Sampling (Lectures 9, 10)

This handout is a complement to the textbook, OWN Chapter 7 (Section 7.1 in particular).

It has been remarked that "A common approach today in signal processing is to convert analog input signals as quickly as possible into digital signals, then to process these digital signals in sophisticated ways before converting the digital output to the desired final analog signal." One key goal of our discussion will precisely be to analyze when and how a desired continuous-time system can be implemented in such a way as the cascade sampler—discrete-time system—conversion to continuous-time signal.

Sampling Of Band-Limited Signals

Definition. *A continuous-time signal $s(t)$ is called* band-limited *(sometimes also* base-band*) if its spectrum (i.e., its Fourier transform) $S(j\omega)$ satisfies*

$$S(j\omega) \quad = \quad 0, \ \textit{for } |\omega| > \omega_M. \tag{1}$$

The key insight of the sampling theorem is to prove that *any* band-limited signal $s(t)$ is — surprisingly — *uniquely* specified by its values sampled at times $t = kT_s$, where $T_s = \pi/\omega_M$ and k goes through all integers, $-\infty < k < \infty$. The constant T_s is called the sampling interval, and we will call $\omega_s = 2\pi/T_s = 2\omega_M$ the sampling frequency. Note that the sampling frequency is *twice* the highest frequency in the sampled signal.

This can be proved in various ways. We discuss two of them in class: the first is explained in this handout, and the second in Section 7.1 of the textbook.

Derivation Of The Sampling Theorem

The key idea of our derivation is to write the spectrum $S(j\omega)$ as a Fourier series.

To do this, consider $\tilde{S}(j\omega)$, which is just the periodic repetition of $S(j\omega)$, with period $2\omega_M$. The original, band-limited spectrum $S(j\omega)$ and the periodic repetition $\tilde{S}(j\omega)$ are sketched in Figure 1 for a simple example. But we have seen in class that any periodic signal can be expressed as a Fourier series. For the case at hand, the fundamental period is $2\omega_M$, and the fundamental frequency is $2\pi/(2\omega_M) = \pi/\omega_M$, and we can write

$$\tilde{S}(j\omega) \quad = \quad \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{\pi}{\omega_M}\omega}, \tag{2}$$

where the coefficients c_n are given, as usual, by

$$c_n \quad = \quad \frac{1}{2\omega_M}\int_{-\omega_M}^{\omega_M} \tilde{S}(j\omega)e^{-jn\frac{\pi}{\omega_M}\omega}d\omega = \frac{1}{2\omega_M}\int_{-\omega_M}^{\omega_M} S(j\omega)e^{-jn\frac{\pi}{\omega_M}\omega}d\omega. \tag{3}$$

The last equality holds because over the interval $-\omega_M < \omega < \omega_M$, we have that $\tilde{S}(j\omega) = S(j\omega)$. For the derivation of sampling, we do not need these formulae for c_n.

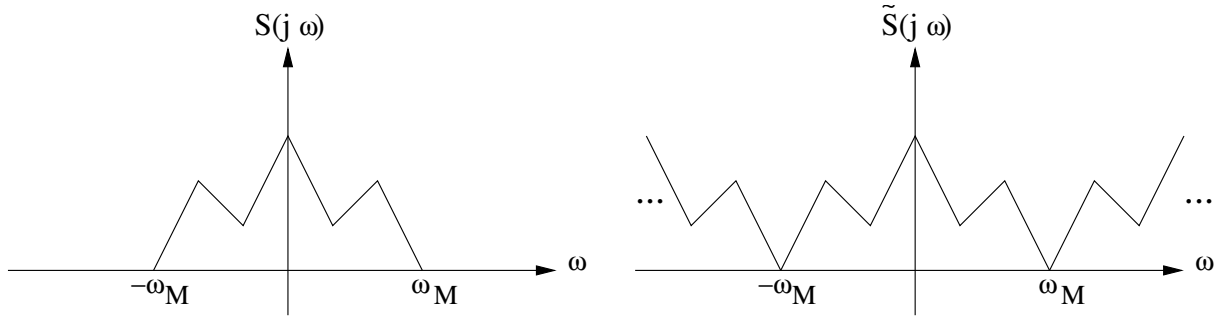

Figure 1: An example of a band-limited signal $s(t)$. Its spectrum $S(j\omega)$ is shown on the left, and the periodic extension $\tilde{S}(j\omega)$ used in our proof of the sampling theorem is shown on the right.

The signal $s(t)$ can, of course, be written as

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\omega)e^{j\omega t}d\omega \tag{4}$$

$$= \frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} S(j\omega)e^{j\omega t}d\omega, \tag{5}$$

since $S(j\omega)$ is zero when $|\omega| > \omega_M$. But over this interval, $S(j\omega)$ and $\tilde{S}(j\omega)$ are equal, and so we can write

$$s(t) = \frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} \tilde{S}(j\omega)e^{j\omega t}d\omega \tag{6}$$

$$= \frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} \left(\sum_{n=-\infty}^{\infty} c_n e^{jn\frac{\pi}{\omega_M}\omega}\right) e^{j\omega t}d\omega \tag{7}$$

The next key step is to swap the summation and the integration to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} c_n \left(\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} e^{jn\frac{\pi}{\omega_M}\omega} e^{j\omega t}d\omega\right). \tag{8}$$

Now, let's look at the signal only at the points $t = k\pi/\omega_M$, where k is any integer, $-\infty < k < \infty$. We find

$$s(t = k\pi/\omega_M) = \sum_{n=-\infty}^{\infty} c_n \left(\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} e^{j\omega\left(\frac{k\pi}{\omega_M}+\frac{n\pi}{\omega_M}\right)}d\omega\right). \tag{9}$$

The integral is zero *unless* $k = -n$. (No magic here: just solve the integral!) When $k = -n$, the integral expression in parentheses evaluates to ω_M/π, and we find

$$s(k\pi/\omega_M) = \frac{\omega_M}{\pi}c_{-k}. \tag{10}$$

That's great! This is the heart of the sampling theorem: Apparently, merely knowing the signal at times $k\pi/\omega_M$ gives you *all* the coefficients c_k. But those coefficients give you the full

spectrum $S(j\omega)$. Of course, once you know the spectrum $S(j\omega)$, it is an easy game to get back the full signal $s(t)$. This proves the sampling theorem:

Theorem (Nyquist-Shannon sampling theorem). *An arbitrary (real or complex) base-band signal $s(t)$ of bandwidth ω_M (i.e., $S(j\omega) = 0$ for $|\omega| > \omega_M$) is uniquely determined by its samples taken at regular intervals of length $T_s = \pi/\omega$, i.e., at a sampling frequency of $\omega_s = 2\pi/T_s = 2\omega_M$.*

Note in particular that one has to sample at *twice* the highest frequency in the considered signal.

Another derivation of the same sampling theorem is the so-called *Impulse-train Sampling*, as described in Section 7.1.1 of the textbook. This will be discussed in Lecture 10.

Reconstruction From The Samples

The theorem says that the base-band signal $s(t)$ is uniquely *specified* by the samples $s(k\pi/\omega_M)$. But, given these samples, how can we reconstruct the signal $s(t)$?

This is very simple. As we have seen above, we can write $s(t)$ as

$$s(t) = \sum_{n=-\infty}^{\infty} c_n \left(\frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} e^{jn\frac{\pi}{\omega_M}\omega} e^{j\omega t} d\omega \right), \tag{11}$$

where the coefficients c_n are simply the samples,

$$c_n = \frac{\pi}{\omega_M} s(-n\pi/\omega_M). \tag{12}$$

To have a nicer expression, let us substitute, in the sum, $k = -n$. Then, the signal can be written as

$$s(t) = \sum_{k=-\infty}^{\infty} \frac{\pi}{\omega_M} s(k\pi/\omega_M) \left(\frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} e^{-jk\frac{\pi}{\omega_M}\omega} e^{j\omega t} d\omega \right). \tag{13}$$

The integral in parentheses is an old acquaintance of ours: It is the inverse Fourier transform of a box function, multiplied by the factor $e^{-jk\frac{\pi}{\omega_M}\omega}$. That is, we have to find the inverse Fourier transform of the following spectrum:

$$e^{-jk\frac{\pi}{\omega_M}\omega} B_{\omega_M}(\omega), \tag{14}$$

where $B_{\omega_M}(\omega)$ is the box function of height 1 and width $2\omega_M$, centered at the origin.

The inverse Fourier transform of the box function is the sinc:

$$b_{\omega_M}(t) = \frac{\omega_M}{\pi} \operatorname{sinc}\left(\frac{\omega_M t}{\pi} \right), \tag{15}$$

where, as defined in class, $\operatorname{sinc}(x) = (\sin(\pi x))/(\pi x)$.

But by the time shifting property of the Fourier transform, the inverse Fourier transform of

$$e^{-jk\frac{\pi}{\omega_M}\omega} B_{\omega_M}(\omega) \tag{16}$$

is simply

$$b_{\omega_M}\left(t - k\frac{\pi}{\omega_M}\right) = \frac{\omega_M}{\pi}\operatorname{sinc}\left(\frac{\omega_M}{\pi}\left(t - k\frac{\pi}{\omega_M}\right)\right) \tag{17}$$

$$= \frac{\omega_M}{\pi}\operatorname{sinc}\left(\frac{\omega_M}{\pi}t - k\right) \tag{18}$$

Using this in Equation (13), we get

$$s(t) = \sum_{k=-\infty}^{\infty} s(k\pi/\omega_M)\operatorname{sinc}\left(\frac{\omega_M}{\pi}t - k\right). \tag{19}$$

Hence, to reconstruct $s(t)$ from the samples, we simply add up shifted copies of the sinc function, each weighted by the corresponding sample.