Chapter 3

Modulation exercises

Each problem is annotated with the letter E, T, C which stands for exercise, requires some thought, requires some conceptualization. Problems labeled E are usually mechanical, those labeled T require a plan of attack, those labeled C usually have more than one defensible answer. Recall the representation of bandlimited signals.

**Definition** $x(t), -\infty < t < \infty$, is a real bandlimited signal with carrier $f_c$ (Hz) if $|X(f)| = 0$, $|f - f_c| > \frac{1}{2}B_x$ and $B_x << f_c$ as shown in Figure 3.1.

![Figure 3.1: FT of bandlimited signal](image)

**Theorem** Let $x$ be a bandlimited signal, and $\hat{x}$ its Hilbert transform. Then $x, \hat{x}$ can be represented as

\[
\forall t, \quad x(t) = A(t) \cos[2\pi f_c t + \theta(t)] \\
\hat{x}(t) = A(t) \sin[2\pi f_c t + \theta(t)]
\]
Moreover, the amplitude $A(t)$ and phase $\theta(t)$ can be obtained from $x, \dot{x}$ and the carrier signal as follows. First obtain the complex baseband signal $z$:

$$z(t) = [x(t) + j\dot{x}(t)]e^{-j2\pi f_c t},$$

then

$$A(t) = |z(t)|, \quad \theta(t) = \arg z(t).$$

Moreover, $|Z(f)| = 0, |f| > \frac{1}{2}B_x.$

In the context of modulation, $A$ and $\theta$ are the modulating signals, and $x$ is the modulated (transmitted) signal with carrier $f_c$. Thus the demodulator, after receiving $x$, must first obtain $\dot{x}$, then $z$, and then recover $A$ and $\theta$.

1. E What is $z$ in the following two cases where we use $f_c = 100$ Hz.

   (a) $x(t) = \cos(2\pi \times 99t) + \sin(2\pi \times 101t)$.

   (b) $x$ is given in Figure 3.2.

![Figure 3.2: Bandlimited signal in Problem 1](image)

2. T Repeat Problem 1 but now use $f_c = 105$ Hz.

3. T Given a modulating signal $f$ that is bandlimited to $W$ Hz, a carrier $\cos(\omega_c t)$ with $\omega_c >> 2\pi W$, a nonlinearity $g$, and a band-pass filter that only passes frequencies withing $-\omega_c \pm 2\pi W$ or $\omega_c \pm 2\pi W$, we want to build a modulator that has output

$$x(t) = A[1 + \beta f(t)] \cos \omega_c t$$

by combining the components as in Figure 3.3. Assume $|f(t)| << 1$. Compute $\beta$ for the following nonlinearities.

   (a) $g(x) = \begin{cases} x, & x < 0 \\ 0, & x \geq 0 \end{cases}$
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![Figure 3.3: Modulation scheme in Problem 2](image)

(b)

\[ g(x) = \begin{cases} 
  x, & |x| < 1/2 \\ 
  0, & |x| \geq 1/2 
\end{cases} \]

4. E Consider a narrow band signal of the form

\[ \forall t, \quad x(t) = \cos(\omega_c t + \theta(t)) \]

Assume \( X(\omega) \) is zero except where \( |\omega| - \omega_c | < 2\pi W \). (Hint. Below you may use \( |\omega| = \arg(j\omega) - j \text{sgn}\omega\).)

(a) Find \( y, z \) in the upper arrangement of Figure 3.4.

(b) Find \( z \) in the lower arrangement. Here the bandpass filter only passes the frequencies \( |\omega| - \omega_c | < 2\pi W \).

![Figure 3.4: Deodulation schemes in Problem 4](image)

5. T The arrangement in Figure 3.5 is a scrambler (i-iv) followed by a descrambler (v-vii). The single sideband modulators generate, for an input signal \( f \), the output signal \( \phi(t) = f(t) \cos W t - \dot{f}(t) \sin W t \). The LPF has frequency response \( H(\omega) = 1, |\omega| \leq W \), and \( H(\omega) = 0 \), otherwise. The signal \( m \) at input i is given by its Fourier transform \( M \) in the lower part of the figure.

(a) Give graphical representations in the frequency domain of the signals at ii-vii.
(b) Explain in one sentence in which sense the signal at iv is a scrambled version of the input signal.

6. Consider the PAM transmission system (you don’t need to know what PAM is) in which the frequency response of the channel and the LPF combined is given by

\[ H(\omega) = \begin{cases} \frac{1}{2}(1 + \cos \frac{\omega}{2}), & |\omega| < 2\pi \\ 0, & |\omega| \geq 2\pi \end{cases} \]

(a) Find the (non-causal) impulse response of this system.
(b) Let the input to the PAM system be

\[ x(t) = 5\delta(t) + \delta(t - 1) + 4\delta(t - 2). \]

Use the result of the previous part to obtain a formula for the output \( y \).
(c) What are the values of \( y(0), y(1), y(2), y(3) \)?
(d) If the clock at the receiving end is delayed by \( \tau \) seconds the sampled output is \( y(\tau), y(1 + \tau), y(2 + \tau), y(3 + \tau) \). Let \( \tau = 0.2 \). Use your calculator to compute one of these values.

7. A test channel for binary transmission has the frequency response \( H(\omega) = \frac{1}{\alpha + i\omega} \). The input to the channel is given in Figure 3.6. The input on the left corresponds to the binary input 01 (case A), and on the right corresponds to 11 (case B). The second binary digit is sampled at the output at \( t = 1.5 \).

(a) The output at \( t = 1.5 \) will depend on whether the first binary digit is 0 or 1, and denote it as \( y_A(1.5) \) and \( y_B(1.5) \), respectively. What is the lowest possible value of \( \alpha \) if we have the requirement

\[ \frac{y_B(1.5) - y_A(1.5)}{y_A(1.5)} \leq 0.01. \]
8. \textbf{T (PSK Spectra)} Consider phase shift keying using phase deviation constant $\phi_\Delta = \pi$. That is a “0” is represented by $\cos(2\pi f_c t)$ and a “1” is represented by $\cos(2\pi f_c t + \pi)$. Assume a carrier frequency of 10.0 MHz. Assume data is sent at the rate of 1000 bits per second.

(a) Sketch the magnitude spectrum of the PSK signal for the periodic bit sequence 01010101... (Hint: the bit sequence can be represented by a square wave. Choose the square wave to be even in time).

(b) For the sequence above, estimate the bandwidth within which 90\% of the signal power can be found. (Numerical methods are appropriate).

9. \textbf{E (Hilbert Transform)} Let

$$m(t) = \frac{\sin100\pi t}{\pi t} + \frac{\sin200\pi t}{\pi t}.$$ 

Let

$$x(t) = m(t)\cos(10^5\pi t) + \left(\frac{1}{\pi t} * m(t)\right)\sin(10^5\pi t).$$

Sketch the spectra $M(f)$ and $X(f)$. What type of modulation does this system provide?

10. \textbf{T (Asynchronous Demodulation)} Consider detection of a tone-modulated AM double-sideband with carrier (DSB-WC) signal

$$p[x_A(t)] = (1 + \cos2\pi f_m t)\cos2\pi f_c t$$

using an ideal diode followed by low pass filter, where the output of the detector is $x_0(t)$. (As shown in figure 3.7, ideal rectification of an AM DSB-WC signal can be considered as a multiplication by a square wave at the carrier frequency $f_c$). For $f_c = 5 \times 10^4$ Hz and $f_m = 2 \times 10^3$, sketch $x_A(t), x_5(t), x_6(t)$ and their spectra. Explain using a time domain sketch of $x_0(t)$ why the carrier is necessary for this asynchronous demodulation scheme.

11. \textbf{T} This Matlab exercise lets you simulate the asynchronous demodulation described above. Download the \texttt{hwk8.zip} file from the class web page, and follow the instructions to create the vector \texttt{radiotest}, which represents 1.5 seconds of a 50kHz centered radio signal.
(DSB-LC) sampled at 400 thousand samples per second ($\Delta T = 2.5\mu s$). The radio band is from 50kHz to 500kHz and contains several stations.

(a) Plot radiotest and a rectified radiotest ($= x_5$) from 1.0 to 1.05 seconds. (Hint: consider using abs.)

(b) What is the impulse response $h_{lp}(t)$ for a windowed, causal, ideal low pass filter with cutoff 20 kHz? (Choose an appropriate window length.) Sketch $H_{lp}(f)$.

(c) Use conv to filter $x_5(t)$ with $h_{lp}(\pi\Delta T)$. Plot the result from 1.0 to 1.05 seconds. Downsample from 400 kHz to 8 kHz, and play with sound. Describe what you hear, and explain any artifacts you hear.