EECS 120 Signals & Systems — University of California, Berkeley: Spring 2004
Gastpar — February 13, 2004

Homework 4
Due: Wednesday, February 25, 2004, at 4pm

Reading OWN Chapter 4: Sections 4.2, 4.3, 4.4, 4.5; Chapter 5: Sections 5.1, 5.2, 5.3.

Practice Problems *(Suggestions.)* OWN 4.11, 4.12, 4.19, 5.8, 5.13, 5.16

Problem 1 *(Fourier transform properties.)* 20 Points

(a) (5 Points) OWN Problem 4.25 (b)

(b) (5 Points) OWN Problem 4.25 (e)

(c) (10 Points) OWN Problem 4.23

Problem 2 *(Connections between Fourier representations.)* 15 Points

(a) (5 Points) OWN Problem 4.28 (a)

(b) (10 Points) OWN Problem 4.28 (b) only Part (x).

Problem 3 *(Discrete-time Fourier transform.)* 25 Points

(a) (5 Points) OWN Problem 5.29 (b) Part (i) only. It is sufficient to give the DTFT of the output.

(b) (5 Points) OWN Problem 5.29 (c)

(c) (15 Points) OWN Problem 5.35 (a), (c), (d)

Problem 4 *(Frequency response of linear time-invariant system.)* 25 Points

Let a system be specified by a differential (or difference) equation. Then, its frequency response can be found easily, as you establish in this homework problem.

(a) (5 Points) In the continuous-time case, suppose that the LTI system is specified by

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) \quad = \quad \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t). \tag{1}$$

Solve the differential equation for the input $x(t) = e^{j\omega t}$. As we have seen in class, if the input to a continuous-time LTI system is $x(t) = e^{j\omega t}$, then the output can be expressed as $y(t) = H(j\omega)e^{j\omega t}$. Plug this into the above differential equation to determine $H(j\omega)$. You will obtain an expression for $H(j\omega)$ in terms of the coefficients $a_k, k = 0, \ldots, N$ and $b_m, m = 0, \ldots, M$.

(b) (5 Points) In the discrete-time case, suppose that the LTI system is specified by

$$\sum_{k=0}^{N} a_k y[n-k] \quad = \quad \sum_{m=0}^{M} b_m x[n-m]. \tag{2}$$

Solve the differential equation for the input $x[n] = e^{j\omega n}$. As we have seen in class, if the input to a discrete-time LTI system is $x[n] = e^{j\omega n}$, then the output can be expressed as $y[n] = H(e^{j\omega})e^{j\omega n}$. Plug this into the above differential equation to determine $H(e^{j\omega})$. You will obtain an expression for $H(e^{j\omega})$ in terms of the coefficients $a_k, k = 0, \ldots, N$ and $b_m, m = 0, \ldots, M$.

(c) (10 Points) In Homework 3, Problem 4, you solved a simple differential equation. For the system specified by the differential equation, i.e.,

$$5\frac{dy(t)}{dt} + 10y(t) \quad = \quad x(t), \tag{3}$$

determine the frequency response $H(j\omega)$, and the corresponding impulse response $h(t)$. Then, find the output when the input is $x(t) = \cos(t/5)$. *Hint:* Write $\cos(t/5)$ in terms of functions of the form $e^{j\omega_0 t}$, and recall from class that for such inputs, the output is imply given by $y(t) = H(j\omega_0)x(t)$.

(d) (5 Points) Draw a block diagram involving only unit time delays, additions, and multiplications by constant factors, of the discrete-time LTI system characterized by the following frequency response:

$$H(e^{j\omega}) \quad = \quad \frac{1 + 2e^{-j\omega} - 3e^{-2j\omega} + e^{-3j\omega}}{2 + 3e^{-j\omega} - 5e^{-2j\omega} + 7e^{-3j\omega} - 9e^{-4j\omega}}. \tag{4}$$

Try using as few delay elements as possible. (Recall that delay elements are memory, and hence somewhat expensive.) *Hint:* Determine the difference equation of the system.

Problem 5 *(Discrete-time periodic signals and LTI systems, continued)* 15 Points

In Homework 3, Problem 5, we found that the processing of *periodic* signals (with period N) through LTI systems can be easily understood as a matrix multiplication in N-dimensional vector space. Namely, the input vector $\mathbf{x}$ is multiplied by an appropriate *circulant* matrix H_N to produce the output vector $\mathbf{y}$.

(a) As we have reviewed in Homework 2, Problem 3, any (square) matrix H_N can be written in terms of its *eigendecomposition* as

$$H_N \quad = \quad ULU^H, \tag{5}$$

where L is a diagonal matrix and U is a unitary matrix, i.e.,

$$U^H U \quad = \quad I_N, \tag{6}$$

where I_N is the $N \times N$ identity matrix. Fix $N = 9$. Take the impulse response $h[n]$ of Homework 3, Problem 5, Part *(b)*,

$$h[n] \quad = \quad \delta[n] + 0.6\delta[n-1] + 0.3\delta[n-2] + 0.1\delta[n-3], \tag{7}$$

and call the corresponding matrix `H9a` in matlab. Use the matlab function `eig` to determine `Ua` and `La`, as follows:

```
> [ Ua, La ] = eig(H9a)
```

Convince yourself that this is indeed the eigendecomposition, as follows:

```
> Ua' * H9a * Ua
```

This should be a diagonal matrix, naemly, precisely the matrix `La`. Moreover, verify

```
> Ua' * Ua
> Ua * Ua'
```

Both of these should be the identity matrix. Now you can be sure that `Ua` and `La` are indeed the eigenvectors and eigenvalues, respectively, of `H9a`.

Then, take the impulse response $h[n]$ of Homework 3, Problem 5, Part *(d)*, use the program given in Homework 3, Problem 5, Part *(d)*, to find the corresponding matrix H_9. Call this matrix `H9b` and determine the matrices `Ub` and `Lb`, and repeat the verification steps described above. Finally, pick an impulse response $h[n]$ yourself, find the corresponding `H9c` and determine the matrices `Uc` and `Lc`.

Hand in printouts of the following three matrix products:

$$\texttt{Ua' * H9a * Ua}\ ,\quad \texttt{Ub' * H9a * Ub}\ ,\quad \texttt{Uc' * H9a * Uc}\ . \tag{8}$$

That is, you apply all the eigendecompositions to the matrix `H9a`. What do you observe? What can you conclude about circulant matrices?

(b) Write a matlab program that produces the matrix F_N whose element in row m and column n is

$$\{F_N\}_{nm} = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}(m-1)(n-1)}. \tag{9}$$

This matrix is often called the *Fourier matrix.* Hand in a printout of the matrix F_9, and verify that H_9 is a unitary matrix. Hand in printouts of the following three matrix products:

$$\texttt{F9' * H9a * F9}\ ,\quad \texttt{F9' * H9b * F9}\ ,\quad \texttt{F9' * H9c * F9}\ . \tag{10}$$

What do you observe? What are the eigenvectors of any circulant matrix? Compare the Fourier matrix to the matrices `Ua, Ub, Uc`.

(c) We started this problem by considering a discrete-time periodic signal $x[n]$ of period N, consisting of the infinite repetition of the vector $\mathbf{x}$. Show that the discrete-time Fourier series $X[k]$ of $x[n]$ is given by the infinite repetition of the vector $\mathbf{X}$, given by

$$\mathbf{X} = \frac{1}{\sqrt{N}} F_N^H \mathbf{x}. \tag{11}$$

Then, we showed that when this N-periodic signal $x[n]$ is the input of an LTI system, then the output $y[n]$ is also N-periodic and can be expressed as the infinite repetition of the vector $\mathbf{y}$, given by

$$\mathbf{y} = H_N \mathbf{x} \tag{12}$$

and we saw that the matrix H_N has a special structure called *circulant*. As we have seen in Parts (a) and (b) of this problem, this implies that H_N can actually be written as

$$H_N = F_N \Lambda F_N^H, \tag{13}$$

where F_N is the Fourier matrix, and Λ is a diagonal matrix.

Express the relationship given in Equation (12) in terms of the vectors $\mathbf{X} = \frac{1}{\sqrt{N}} F_N^H \mathbf{x}$ and $\mathbf{Y} = \frac{1}{\sqrt{N}} F_N^H \mathbf{y}$. What is the computational advantage of the resulting form of Equation (12)?

Hint: Note that you can rewrite as $\mathbf{x} = \sqrt{N} F_N \mathbf{X}$, and $\mathbf{y} = \sqrt{N} F_N \mathbf{Y}$, and simply plug this into Equation (12).