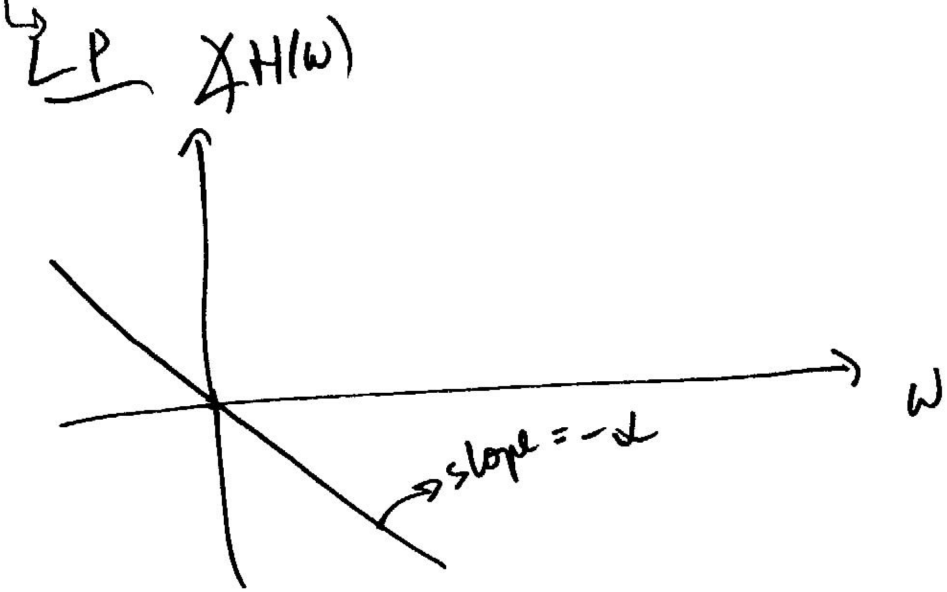
Oct 1, 2003

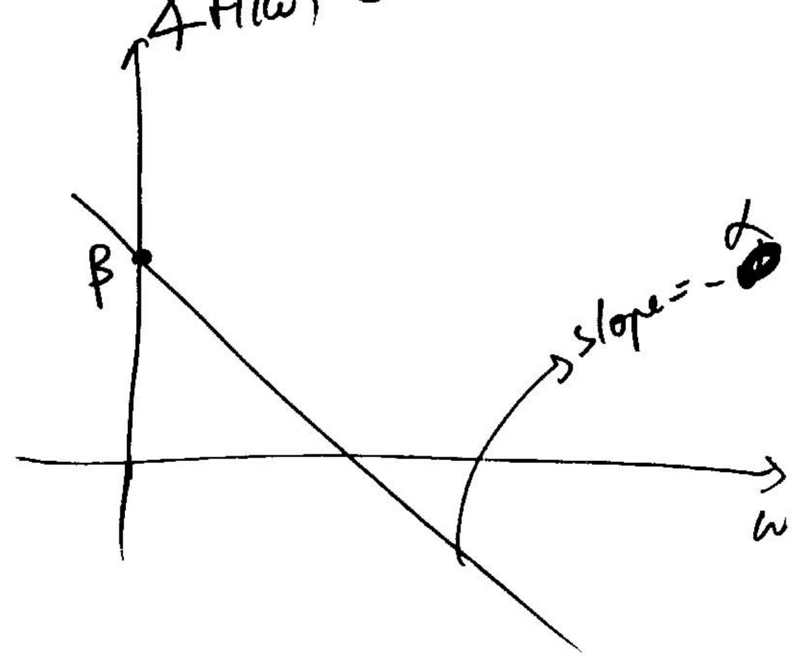
# Linear Phase filtering

**Def:** $H(\omega) \equiv \underbrace{H_m(\omega)}_{\text{Real}} e^{-j\alpha\omega}$ linear phase.

Generalized linear phase:

$$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} e^{j(\beta - \alpha\omega)}$$

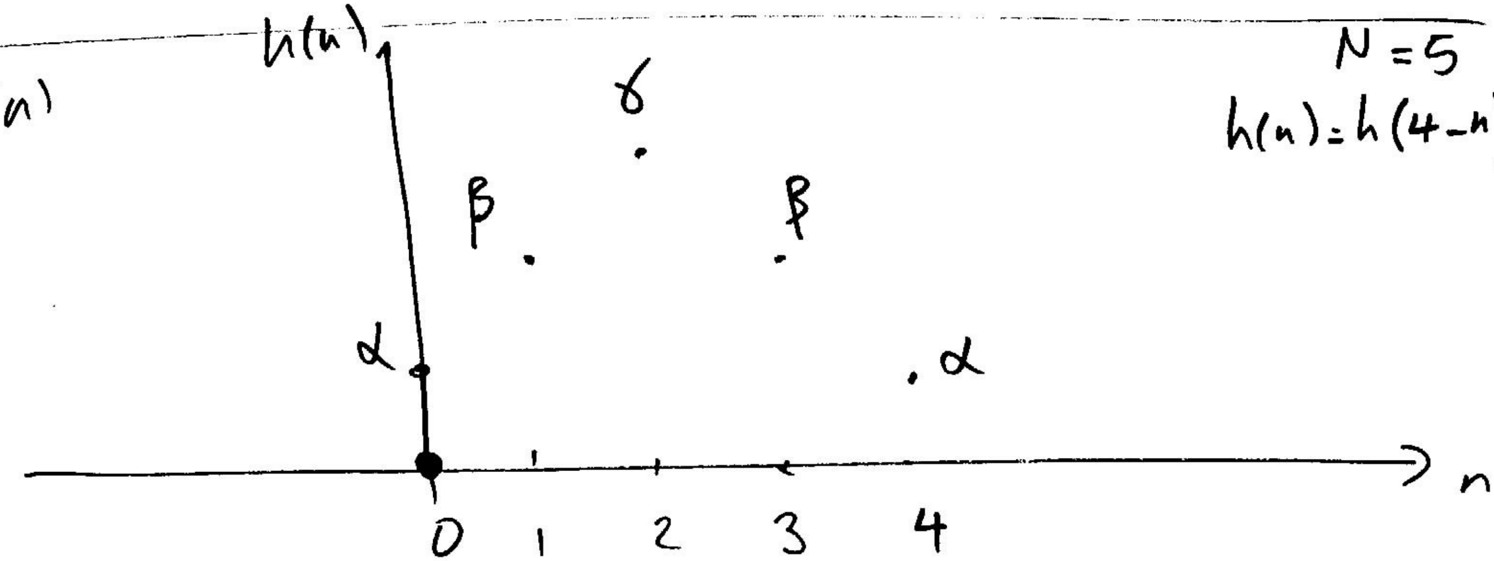


Theme: Impose symmetry or anti symmetry on filter coefficients of FIR $\Longrightarrow$ get linear phase.

Show: If $h(n) = h(N-1-n)$ for FIR filter with real coefficients $\Rightarrow$ Then linear phase.

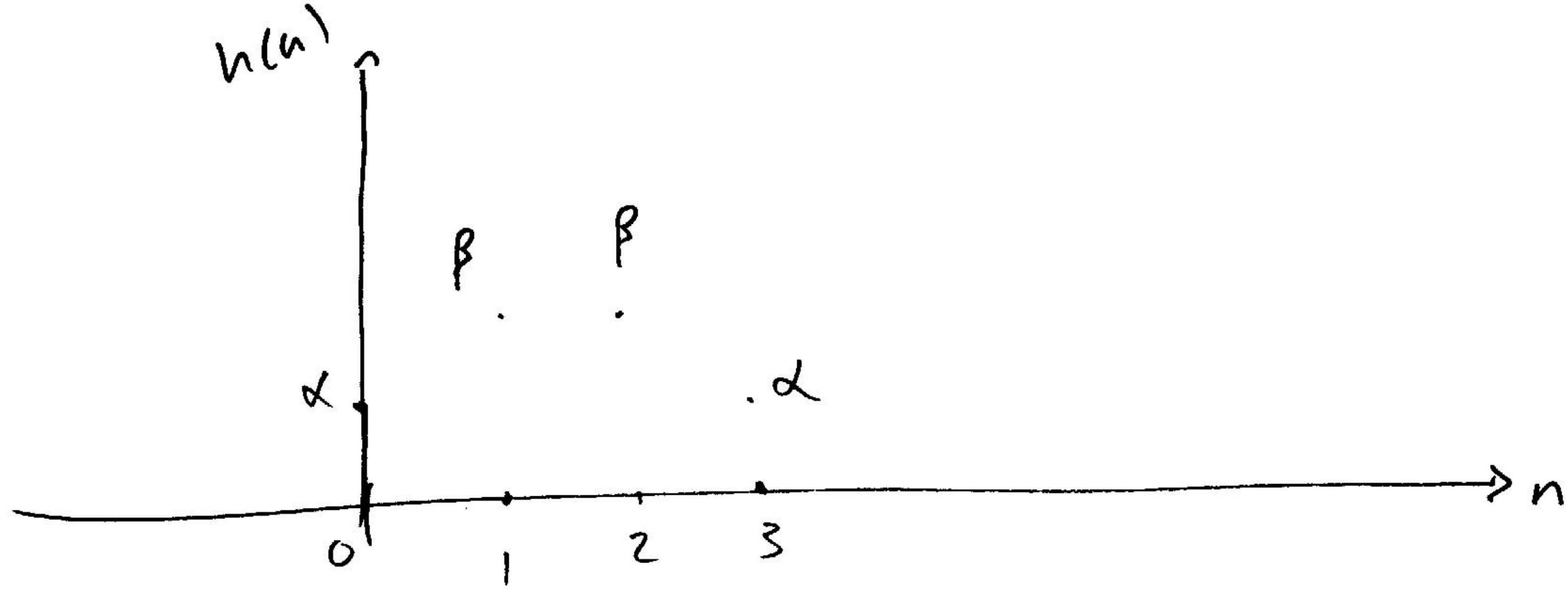
EX $h(n)$



$N = 5$

$h(n) = h(4-n)$

EX $\quad N = 4 \qquad h(n) = h(3-n)$




Proof:
$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

change of variable.
$m = N-1-n$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) e^{-j\omega n} + \sum_{m=0}^{N/2-1} h(N-1-m) e^{-j\omega(N-1-m)}$$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) e^{-j\omega n} + \sum_{n=0}^{N/2-1} \underbrace{h(N-1-n)}_{\substack{\text{assumption} \\ h(n)}} e^{-j\omega(N-1-n)}$$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) \left[ e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[ e^{-j\omega n} e^{+j\omega\left(\frac{N-1}{2}\right)} + e^{j\omega n} e^{-j\omega\left(\frac{N-1}{2}\right)} \right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[ 2 \cos\left(\omega n - \frac{\omega}{2}(N-1)\right) \right]$$

$H_m(\omega) =$ Real quantity.

Compare: $H(\omega) = H_m(\omega)\, e^{-j\alpha\omega}$

$\Longrightarrow \alpha = \dfrac{N-1}{2} \Longrightarrow$ QED

$\Rightarrow$ linear phase

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What is $\measuredangle H(\omega)$?

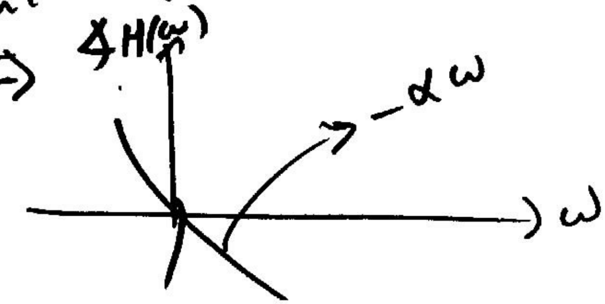
$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real.}}\, e^{-j\alpha\omega}$

$H(\omega) = |H(\omega)|\, e^{j\measuredangle H(\omega)}$

Consider two cases:

① $H_m(\omega) > 0 \Rightarrow |H_m(\omega)| = |H(\omega)|$

$\Longrightarrow$

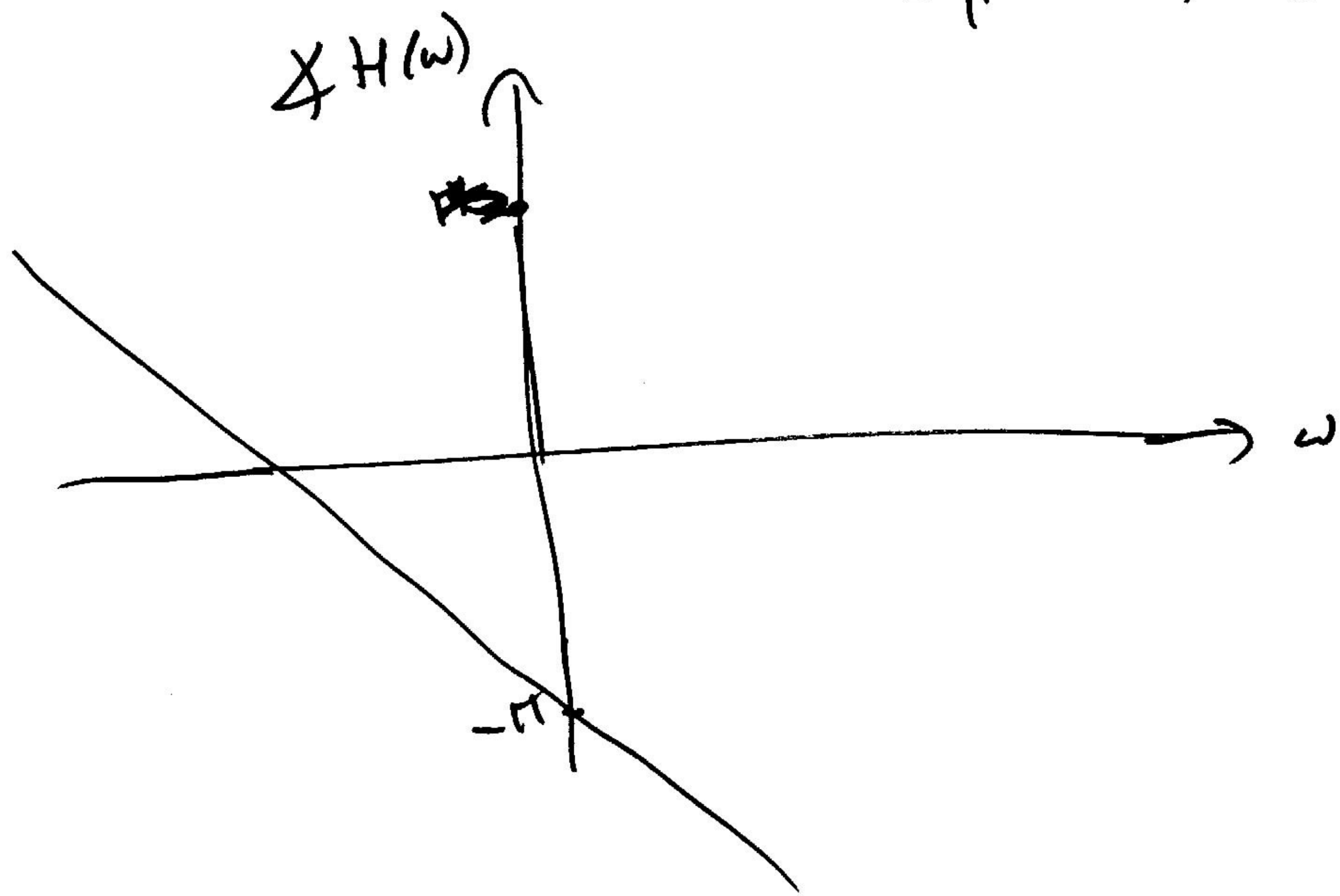


Case ②: $H_m(\omega) < 0 \implies$

$$H(\omega) = |H_m(\omega)|\,(-1)\,e^{-j\alpha\omega}$$

$$= |H_m(\omega)|\,e^{-j\pi}\,e^{-j\alpha\omega}$$

$$= |H_m(\omega)|\,e^{-j(\alpha\omega+\pi)}$$



<u>Show</u>: Symmetry Condition That makes FIR filters become linear phase, also reduces implementation complexity

$h(n) = h(N-1-n)$

$$x(n) \rightarrow \boxed{\begin{array}{l} h(n) \\ \text{FIR} \\ \text{linear phase} \end{array}} \rightarrow y(n)$$

$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{k=N/2}^{N-1} h(k)\, x(n-k)$$

change of variable
$k = N-1-m$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{m=0}^{N/2-1} \underbrace{h(N-1-m)}_{h(m)}\, x(n-N+1+m)$$

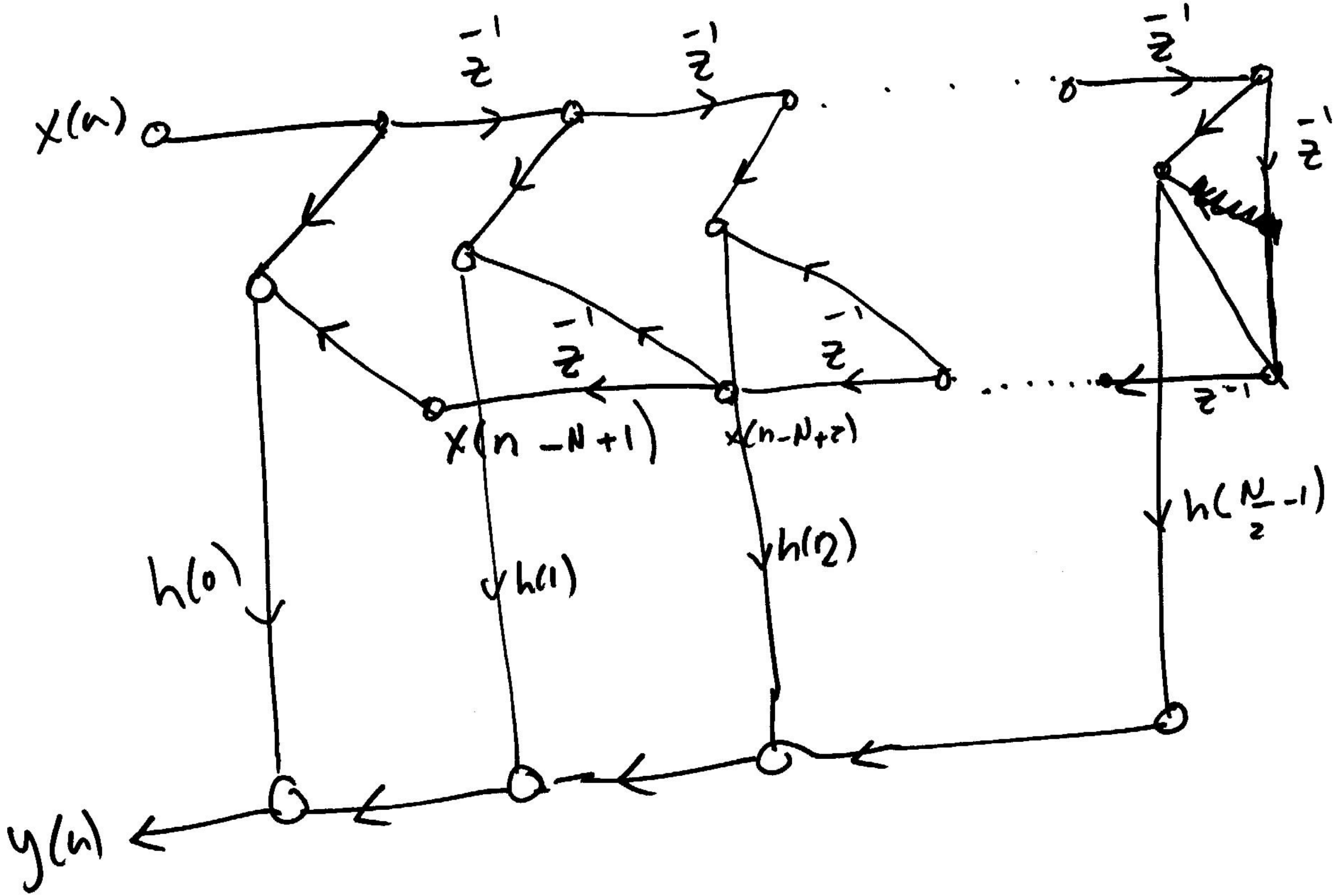
$$y(n) = \sum_{k=0}^{N/2-1} h(k) \left[ \underbrace{x(n-k) + x(n-N+1+k)}_{w(n)} \right]$$

each $n$: $\left.\begin{array}{l} N/2 \text{ mults.} \\ N \text{ adds} \end{array}\right\}$ linear phase.

each $n$: $\left.\begin{array}{l} N \text{ mults} \\ N \text{ adds} \end{array}\right\}$ non linear phase

$\Rightarrow$ Linear phase: $\frac{1}{2}$ as many mults.

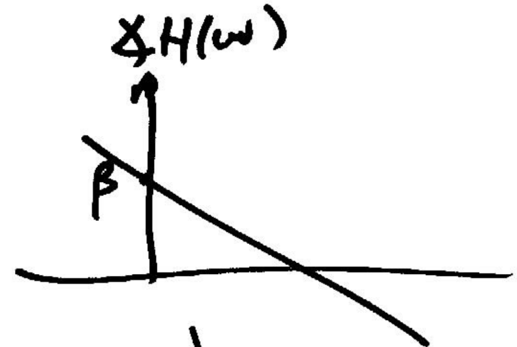
$\Rightarrow$ Fixed point Arith. mults are more expensive than adds.



$x(n)$
$z^{-1}$
$z^{-1}$
$z^{-1}$
$z^{-1}$
$z^{-1}$
$z^{-1}$
$z^{-1}$
$x(n-N+1)$
$x(n-N+2)$
$h(0)$
$h(1)$
$h(2)$
$h(\frac{N}{2}-1)$
$y(n)$

Under what other conditions do we get Linear phase?

GLP: $H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}} \, e^{j(\beta - \alpha\omega)}$




$H(\omega) = H_m(\omega) \cos(\beta - \alpha\omega) + j\, H_m(\omega) \sin(\beta - \alpha\omega)$

$\tan \measuredangle H(\omega) = \dfrac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \tan(\beta - \alpha\omega)$ ← equ (1)

$H(\omega) = \sum_n h(n)\, e^{-j\omega n}$

$H(\omega) = \sum_n h(n) \cos \omega n - j \sum_n h(n) \sin \omega n$

$$\tan \measuredangle H(\omega) = \frac{-\sum_n h(n) \sin \omega n}{\sum_n h(n) \cos \omega n} \quad \leftarrow \text{eqn 2}$$

Equate eqns 1 & 2 so that our filther $h(n)$ becomes GLP filter.

$$\Rightarrow \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)} = \frac{-\sum_n h(n) \sin \omega n}{\sum_n h(n) \cos \omega n}$$

$$\therefore \quad \boxed{\sum_n h(n) \sin\left[\omega(n-\alpha) + \beta\right] = 0}$$

necessary condition for $h(n)$ to be GLP.

2 Cases:

Case 1: $\beta = 0$ or $\pi$

$$\sum_n h(n) \sin(\omega(n-\alpha)) = 0$$

Can show: If $\underline{h(n) = h(2\alpha - n)}$ Then

This is satisfied

Case 2 $\beta = \pi/2$ or $3\pi/2$

$$\sum_n h(n) \cos(\omega(n-\alpha)) = 0$$

Can show: if $\underline{h(2\alpha - n) = -h(n)}$

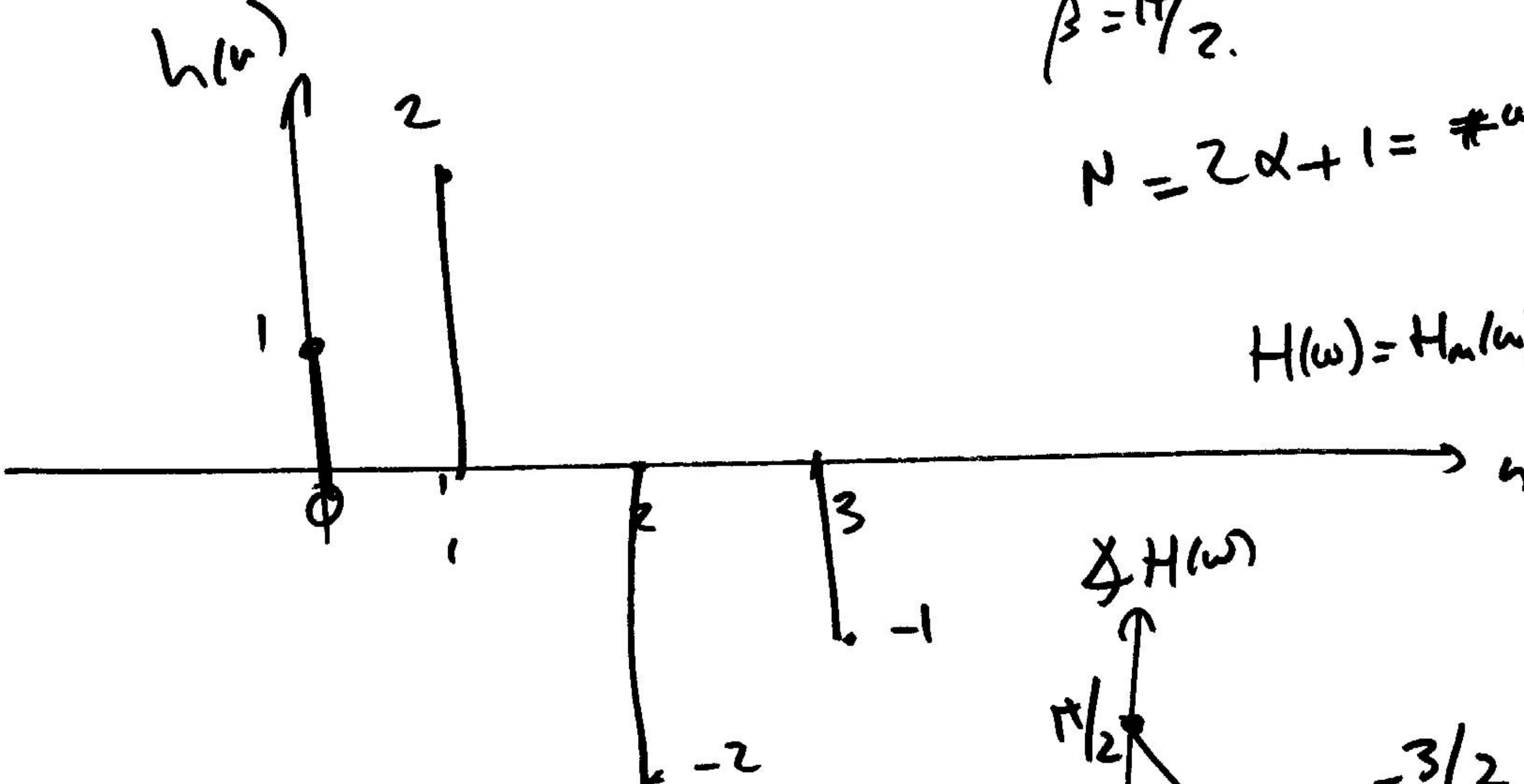
Then satisfy

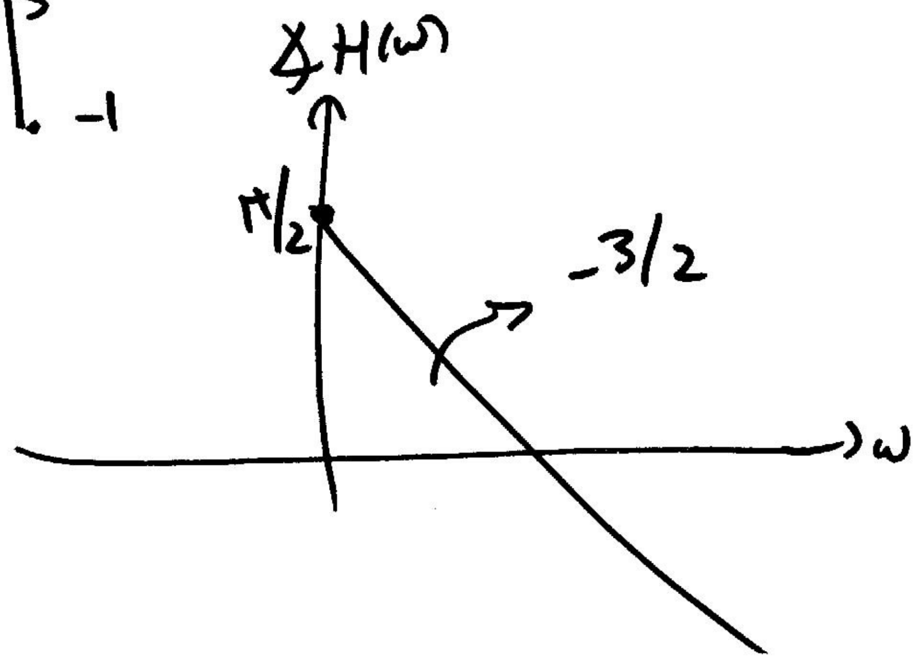
Ex  # of taps: $4 = N$  $-h(n) = h(3-n)$

$\beta = \pi/2.$

$N = 2\alpha + 1 = $ # of Taps

$H(\omega) = H_m(\omega) e^{-j\omega \frac{3}{2}}$



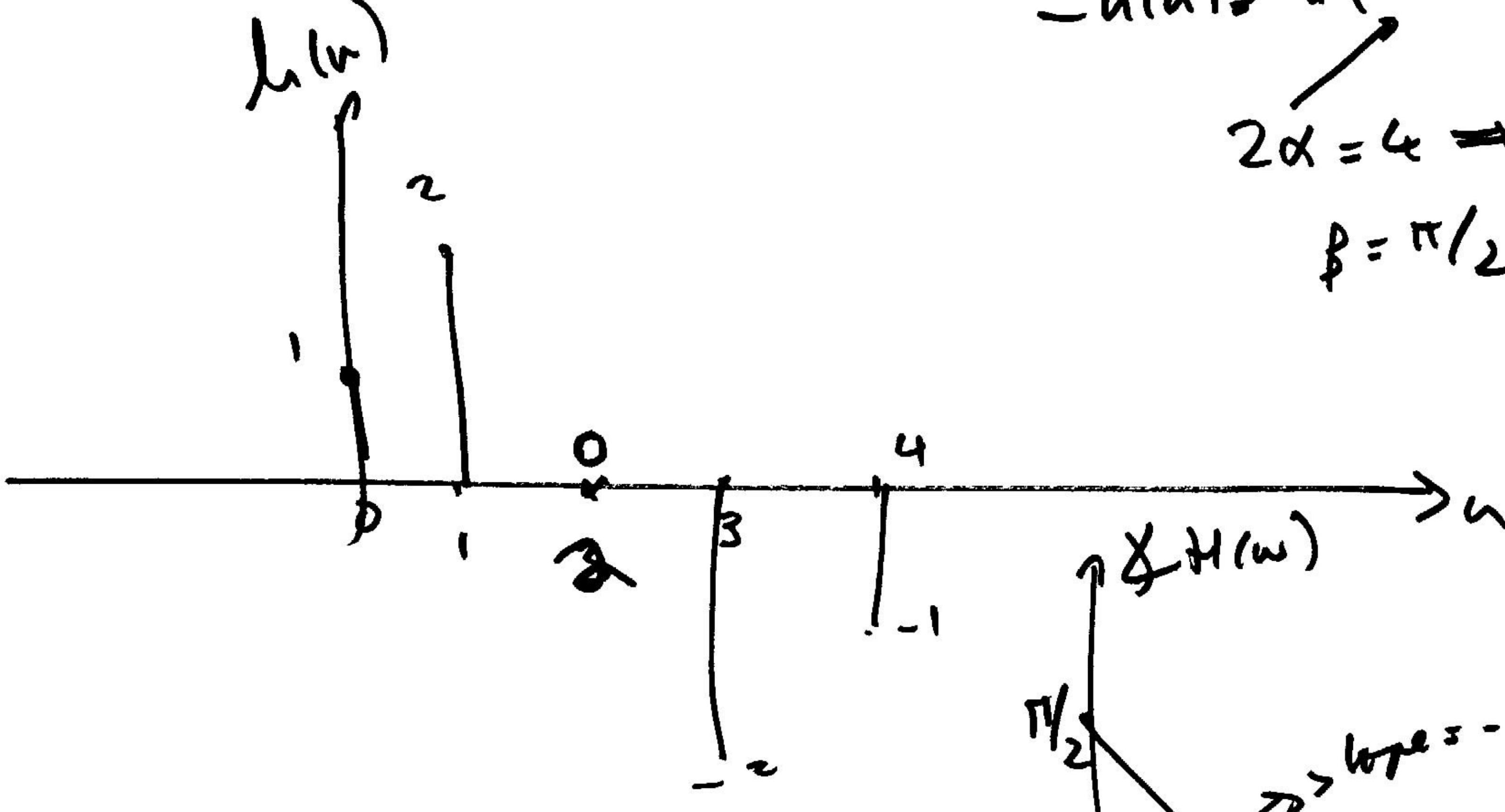



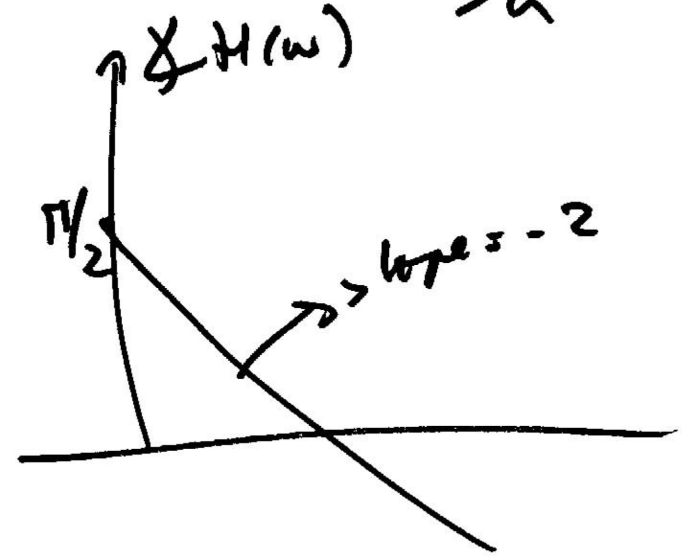
**EX 2** $h(n)$ 5 pts.

$-h(n) = h(4-n)$

$2\alpha = 4 \Rightarrow \alpha = 2$

$\beta = \pi/2$



Can show 2 Things.

Case 1

$$h(n) = \begin{cases} h(N-1-n) & 0 \leq n < N \\ 0 & \text{otherwise.} \end{cases}$$

$$H(\omega) = H_m(\omega)\, e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\boxed{\alpha = \frac{N-1}{2}}$$

Case 2

$$h(n) = \begin{cases} -h(N-1-n) & 0 \le n < N \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{\alpha = \frac{N-1}{2}}$$

$$H(\omega) = H_r(\omega)\, e^{+j\pi/2}\, e^{-j\omega\left(\frac{N-1}{2}\right)}$$

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Either Case 1 or Case 2

odd or even # of taps. N even / odd.

Case 1

even odd

Case 2

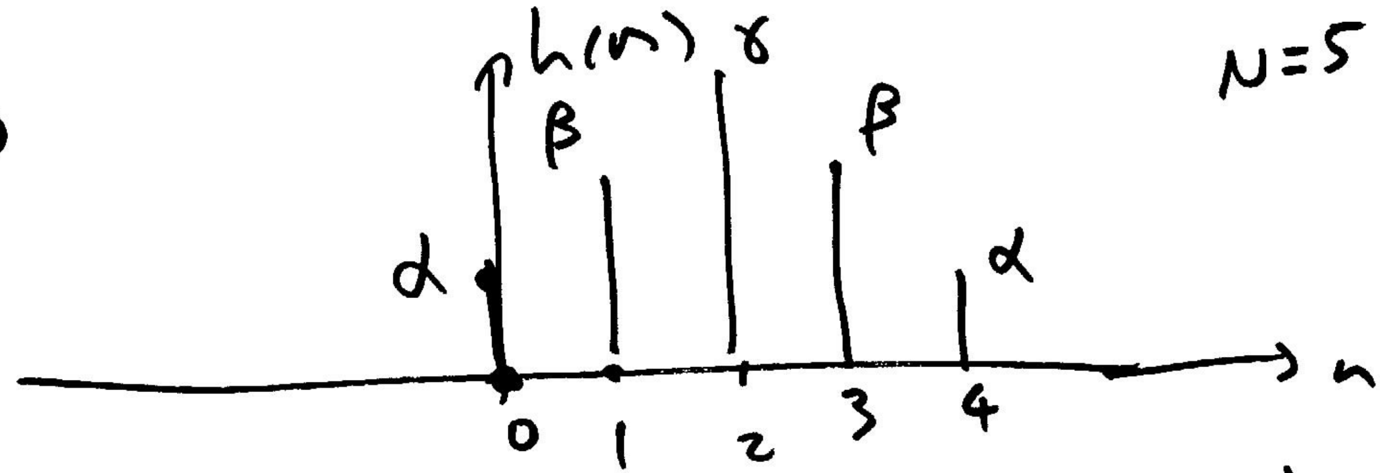
even odd.

$\Longrightarrow$ 4 types of GLP. FIR filter

**Type 1**: Symmetry $h(n) = h(N-1-n)$

\# of Taps is odd

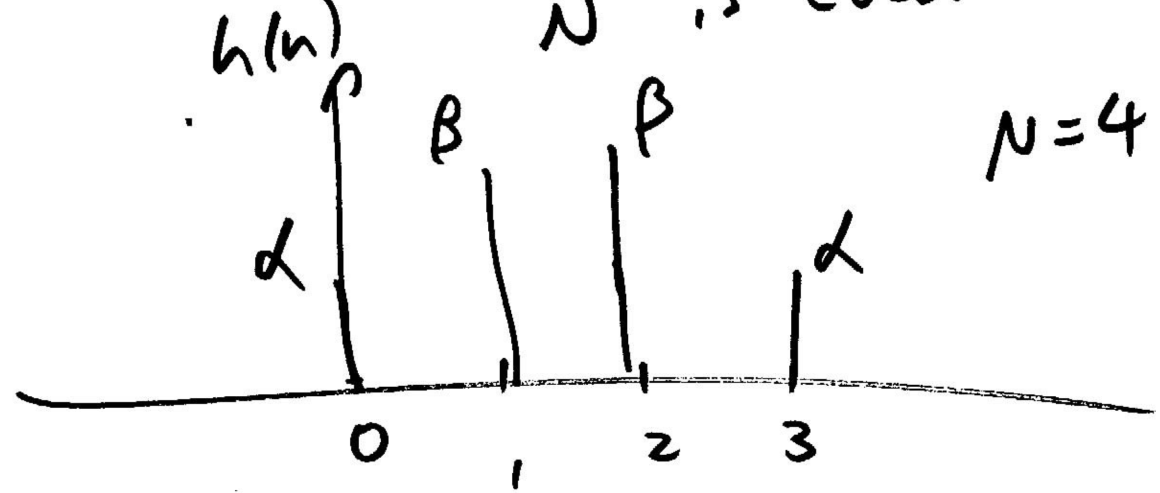
$\beta = 0$




**Type II**: symmetry $h(n) = h(N-1-n)$

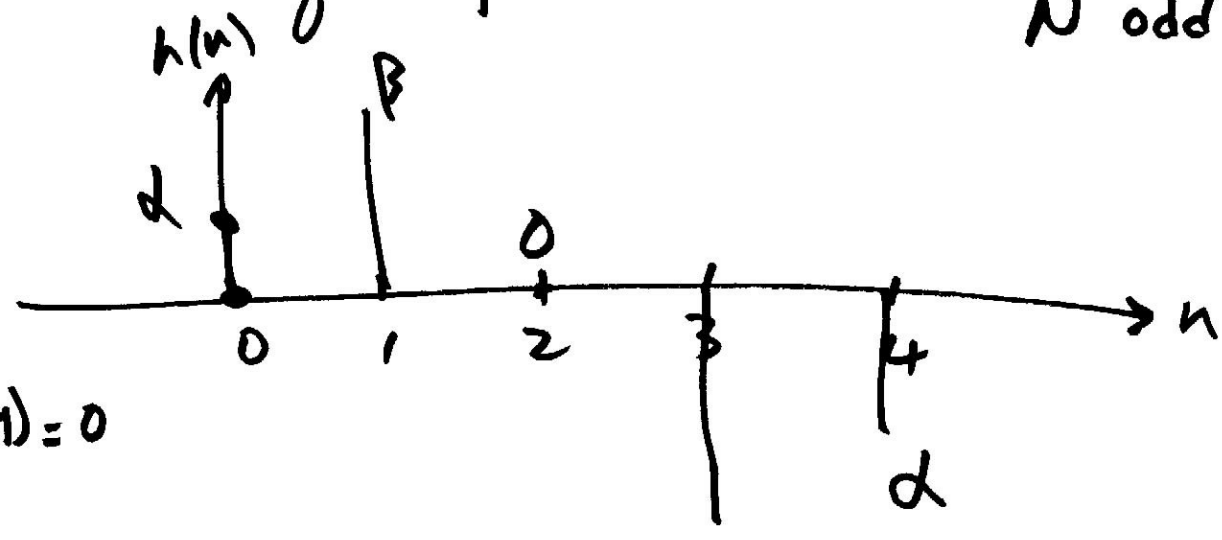
$N$ is even

$\beta = 0$



**Type III**: anti symmetry $h(n) = -h(N-1-n)$, $N$ odd.

$\beta = \pi/2$

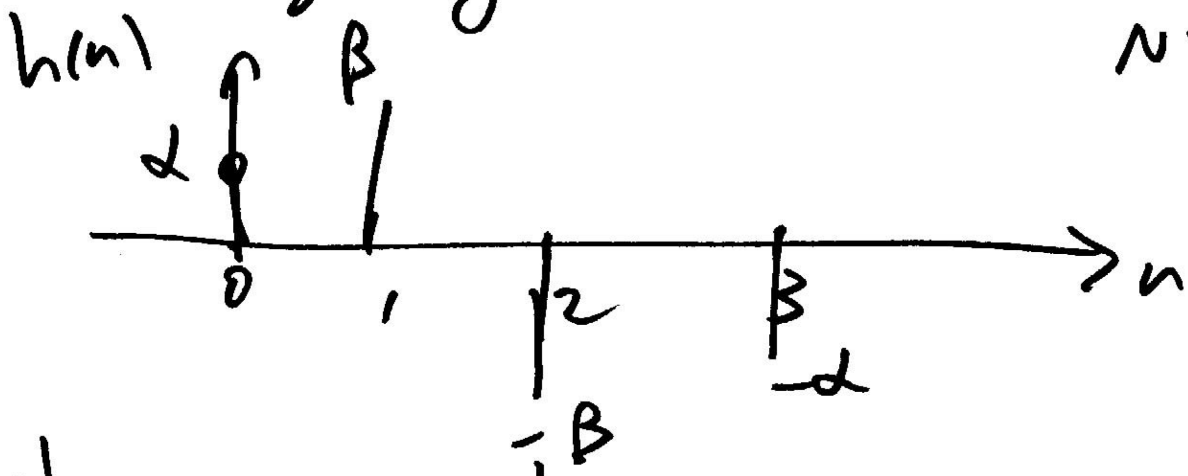



$\Rightarrow H(0) = H(\pi) = 0$

**Type IV** anti symmetry $N$ even

$N = 4$

$\beta = \pi/2$




$\alpha = \frac{N-1}{2}$