9/05/03

Z. Transform

Motivation: F.T. doesn't exist for some signal.

$x(n) = 2^n u(n)$.

or

$x(n) = a^n u(n)$ $\quad |a| > 1$

F.T. doesn't exist.

complex variable.

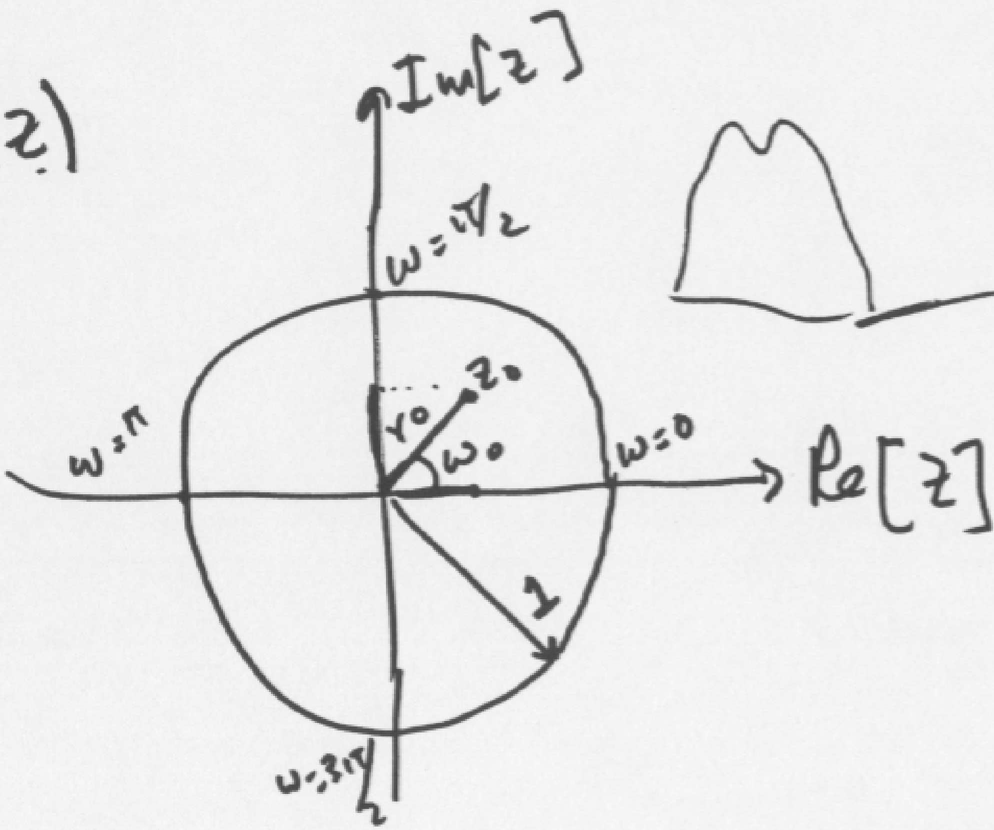
$x(n) \longleftrightarrow Z\{x(n)\} = X(z) \triangleq \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$

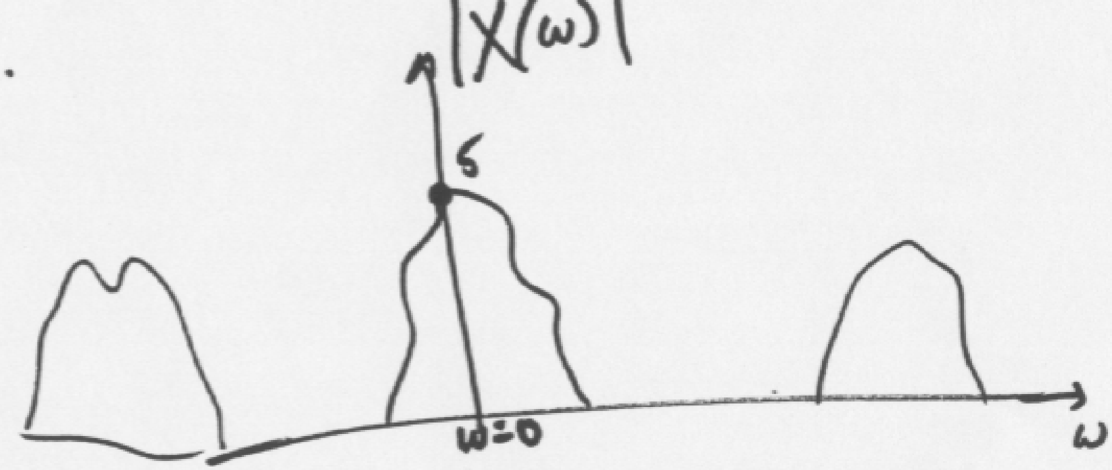
$X(\omega) = \text{D.T.F.T.}\{x(n)\} = \sum_n x(n) e^{-j\omega n}$

$$\left[X(z)\right]_{z=e^{j\omega}} = X(\omega)$$ → Real variable.

Z.T. evaluated at unit circle corresponds to D.T.F.T.

$X(z)$





z = complex variable
= Re + j Im.

$z = re^{j\omega}$

$z_0 = r_0 e^{j\omega_0}$

$\Rightarrow |z_0| = r_0 \quad \measuredangle z_0 = \omega_0$

Does $X(z)$ converge? For what values of z does $X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$ converge?

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) \left(re^{+j\omega}\right)^{-n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \underbrace{\left(x(n)\, r^{-n} \right)}_{\text{D.T.F.T of } x(n) r^{-n}} e^{-j\omega n}$$

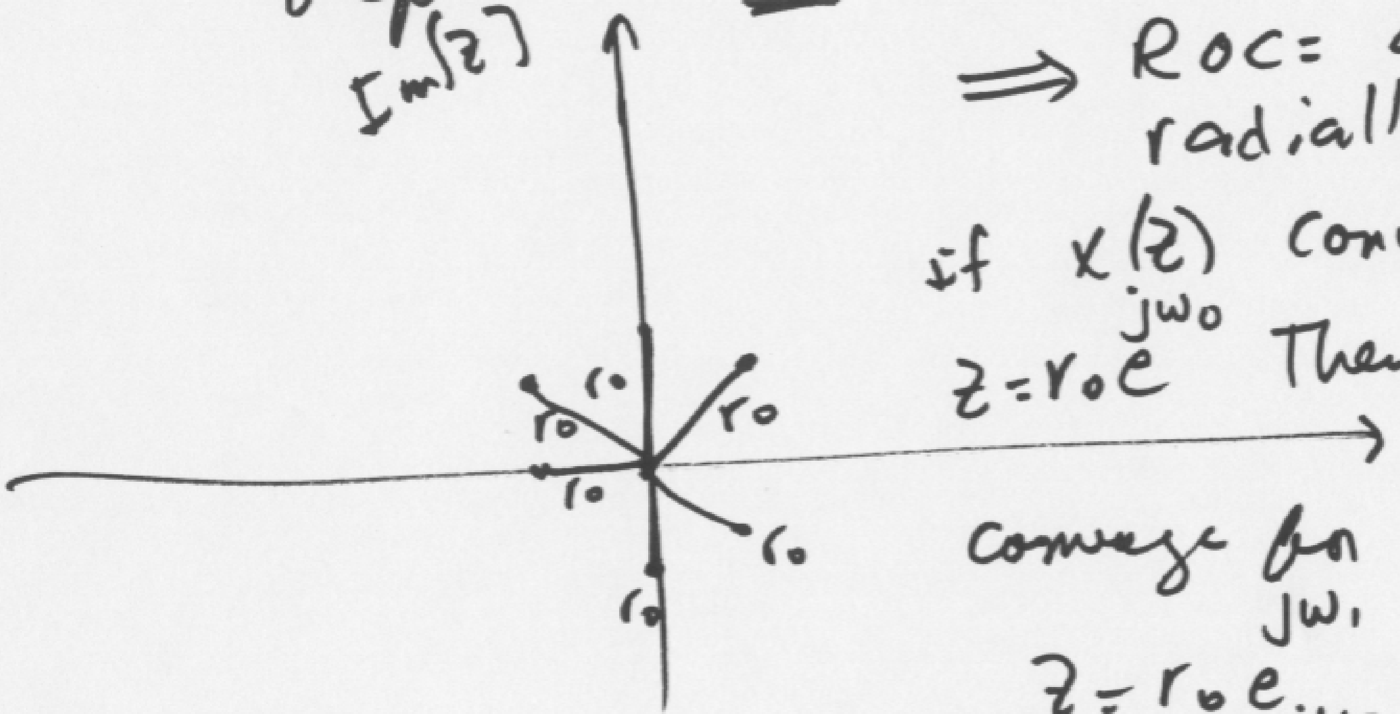
Remember: D.T.F.T. of $x(n) r^{-n}$ exists (converges)

if $x(n) r^{-n}$ is abs. summable.

i.e. $\sum_n \left| x(n) r^{-n} \right| < \infty$

If $\sum_n \left| x(n) r^{-n} \right| < \infty \Longrightarrow$ Z.T. exists & converges.

$\Longrightarrow$ Conclusion: Convergence only depends on $\underline{r}$ & not ω.



$\Longrightarrow$ ROC = always radially symmetric

if $X(z)$ converges for $z = r_0 e^{j\omega_0}$ Then it also converges for $z = r_0 e^{j\omega_1}$, $r_0 e^{j\omega_2}$

$\Longrightarrow$ NOT angle dependent.

ROC = Region of convergence.

Region in z domain for which

$\sum_n x(n) z^{-n}$ converges.

$\Rightarrow$ ROC possibilites :

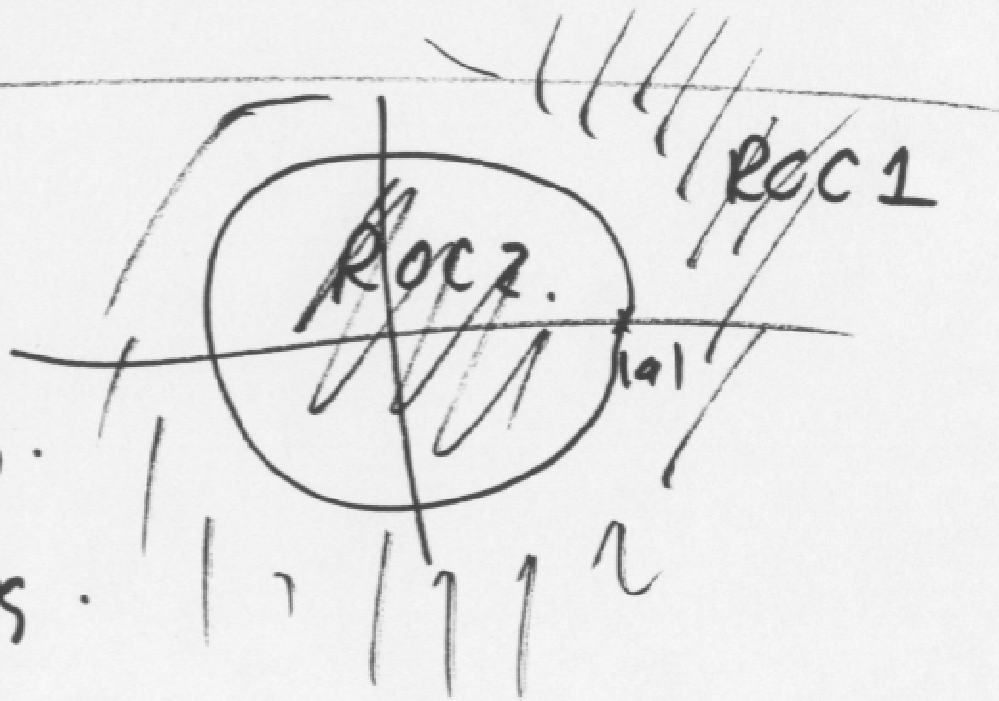
- inside of some circle
- outside of some circle.
- Between 2 circles $\rightarrow$ ring.
- A point i.e. origin

Claim $X(z)$ by itself does not specify uniquely a sequence. Always need ROC.

$$X(z) + \text{ROC1} \longleftrightarrow x_1(n)$$

$$X(z) + \text{ROC2} \longleftrightarrow x_2(n)$$

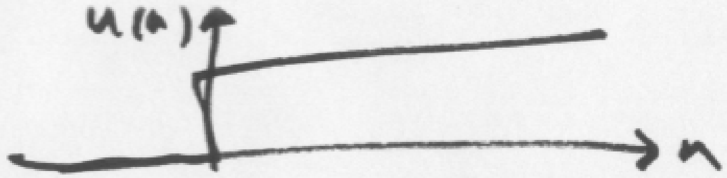
$$X(z) = \frac{1}{1 - a z^{-1}}$$



ROC1: Right handed seq.

ROC2: Left handed seq.

$\underline{Ex}$ $x(n) = a^n u(n)$



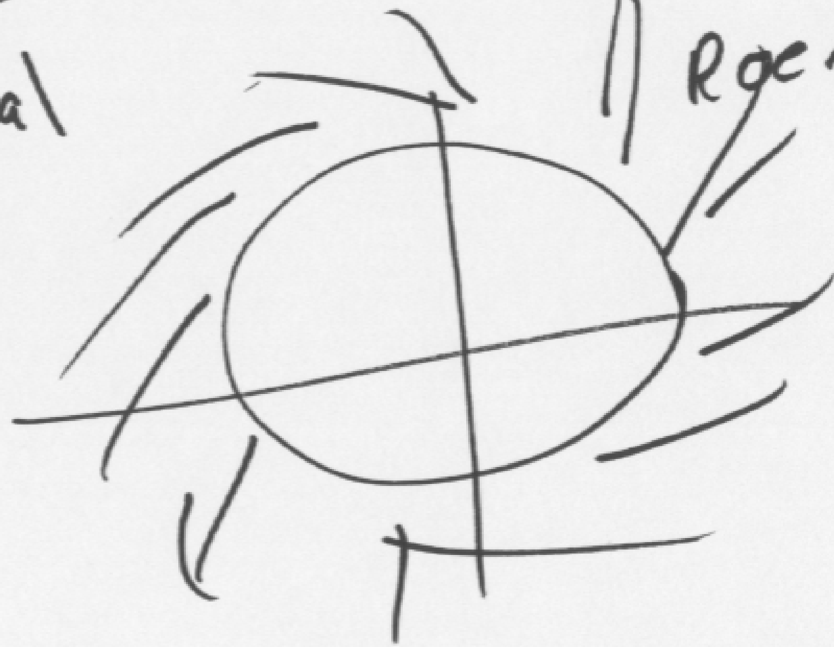
$$X(z) = \sum_{-\infty}^{\infty} a^n z^{-n} u(n)$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= \frac{1}{1 - a z^{-1}}$$

Converges if $|a z^{-1}| < 1$

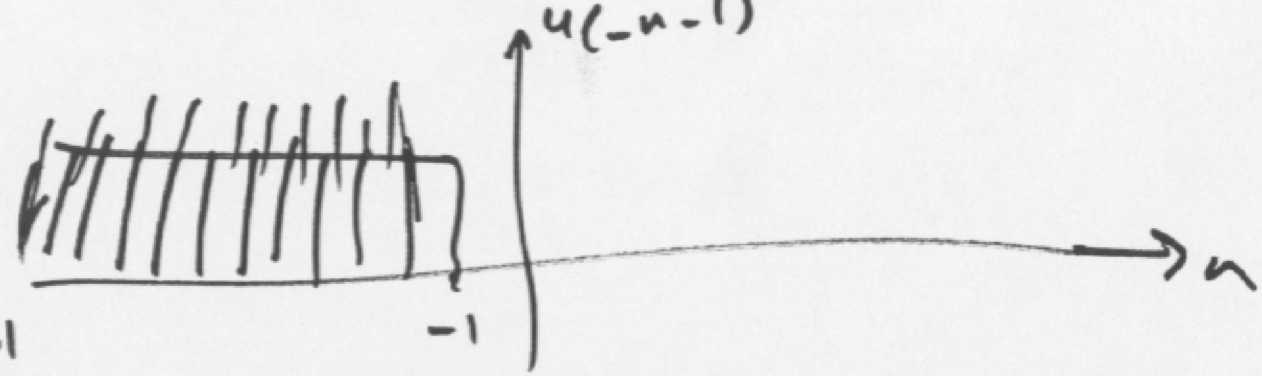
$\Rightarrow |z| > |a|$



ROC 1

<u>Ex</u> $\quad x(n) = -a^n u(-n-1)$

Left handed seq. ←



$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

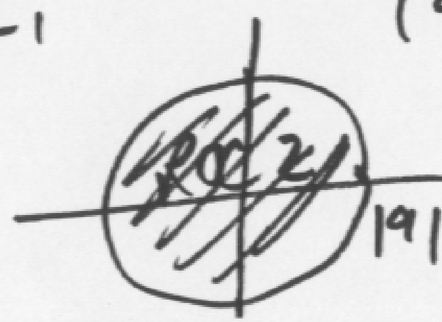
$\boxed{m = -n} \rightarrow$ chge of variable

$$X(z) = 1 - \underbrace{\sum_{m=0}^{\infty} (a^{-1}z)^m}_{\text{converges if}}$$

$|a^{-1}z| < 1 \Longrightarrow$

$|z| < |a|$

$$X(z) = \frac{1}{1 - az^{-1}}$$



Ex $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

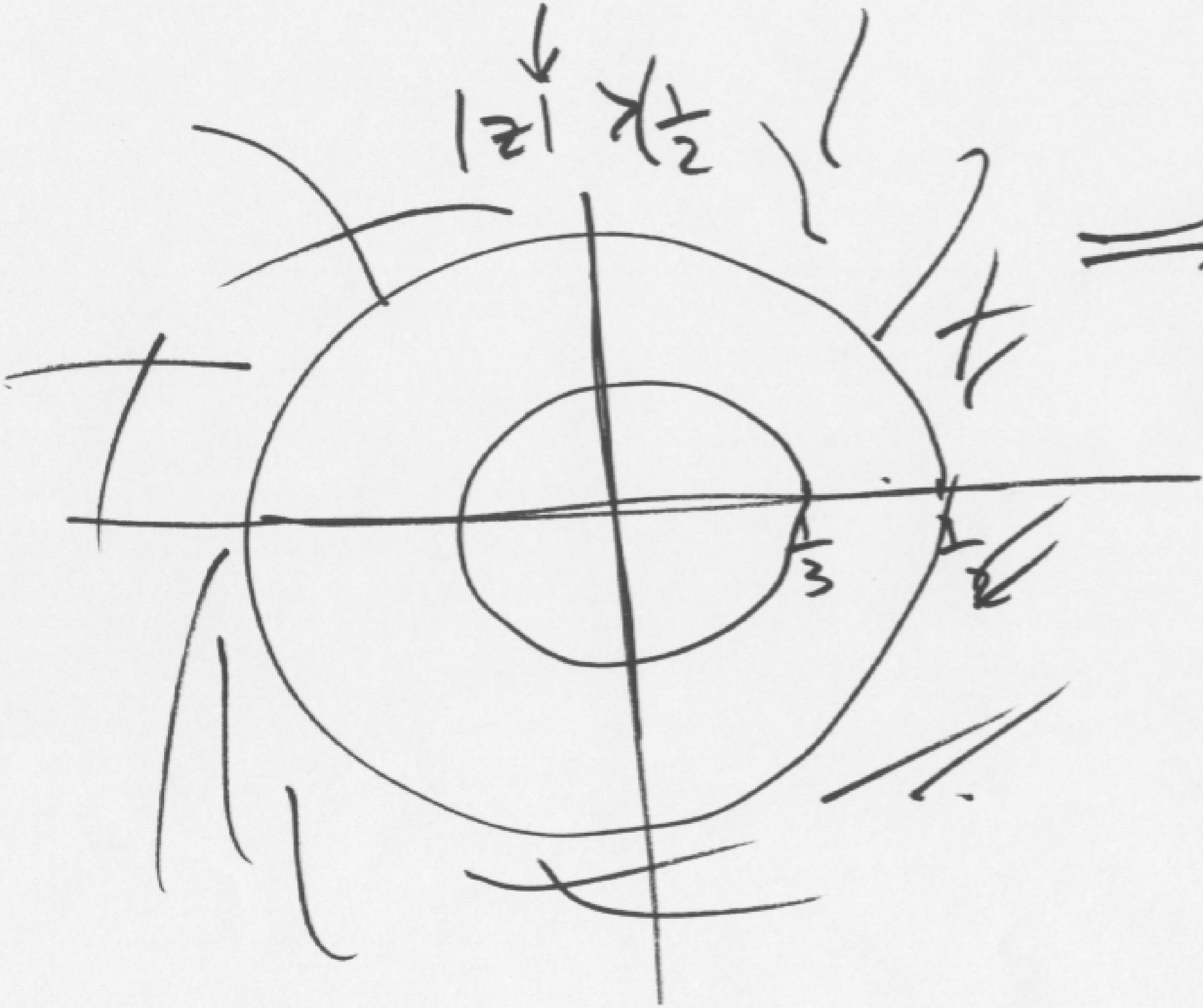
$|z| > \frac{1}{2}$

$|z| > \frac{1}{3}$

$\Longrightarrow$ intersect.

ROC:

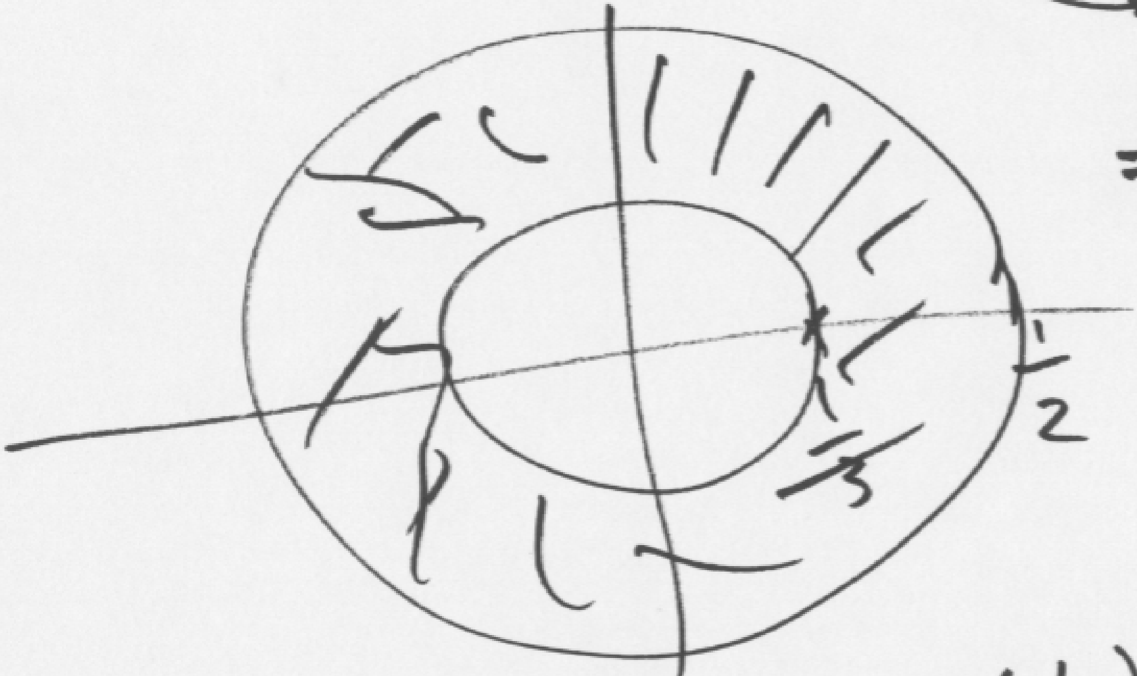
$|z| > \frac{1}{2}$.



Ex $x(n) = \underbrace{\left(-\frac{1}{3}\right)^n u(n)}_{|z| > \frac{1}{3}} - \underbrace{\left(\frac{1}{2}\right)^n u(-n-1)}_{|z| < \frac{1}{2}}$

$\to$ Both handed seq.

$\Rightarrow$ ROC is ring.



Ex $x(n) = \underbrace{\left(-\frac{1}{2}\right)^n u(n)}_{|z| > \frac{1}{2}} - \underbrace{\left(\frac{1}{3}\right)^n u(-n-1)}_{|z| < \frac{1}{3}}$

$\Rightarrow$ Z.T. does not exist.

~~Ex~~

Observati

RHS $\Longrightarrow$ ROC is outside of some circle

LHS $\Longrightarrow$ ROC is inside of some circle.

BHS $\Longrightarrow$ Ring. or not exist.

Finite length seq:

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

Ex $x(n) = \begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$

$$X(z) = \frac{1 - (a z^{-1})^N}{1 - a z^{-1}}$$

$$X(z) = 1 + a z^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-N+1}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots \frac{a^{N-1}}{z^{N-1}}$$

$\Rightarrow$ ROC of $X(z)$ is everywhere except at origin. $z=0$

~~Properties of z.~~

$$x(n) = \begin{cases} a^{n} & -N \leq n \leq 0 \\ 0 & \end{cases}$$

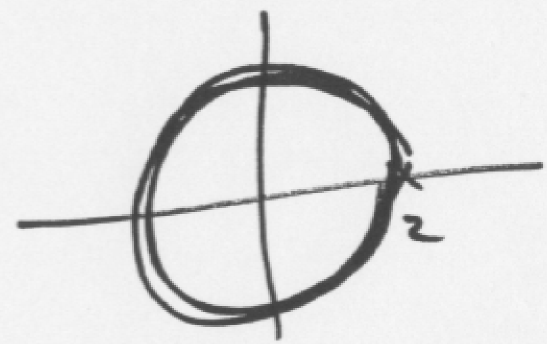
$$X(z) = \sum_{n=0}^{-N} a^{n} z^{-n} = 1 + a^{-1} z + a^{-2} z^{2} + \dots + a^{-N} z^{+N}$$

Converges everywhere except at $z=\infty$

Properties of Z.T. ROC

1. ROC is fn of r and not ω
2. F.T. exists if ROC includes unit circle.
3. ROC cannot include a pole.

$$\frac{1}{1-\frac{1}{2}z}$$



4. Finite length seq

ROC $\Rightarrow$ everywhere except at $z=0$ or $z=\infty$.

Show: If RHS $\Longrightarrow$ ROC outside of some circle.

$x(n)$ RHS: ~~if nonzero for n_0, then also nonzero for $n > n_0$.~~ nonzero for n_0 and larger values of n.

If $X(z)$ converges for r_0, it also converges for $\forall\, r > r_0$.

Assume. $X(z)$ converges for some r_0

Show that it also converges for $r > r_0$.

$$\Rightarrow \sum_{n=-\infty}^{+\infty} \left| x(n)\, r_0^{-n} \right| < \infty \quad \text{True}$$

RHS

$$\sum_{n=n_0}^{\infty} \left| x(n)\, r_0^{-n} \right| < \infty \quad \text{True.}$$

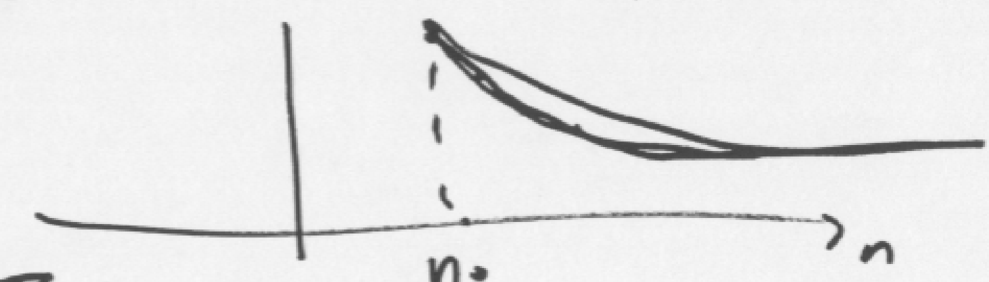
↳ show $\quad r > r_0 \quad \sum_{n=n_0}^{\infty} |x(n) r^{-n}| < \infty$

$$|x(n) r^{-n}| = \left| x(n) r_0^{-n} \left(\frac{r}{r_0}\right)^{-n} \right|$$

$$= |x(n) r_0^{-n}| \left| \left(\frac{r}{r_0}\right)^{-n} \right|$$

↳ Show. $r > r_0 \quad \sum_{n=n_0}^{\infty} |x(n) r_0^{-n}| \left| \left(\frac{r}{r_0}\right)^{-n} \right| < \infty$

$r > r_0 \Rightarrow \left|\frac{r}{r_0}\right| < 1$

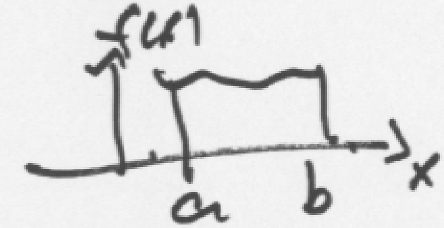


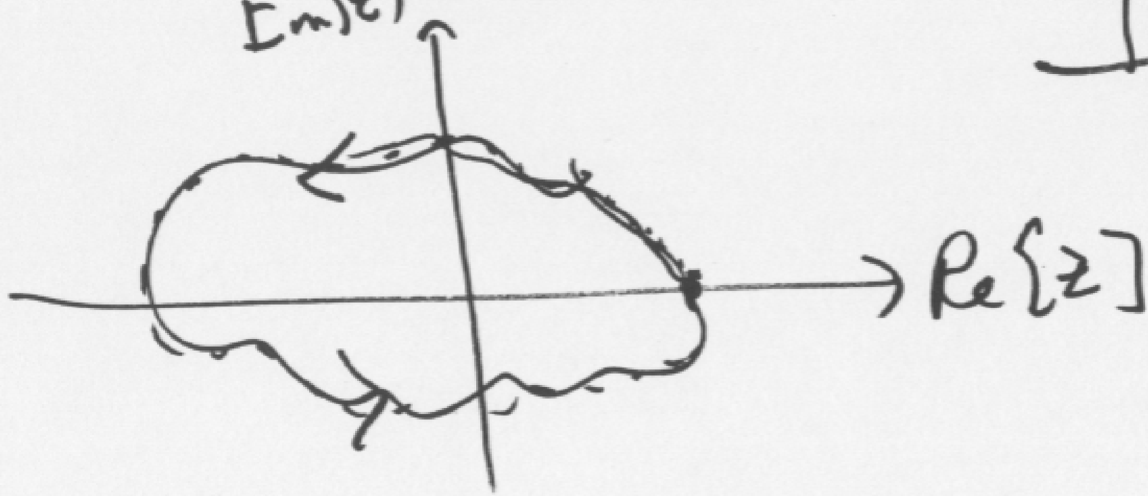
Inverse z. Trafom

$$x(n) \longrightarrow X(z) = \sum_n x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz$$

Any closed contour in ROC of $X(z) z^{n-1}$



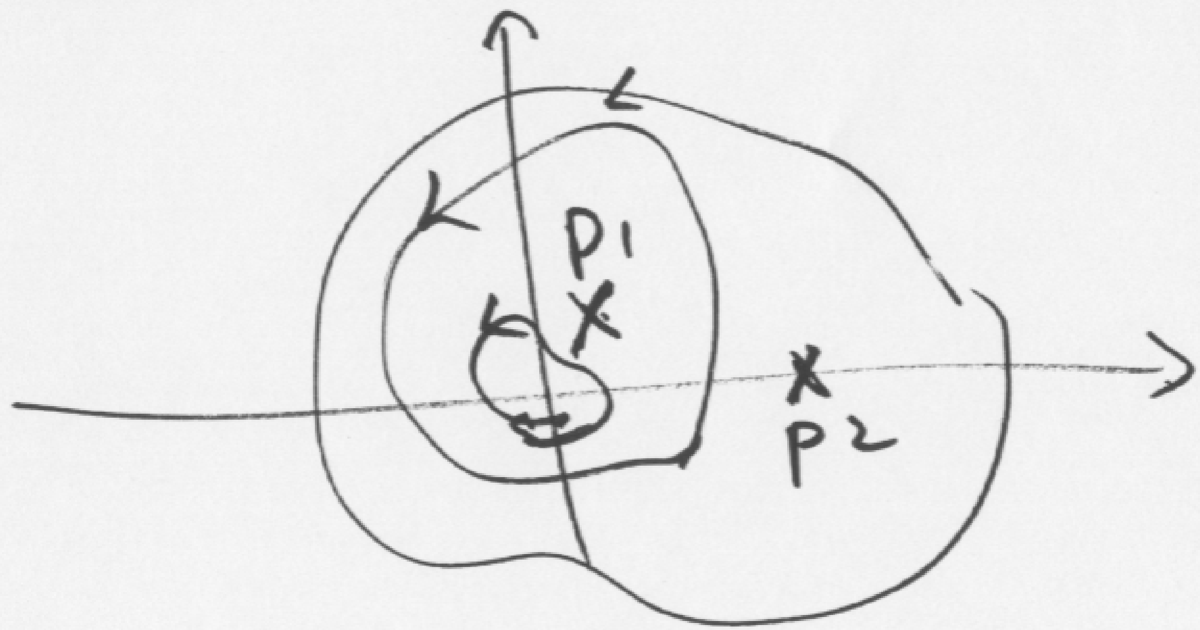


Cauchy Residue Theorem:

$$\frac{1}{2\pi j} \oint_C F(z)\, dz = \sum_{\text{poles}} \text{Residues of } F(z) \text{ at each pole of } F(z) \text{ inside contour } C.$$

C closed contour

F(z)



If $F(z) = \dfrac{\phi(z)}{(z - z_0)^s}$ Then.

$$\text{Residue}\left[F(z)\right]_{z=z_0} \triangleq \frac{1}{(s-1)!}\left[\frac{d^{s-1}\phi(z)}{dz^{s-1}}\right]_{z=z_0}$$

$$F(z) = \frac{1}{(z-2)^2}$$

What is residue of $F(z)$ at $z = 2$.

$\phi(z) = 1$ $s = 2$, $z_0 = 2$

$$\text{Reside}\left[F(z)\right]_{z=2} = \frac{1}{1}\left[\frac{d}{dz}(1)\right] = 0$$

$\underline{Ex}$ $F(z) = \dfrac{z^3}{z-2}$ $\qquad \phi(z) = z^3$

$z_0 = 2$

$s = 1$

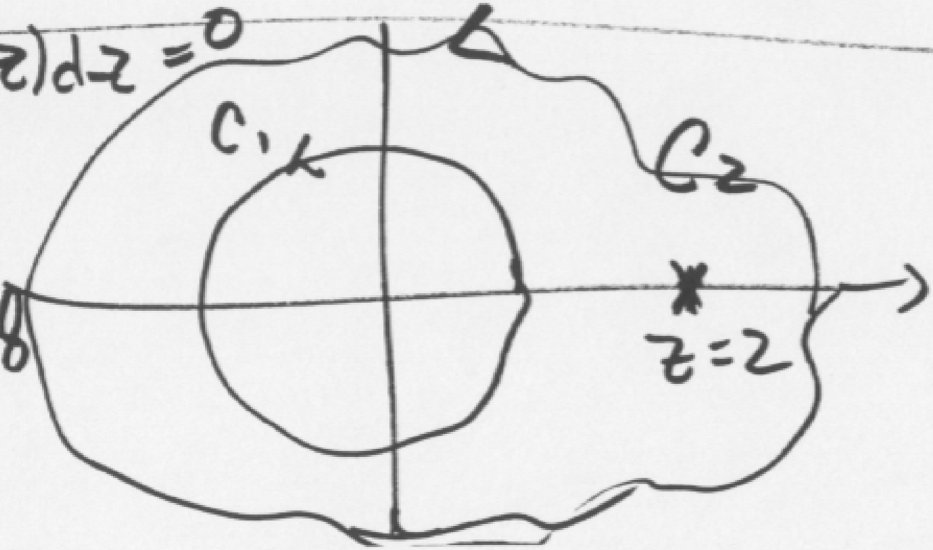
$$\Big[\text{Reside } F(z)\Big]_{z=z_0} = \frac{1}{1}\Big[\phi(z)\Big]_{z=z_0}$$

$$= \Big(z^3\Big]_{z=2}$$

$$= 8$$

$$\frac{1}{2\pi j}\oint_{C_1} F(z)\,dz = 0$$

$$\frac{1}{2\pi j}\oint_{C_2} F(z)\,dz = 8$$



z plane.