^k + \cdots + b_1 i\omega + b_0}{(i\omega)^m + \cdots + a_1 i\omega + a_0}.$$

We saw this relation in (**??**).

Example 0.2: We find the response $y(t), t \geq 0$, for the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 3x(t) + \frac{dx}{dt},$$

when the input is a unit step $x(t) = u(t)$ and the initial conditions are $y(0) = 1, y^{(1)}(0) = 2$. Taking Laplace transforms of both sides yields

$$[s^2\hat{Y}(s) - s\overline{y}(0) - \overline{y}^{(1)}(0)] + 3[s\hat{Y}(s) - \overline{y}(0)] + 2\hat{Y}(s) = 3\hat{X}(s) + s\hat{X}(s).$$

Therefore,

$$\hat{Y}(s) = \frac{s+3}{s^2+3s+2}\,\hat{X}(s) + \frac{s\overline{y}(0) + \overline{y}^{(1)}(0) + 3\overline{y}(0)}{s^2+3s+2}.$$

Substituting $\hat{X}(s) = 1/s$, $\overline{y}(0) = 1, \overline{y}^{(1)} = 2$, yields

$$\begin{aligned} \hat{Y}(s) &= \frac{s+3}{s(s^2+3s+2)} + \frac{s+5}{s^2+3s+2} \\ &= [\frac{3/2}{s} - \frac{2}{s+1} + \frac{1/2}{s+2}] + [\frac{4}{s+1} - \frac{3}{s+2}]. \end{aligned}$$

Taking inverse Laplace transforms gives

$$\begin{aligned} y(t) &= y_{zs}(t) + y_{zi}(t) \\ &= [\frac{3}{2}u(t) - 2e^{-t}u(t) + \frac{1}{2}e^{-2t}u(t)] + [4e^{-t}u(t) - 3e^{-2t}u(t)] \\ &= \frac{3}{2}u(t) + [2e^{-t} - \frac{5}{2}e^{-2t}]u(t) \\ &= y_{ss}(t) + y_{tr}(t). \end{aligned}$$

As in the case of difference equations, the decomposition of the response into zero-state and zero-input responses is different from the decomposition into transient and steady-state responses. (Indeed, the steady-state response does not exist if the system is unstable, whereas the former decomposition always exists.)

0.1 State-space models

Section **??** introduced single-input, single-output (SISO) multidimensional of discrete-time and continuous-time LTI systems. We use transform techniques to understand the behavior of these models.

The discrete-time SISO state-space model is

$$\forall\, n \geq 0, \quad s(n+1) = As(n) + bx(n) \tag{12}$$
$$y(n) = c^T s(n) + dx(n) \tag{13}$$

where $s(n) \in Reals^N$ is the state, $x(n) \in Reals$ is the input, and $y(n) \in Reals$ is the output at time n. In this $[A, b, c, d]$ representation, A is an $N \times N$ (square) matrix, b, c are N-dimensional column vectors, and d is a scalar. If the initial state is $s(0)$, the state response and the output response of this system to an input sequence $x(0), x(1), \cdots$ are, respectively,

$$s(n) = A^n s(0) + \sum_{m=0}^{n-1} A^{n-1-m} b x(m) \tag{14}$$

$$y(n) = c^T A^n s(0) + \{\sum_{m=0}^{n-1} c^T A^{n-1-m} b x(m) + d x(n)\} \tag{15}$$

for all $n \geq 0$. The state-space model may be used to calculate these responses recursively. We study how to obtain their Z transforms. The key is to compute the Z transform of the entire $N \times N$ matrix sequence $A^n, n \geq 0$.

Observe that

$$\boxed{\sum_{n=0}^{\infty} z^{-n} A^n = [I - z^{-1} A]^{-1}.} \tag{16}$$

Here z is a complex number and I is the $N \times N$ identity matrix. The series on the left is an infinite sum of $N \times N$ matrices which converges to the $N \times N$ matrix on the right, for $z \in RoC$. RoC is determined later.

Assuming the series converges, it is easy to check the equality (16): Just multiply both sides by $[I - z^{-1}A]$ and verify that

$$[I - z^{-1}A] \sum_{n=0}^{\infty} z^{-n} A^n = \sum_{n=0}^{\infty} z^{-n} A^{-n} - \sum_{n=0}^{\infty} z^{-(n+1)} A^{n+1} = z^0 A^0 = I.$$

Next, denote by F the matrix inverse,

$$F(z) = [I - z^{-1}A]^{-1} = z[zI - A]^{-1}, \tag{17}$$

and the coefficients of A^n and $F(z)$ by

$$A^n = [a_{ij}(n) \mid 1 \leq i, j \leq N], \quad F(z) = [f_{ij}(z) \mid 1 \leq i, j \leq N].$$

Then $f_{ij}(z) = \sum_{n=0}^{\infty} z^{-n} a_{ij}(n)$ is the Z transform of the sequence $a_{ij}(n), n \geq 0$, $1 \leq i, j \leq N$. So we can obtain $A^n, n \geq 0$, by taking the inverse Z transform of the coefficients of $F(z)$. Consider an example.

Example 0.3: Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix},$$

so

$$[zI - A]^{-1} = \begin{bmatrix} z-2 & -1 \\ -3 & z-4 \end{bmatrix}^{-1} = \frac{1}{\det[zI - A]} \begin{bmatrix} z-4 & 1 \\ 3 & z-2 \end{bmatrix},$$

in which $\det[zI - A]$ denotes the determinant of $[zI - A]$,

$$\det[zI - A] = (z-2)(z-4) - 3 = z^2 - 6z + 5 = (z-1)(z-5).$$

Hence,

$$F(z) = z[zI-A]^{-1} = \frac{z}{(z-1)(z-5)}\left[\begin{array}{cc} z-4 & 1 \\ 3 & z-2 \end{array}\right] = \left[\begin{array}{cc} \frac{z(z-4)}{(z-1)(z-5)} & \frac{z}{(z-1)(z-5)} \\ \frac{3z}{(z-1)(z-5)} & \frac{z(z-2)}{(z-1)(z-5)} \end{array}\right].$$

The partial fraction expansion of the coefficients of F is

$$F(z) = \left[\begin{array}{cc} \frac{(3/4)z}{z-1} + \frac{(1/4)z}{z-5} & \frac{(-1/4)z}{z-1} + \frac{(1/4)z}{z-5} \\ \frac{(-3/4)z}{z-1} + \frac{(3/4)z}{z-5} & \frac{(1/4)z}{z-1} + \frac{(3/4)z}{z-5} \end{array}\right].$$

From table **??** we find the inverse Z transform of $F(z)$: for all $n \geq 0$,

$$A^n = \left[\begin{array}{cc} \frac{3}{4}u(n) + \frac{1}{4}5^n u(n) & -\frac{1}{4}u(n) + \frac{1}{4}5^n u(n) \\ -\frac{3}{4}u(n) + \frac{3}{4}5^n u(n) & \frac{1}{4}u(n) + \frac{3}{4}5^n u(n) \end{array}\right],$$

which is more revealingly expressed as

$$A^n = \left[\begin{array}{cc} 3/4 & -1/4 \\ -3/4 & 1/4 \end{array}\right] + 5^n \left[\begin{array}{cc} 1/4 & 1/4 \\ 3/4 & 3/4 \end{array}\right], \quad n \geq 0,$$

because it shows that the variation in n of A^n is determined by the two poles, at $z = 1$ and $z = 5$, in the coefficients of $F(z)$. Moreover, these two poles are the zeros of

$$\det[zI - A] = (z-1)(z-5).$$

This determinant is called the **characterstic polynomial** of the matrix A and its zeros are called the **eigenvalues** of A. The domain of convergence is $RoC = \{z \in Complex \mid |z| > 5\}$.

We return to the general case in (17). Denote the matrix inverse as

$$[zI - A]^{-1} = \frac{1}{\det[zI - A]} G(z),$$

in which $G(z)$ is the $N \times N$ matrix of co-factors of $[zI - A]$. Hence each coefficient $f_{ij}(z)$ is a rational polynomial whose denominator is the characteristic polynomial of A, $\det[zI - A]$. So all coefficients of $F(z)$ have the same poles, namely, the eigenvalues of A. In order for the system (12), (13) to be stable the poles of F—that is, the eigenvalues of A—must have magnitudes strictly smaller than 1.

Suppose the characteristic polynomial of A has N distinct zeros $p_1, \cdots, p_N$:

$$\det[zI - A] = (z - p_1) \cdots (z - p_N).$$

Then the partial fraction expansion of $F(z)$ has the form

$$F(z) = \frac{z}{z - p_1} R_1 + \cdots + \frac{z}{z - p_N} R_N,$$

in which R_i is the matrix of residues of the coefficients of F. at the pole p_i. R_i is a constant matrix, possibly with complex coefficients if p_i is complex. Recalling that $\frac{z}{z-p_i}$ is the inverse Z transform of $p_i^n u(n)$, we can take the inverse Z transform of $F(z)$ to conclude that

$$\boxed{A^n = p_1^n R_1 + \cdots p_N^n R_N, \quad n \geq 0.} \tag{18}$$

Thus A^n is a linear combination of $p_1^n, \cdots, p_N^n$.

We can decompose the response (15) into the zero-input and zero-state responses, expressing the latter as a convolution sum,

$$y(n) = c^T A^n s(0) + \sum_{m=0}^{n} h(n-m)x(m),\ n \geq 0,$$

where the (zero-state) impulse response is

$$h(n) = \begin{cases} d, & n = 0 \\ c^T A^{n-1} b, & n \geq 1 \end{cases}.$$

Let $\hat{X}, \hat{Y}, \hat{H}, \hat{Y}_{zi}$ be the Z transforms

$$\hat{X}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad \hat{Y}(z) = \sum_{n=0}^{\infty} y(n)z^{-n}, \quad \hat{H}(z) = \sum_{n=0}^{\infty} h(n)z^{-n}, \quad \hat{Y}_{zi}(z) = \sum_{n=0}^{\infty} c^T z^{-n} A^n s(0).$$

Then

$$\hat{Y} = \hat{H}\hat{X} + \hat{Y}_{zi}.$$

Because $\sum_{n=0}^{\infty} z^{-n} A^n = z[zI - A]^{-1}$, we obtain

$$\boxed{\hat{H}(z) = c^T [zI - A]^{-1} b + d,}$$

and

$$\boxed{\hat{Y}_{zi}(z) = zc^T [zI - A]^{-1} s(0).}$$

We continue with the previous example.

Example 0.4: Suppose A is as in example 0.3, $b^T = [1 \quad 1], c^T = [2 \quad 0], d = 3$, and $(s(0))^T = [0 \quad 4]$. Then the transfer function is

$$\hat{H}(z) = [2 \quad 0] \begin{bmatrix} \frac{(z-4)}{(z-1)(z-5)} & \frac{1}{(z-1)(z-5)} \\ \frac{3}{(z-1)(z-5)} & \frac{(z-2)}{(z-1)(z-5)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 = \frac{2(z-4)+2}{(z-1)(z-5)} + 3,$$

and the Z transform of the zero-input response is

$$\hat{Y}_{zi}(z) = [2 \quad 0] \begin{bmatrix} \frac{z(z-4)}{(z-1)(z-5)} & \frac{z}{(z-1)(z-5)} \\ \frac{3z}{(z-1)(z-5)} & \frac{z(z-2)}{(z-1)(z-5)} \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \frac{8z}{(z-1)(z-5)}.$$

The continuous-time $[A,b,c,d]$ SISO state-space model is

$$\dot{v}(t) = Av(t) + bx(t), \tag{19}$$
$$y(t) = c^T v(t) + dx(t), \tag{20}$$

where, for $t \in Reals_+$, $v(t) \in Reals^N$ is the state, $x(t) \in Reals$ is the input, and $y(t) \in Reals$ is the output. A is an $N \times N$ matrix, and b, c are N-dimensional column vectors, and d is a scalar. (We use v instead of s to denote the state, because s is reserved for the Laplace transform variable.)

Given the initial state $v(0)$ and the input signal $x(t), t \geq 0$, we will show that the state response and the output response are determined by the formulas

$$v(t) = e^{tA}v(0) + \int_0^t e^{(t-\tau)A}bx(\tau)d\tau, \tag{21}$$
$$y(t) = c^T e^{tA}v(0) + [\int_0^t c^T e^{(t-\tau)A}bx(\tau)d\tau] + dx(t). \tag{22}$$

In these formulas, e^{tA} or $\exp(tA)$ is the name of the $N \times N$ matrix

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \cdots, \tag{23}$$

where $(tA)^k$ is the matrix tA multiplied by itself k times, and $(tA)^0 = I$, the $N \times N$ identity matrix. Definition (23) of the matrix exponential is the natural generalization of the exponential of a real or complex number. (The series in (23) is absolutely summable.)

Unlike in the discrete-time case, there is no recursive procedure to compute the responses (21), (22). This is because time is continuous, and the difficulty has to do with the integrals in these formulas. For numerical calculation, one resorts to a finite sum approximation of the integrals, as we indicated in section **??**. The Laplace transform provides an alternative approach that is exact.

The key to proving (21) is the fact that $e^{tA}, t \geq 0$ is the solution to the differential equation

$$\frac{d}{dt}e^{tA} = Ae^{tA}, \quad t \geq 0, \tag{24}$$

with initial condition $e^{0A} = I$. Note that (22) follows immediately from (21) and (20).

To verify (24) we substitute for e^{tA} from (23) and differentiate the sum term by term,

$$\frac{d}{dt}e^{tA} = \sum_{k=0}^{\infty} \frac{d}{dt}\frac{(tA)^k}{k!} = \sum_{k=1}^{\infty} \frac{kA}{k!}(tA)^{k-1} = A\sum_{k=1}^{\infty} \frac{(tA)^{k-1}}{(k-1)!} = Ae^{tA}.$$

We can now verify that (21) is indeed the solution of (19) by taking derivatives of both sides and using (24):

$$\begin{aligned}\dot{v}(t) &= Ae^{tA}v(0) + e^{0A}bx(t) + \int_0^t Ae^{(t-\tau)A}bx(\tau)d\tau \\ &= A[e^{tA}v(0) + \int_0^t Ae^{(t-\tau)A}bx(\tau)d\tau] + bx(t) \\ &= Av(t) + bx(t).\end{aligned}$$

We turn to the main difficulty in calculating the terms on the right in the responses (21), (22), namely the calculation of the $N \times N$ matrix $e^{tA}, t \geq 0$. We determine its Laplace transform, denoting it by

$$G(s) = \int_0^\infty e^{tA} e^{-st} dt.$$

This means that $g_{ij}(s)$ is the Laplace transform of $a_{ij}(t), t \geq 0$, if we denote by $a_{ij}(t), g_{ij}(s)$ the coefficients of the $N \times N$ matrices e^{tA} and $G(s)$. The region of convergence of G, RoC, is determined later.

Using the derivative formula (8) in (24) we see that

$$sG(s) - I = AG(s),$$

so that

$$\boxed{G(s) = \int_0^\infty e^{tA} e^{-st} dt = [sI - A]^{-1}.} \tag{25}$$

Example 0.5: Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix},$$

so

$$[sI - A]^{-1} = \begin{bmatrix} s-1 & -2 \\ 2 & s-1 \end{bmatrix}^{-1} = \frac{1}{\det[sI - A]} \begin{bmatrix} s-1 & 2 \\ -2 & s-1 \end{bmatrix}.$$

The determinant is

$$\det[sI - A] = (s-1)^2 + 4 = (s - 1 + 2i)(s - 1 - 2i),$$

so

$$\begin{aligned} [sI - A]^{-1} &= \begin{bmatrix} \frac{s-1}{(s-1+2i)(s-1-2i)} & \frac{2}{(s-1+2i)(s-1-2i)} \\ \frac{-2}{(s-1+2i)(s-1-2i)} & \frac{s-1}{(s-1+2i)(s-1-2i)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1/2}{s-1+2i} + \frac{1/2}{s-1-2i} & \frac{i/2}{s-1+2i} + \frac{-i/2}{s-1-2i} \\ \frac{-i/2}{s-1+2i} + \frac{i/2}{s-1-2i} & \frac{1/2}{s-1+2i} + \frac{1/2}{s-1-2i} \end{bmatrix}. \end{aligned}$$

The region of convergence $RoC = \{s \in Complex \mid Re\{s\} > 1\}$. We can now find the inverse Laplace transform using table **??** and express it in two ways: for all $t \geq 0$,

$$\begin{aligned} e^{tA} &= e^{(1-2i)t} \begin{bmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} + e^{(1+2i)t} \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} \\ &= e^t \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}. \end{aligned}$$

The first expression shows e^{tA} as a linear combination of the exponentials $e^{(1-2i)t}$ and $e^{(1+2i)t}$, in which the exponents, $1 - 2i$ and $1 + 2i$, are the two eigenvalues of A—that is, the zeros of its characteristic polynomial, $\det[sI - A]$. The second expression shows that e^{tA} is sinusoidal with frequency 2 radians/sec equal to the imaginary part of the eigenvalues whose amplitude grows exponentially corresponding to the real part of the eigenvalues.

We return to the general case in (25). Denote the matrix inverse as

$$G(s) = [sI - A]^{-1} = \frac{1}{\det[sI - A]} K(s),$$

in which $K(s)$ is the $N \times N$ matrix of co-factors of $[sI - A]$. Each coefficient $g_{ij}(s)$ of $G(s)$ is a rational polynomial of A whose denominator is the characterstic polynomial of A, $\det[sI - A]$. So all coefficients of $G(s)$ have the same poles—the eigenvalues of A. For the system (19), (20) to be stable, the poles of $G(s)$ must have strictly negative real parts. The system of example 0.5 is unstable, because the real part of the eigenvalues is $+1$.

Suppose the characteristic polynomial has N distinct zeros $p_1, \cdots, p_N$,

$$\det[sI - A] = (s - p_1) \cdots (s - p_N).$$

Then the partial fraction expansion of $G(s)$ has the form

$$G(s) = [sI - A]^{-1} = \frac{1}{s - p_1} R_1 + \cdots + \frac{1}{s - p_N} R_N,$$

in which R_i is the matrix of residues at the pole p_i of the coefficients of $G(s)$. R_i is a constant matrix, possibly with complex coefficients, if p_i is complex. Because the inverse Laplace transform of $\frac{1}{s - p_i}$ is $e^{p_i t} u(t)$, the inverse Laplace transform of $[sI - A]^{-1}$ is

$$\boxed{e^{tA} u(t) = [e^{p_1 t} R_1 + \cdots + e^{p_N t} R_N] u(t).} \tag{26}$$

Thus the matrix e^{tA} as a function of t is a linear combination of $e^{p_1 t}, \cdots, e^{p_N t}$, where the p_i are the eigenvalues of A—that is the zeros of $\det[sI - A]$.

We decompose the response (22) into the sum of the zero-input and zero-state responses, expressing the latter as a convolution integral,

$$y(t) = c^T e^{tA} v(0) + \int_0^t h(t - \tau) x(\tau) d\tau, \ t \geq 0,$$

in which the (zero-state) impulse response is: for all $t \in Reals$,

$$h(t) = c^T e^{tA} b u(t) + d\delta(t).$$

(Here δ is of course the Dirac delta function.) Let $\hat{X}, \hat{Y}, \hat{H}, \hat{Y}_{zi}$ be the Laplace transforms

$$\hat{X}(s) = \int_0^\infty x(t) e^{-st} dt, \ \hat{Y}(s) = \int_0^\infty y(t) e^{-st} dt, \ \hat{H}(s) = \int_{-\infty}^\infty h(t) e^{-st} dt, \ \hat{Y}_{zi}(s) = \int_0^\infty c^T e^{tA} v(0) e^{-st} dt.$$

Then

$$\hat{Y} = \hat{H}\hat{X} + \hat{Y}_{zi},$$

in which

$$\boxed{\hat{H}(s) = c^T [sI - A]^{-1} b + d,}$$

and

$$\boxed{\hat{Y}_{zi}(s) = c^T [sI - A]^{-1} v(0).}$$

We continue with example 0.5.

Example 0.6: Suppose A is as in example 0.5, $b^T = [1\ \ 1]^T, c^T = [2\ \ 0]^T, d = 3$, and $v(0)^T = [0\ \ 4]^T$. Then the transfer function is

$$\hat{H}(s) = [2\ \ 0]\begin{bmatrix} \frac{s-1}{(s-1)^2-4} & \frac{-2}{(s-1)^2-4} \\ \frac{2}{(s-1)^2-4} & \frac{s-1}{(s-1)^2-4} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 = \frac{2s-6}{(s-1)^2-4} + 3,$$

and the Laplace transform of the zero-input response is

$$\hat{Y}_{zi}(s) = [2\ \ 0]\begin{bmatrix} \frac{s-1}{(s-1)^2-4} & \frac{-2}{(s-1)^2-4} \\ \frac{2}{(s-1)^2-4} & \frac{s-1}{(s-1)^2-4} \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \frac{-16}{(s-1)^2-4}.$$

Exercises

Each problem is annotated with the letter **E, T, C** which stands for exercise, requires some thought, requires some conceptualization. Problems labeled **E** are usually mechanical, those labeled **T** require a plan of attack, those labeled **C** usually have more than one defensible answer.

1. **E** Determine the zero-input and zero-state responses, and the transfer function for the following. In both cases take $y(-1) = y(-2) = 0$ and $x(n) = u(n)$.
   (a) $y(n) + y(n-2) = x(n), n \geq 0$.
   (b) $y(n) + 2y(n-1) + y(n-2) = x(n), n \geq 0$.

2. **E** Determine the zero-input and the zero-state responses for the following.
   (a) $5\dot{y} + 10y = 2x, y(0) = 2, x(t) = u(t)$.
   (b) $\ddot{y} + 5\dot{y} + 6y = -4x - 3\dot{x}, y(0) = -1, \dot{y}(0) = 5, x(t) = e^{-t}u(t)$.
   (c) $\ddot{y} + 4y = 8x, y(0) = 1, \dot{y}(0) = 2, x(t) = u(t)$.
   (d) $\ddot{y} + 2\dot{y} + 5y = \dot{x}, y(0) = 2, \dot{y}(0) = 0, x(t) = e^{-t}u(t)$.

3. **T** Consider the circuit of figure 1. The input is the voltage x, the output is the capacitor voltage v. The inductor current is called i.

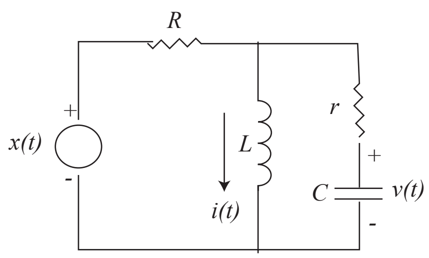

Figure 1: Circuit of problem 3

   (a) Derive the $[A, b, c, d]$ representation for this system using $s(t) = [i(t), v(t)]^T$ as the state.

(b) Obtain an $[F,g,h,k]$ representation for a discrete-time model of the same circuit by sampling at times $kT, k = 0,1,\cdots$ and using the approximation $\dot{s}(kT) = 1/T(s((k+1)T) - s(kT))$. (This is called a forward-Euler approximation.)

4. **E** For the matrix A in example 0.3, determine $e^{tA}, t \geq 0$.

5. **E** For the matrix A in example 0.5, determine $A^n, n \geq 0$.

6. **T** A continuous-time SISO system has $[A,b,c,d]$ representation with

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

in which a,b are real constants.

(a) Find the eigenvalues of A.

(b) For what values of a,b is the SISO system stable?

(c) Calculate $e^{tA}, t \geq 0$.

(d) Suppose $b = c = [1 \;\; 0]^T$, and $d = 0$. Find the transfer function.

7. **T** Let A be an $N \times N$ matrix. Let p be an eigenvalue of A. An N-dimensional (column) vector e, possibly complex-valued, is said to be an **eigenvector** of A corresponding to p if $e \neq 0$ and $Ae = pe$. Note that an eigenvector always exists since $\det[pI - A] = 0$. Find eigenvectors for each of the two eigenvalues of the matrices in examples 0.3 and 0.5.

8. **E** Let A be a square matrix with eigenvalue p and corresponding eigenvector e. Determine the response of the following.

(a) $s(k+1) = As(k), k \geq 0;\; s(0) = e$.

(b) $\dot{s}(t) = As(t), t \geq 0;\; s(0) = e$.