Homework 2
Due: Thursday, February 3, 2005, at 11:30am

Reading OWN Chapter 2.

You may work in (small) groups to do the homework, but each person must write up their own answers. Note that working together does not mean dividing up the problems and sharing answers later.

For any Matlab problems, submit computer generated plots and code.

This homework may look long, but don't get discouraged! A lot of the length comes from hints and explanations. Most questions are not computationally intensive.

Problem 1 *(Graphical Convolution.)*

Graphically convolve (flip-and-drag method) the following pairs of signals. You do not need to write the equations for your results, but clearly label and scale your axes.

(a) $x_1(t) = 5\delta(t+3) + 9\delta(t-6)$

$$x_2(t) = \begin{cases} 1, & -1 \le t \le 0 \\ 1-t, & 0 \le t \le 1, \\ 0, & |t| > 1 \end{cases}$$

(b) $x_1(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

$x_2(t) = \sum_{k=-\infty}^{\infty} x_1(t+4k)$

(c) $x_1(t) = u(t)$

$$x_2(t) = \begin{cases} 1-|t|, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$

Problem 2 *(Continuous time convolution.)*

Disclaimer: This problem is not about probability. It is just stated in terms of a probability question to show that convolution is an important operator that shows up in places other than LTI systems. Probability is by no means a part of this course or a prerequisite.

If X and Y are independent random variables, then their sum, $Z = X + Y$, is also a random variable. A random variable has a function called a probability density function (pdf) from which all of its statistical properties can be derived. You can think of a pdf as just being a non-negative signal that integrates to 1. If X has the pdf $f_x(x)$ and Y has the pdf $f_y(y)$, then Z has the pdf $f_z(z) = f_x(x) \star f_y(y)$.

(a) Consider the following independent random variables:

W= either 2 or -2, with equal probability
(consider a fair coin flip)
X=a random real number between 0 and 1, all with equal probability
(We say X is distributed uniformly between 0 and 1. This is what most programming languages give you if you ask for a random number.)
Y=another random real number between 0 and 1

These variables have the following pdfs:

$$\begin{aligned} f_W(w) &= \frac{1}{2}\delta(w-2) + \frac{1}{2}\delta(w+2) \\ f_X(x) &= \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \\ f_Y(y) &= \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Sketch the pdfs of W, X, Y, $W+X$, and $X+Y$. No formal derivation is necessary, just sketches. Please put a scale on your pdfs, however.

(b) We say X has a normal (or Gaussian) distribution with mean 0 and variance $(\sigma_1)^2$ if its pdf is:

$$f_x(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma_1^2}\right)$$

We signify this by writing $X \sim \mathcal{N}(0, \sigma_1^2)$. Let $Y \sim \mathcal{N}(0, \sigma_2^2)$. Convolve (show your work!) the pdfs of X and Y to find the pdf of $X + Y$. (HINT: complete the square.) What is this pdf called? What is its mean and variance?

Note: this result is one of the reasons Gaussians are one of the "nicest" random variables to work with.

Problem 3 *(Discrete Time Convolution.)*

(a) OWN 2.21a

(b) OWN 2.21b

(c) OWN 2.21d

Problem 4 *(Differential and Difference Equations.)*

(a) OWN 2.32

(b) OWN 2.61

Problem 5 *(LTI systems and Impulse responses.)*

Disclaimer: Although almost all parts of this problem pertain to both continuous time and discrete time LTI systems, we will focus on continuous time LTI systems to be concrete.

LTI systems, convolution and impulse responses are intimately connected. In class, you have seen that any LTI system can be viewed as a box which convolves the input signal with the system's 'impulse response'. That is, if $x(t)$ is the input, $y(t)$ is the output, and $h(t)$ is the impulse response, we must have that $y(t) = x(t) \star h(t)$. Conversely, if we were given an impulse response $h(t)$, there is an LTI system that has this impulse response. In this problem, we will explore some aspects of the relation between the Dirac delta function, impulse responses, LTI systems, and convolution.

(a) Define the system that differentiates its input as follows.

$$\begin{aligned} y(t) &= \frac{d}{dt}x(t) \\ &\triangleq \lim_{\Delta t \to 0^+} \frac{x(t) - x(t - \Delta t)}{\Delta t} \end{aligned}$$

Show that this system is LTI. (The impulse response of this system is called the unit doublet (OWN 2.5.3.)

(b) The Dirac delta function[1] $\delta(t)$ is a special signal to convolve with another signal. As done in OWN, we can give an operational definition of the Dirac delta function.

Namely, if $|t| > 0$, then $\delta(t) = 0$, and $\int_{-\infty}^{\infty} \delta(t)dt = 1$. Using this operational definition, prove the so-called sifting property of the delta function. That is if t_0 is a real number, show the following.

$$x(t) \star \delta(t - t_0) = x(t - t_0)$$

(c) Define $f(t)$ as below. What function is this?

$$f(t) \triangleq \begin{cases} \int_{-\infty}^{t} \delta(\tau)d\tau, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

(d) Sometimes it is difficult or even impossible to get an accurate approximation for the impulse response of an LTI system because of physical characteristics of the system (such as extremely high resonant frequencies in a circuit). An old hermit you meet while hiking in Tilden Park tells you that you can get the impulse response of an LTI system from the step response of the system (the step response is the output of the system when the input of the system is a unit step). Even though he doesn't look like he knows anything about signals and systems, justify why we should believe him.

[1] Note that the Dirac delta function is not really a function. When we speak of the Dirac delta, we are talking about a generalized distribution. However, we can think of it being a pulse of infinitesimal width Δ and height $1/\Delta$ centered at 0. If you like, you may do this problem using this as your Dirac delta.

Problem 6 *(Matlab, vector bases, matched filters, transforms, and more!)*

Record the answers to each part in your homework. Turn in all m-files.

Go to the ee120 homepage (`http://inst.eecs.berkeley.edu/~ee120/`). Download basis.mat. While you're at it, go to the newsgroup (`ucb.class.ee120`). That's where we post the homework hints.

Import all the variables in basis.mat into Matlab. Use the command `load('basis.mat')`.

Recall that any four linearly independent vectors form a basis for $\mathbb{R}^4$. The file basis.mat contains 4 such vectors, $\texttt{v1}, \texttt{v2}, \texttt{v3}, \texttt{v4}$.

(a) Use Matlab to compute $\|\vec{v}_1\|, \|\vec{v}_2\|, \|\vec{v}_3\|, \|\vec{v}_4\|$. Throughout this problem, assume that any values less than 10^{-10} are rounding errors.

Compute the dot product between every pair of vectors. Using the formula

$$\vec{v}_1^T\vec{v}_2 = \sum_{i=1}^{4} v_1(i) * v_2(i) = \vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\|\|\vec{v}_2\|\cos(\theta)$$

where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$, find the angle between every pair of the given vectors.

You should have discovered that the vectors form a special kind of basis. What is the term for this type of basis?

(b) Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ form a basis for $\mathbb{R}^4$, any vector in $\mathbb{R}^4$ can be expressed as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3,$ and $\vec{v}_4$. In fact, because of your answer to (a), any such expression will be unique.

Let $\vec{u} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4$. Using the linearity of vector multiplication and transposition, and your results from *(a)*, determine $\vec{u}^T\vec{v}_1$, $\vec{u}^T\vec{v}_2$, $\vec{u}^T\vec{v}_3$, and $\vec{u}^T\vec{v}_4$. Justify your answer.

(c) The workspace basis.mat also contains three more vectors, $\vec{x}_1, \vec{x}_2, \vec{x}_3$, in $\mathbb{R}^4$. We know that each of these can be expressed in the form $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4$. Write a program to find the alphas for each of the x_i vectors.

(d) The workspace basis.mat also contains the vectors $\vec{d}_1, \vec{d}_2, ..., \vec{d}_{32}$. These complex vectors form an orthonormal basis for $\mathbb{C}^{32}$. Multiply a few of the vectors together and verify that the basis is, in fact, orthonormal. Plot the real and imaginary parts of a few of these basis vectors. What do they look like?

Now suppose I told you I had a signal vector of length 32. We know we can express it as a linear combination of our basis vectors. Write a few sentences explaining how you would go about finding the coefficient of each basis vector. HINT: consider your answer to part *(c)*.

(e) Look at the formula (eqn. 3.95) for the discrete-time Fourier series coefficients in OWN section 3.6. Intuitively, what's going on there? HINT: don't worry about the negative sign in the exponent; it comes from a conjugation that we have to perform when we're dealing with complex vectors.