Sept. 12, 2003

LTI

$\downarrow$

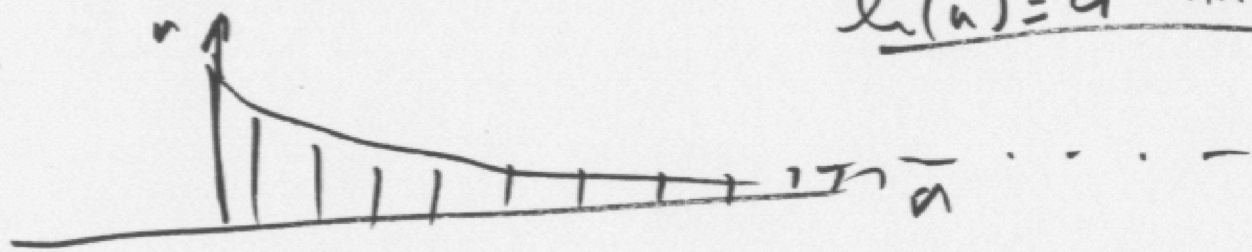
impulse response $h(n)$

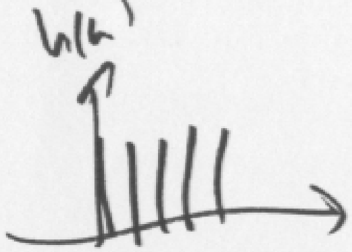
$$x \longrightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \longrightarrow y$$

$$x * h = y.$$

$h(n) = a^n u(n)$

$h(n)$:





LTI
$h(n)$

FIR

$y = x * h$

$y(n) = \sum_k x(k)h(n-k)$

$h(n)$ has finitely many nonzero value.

IIR
infinite impulse response

infinitely many non zero values.

ratio of z-polynomials

Rational Transfer fn

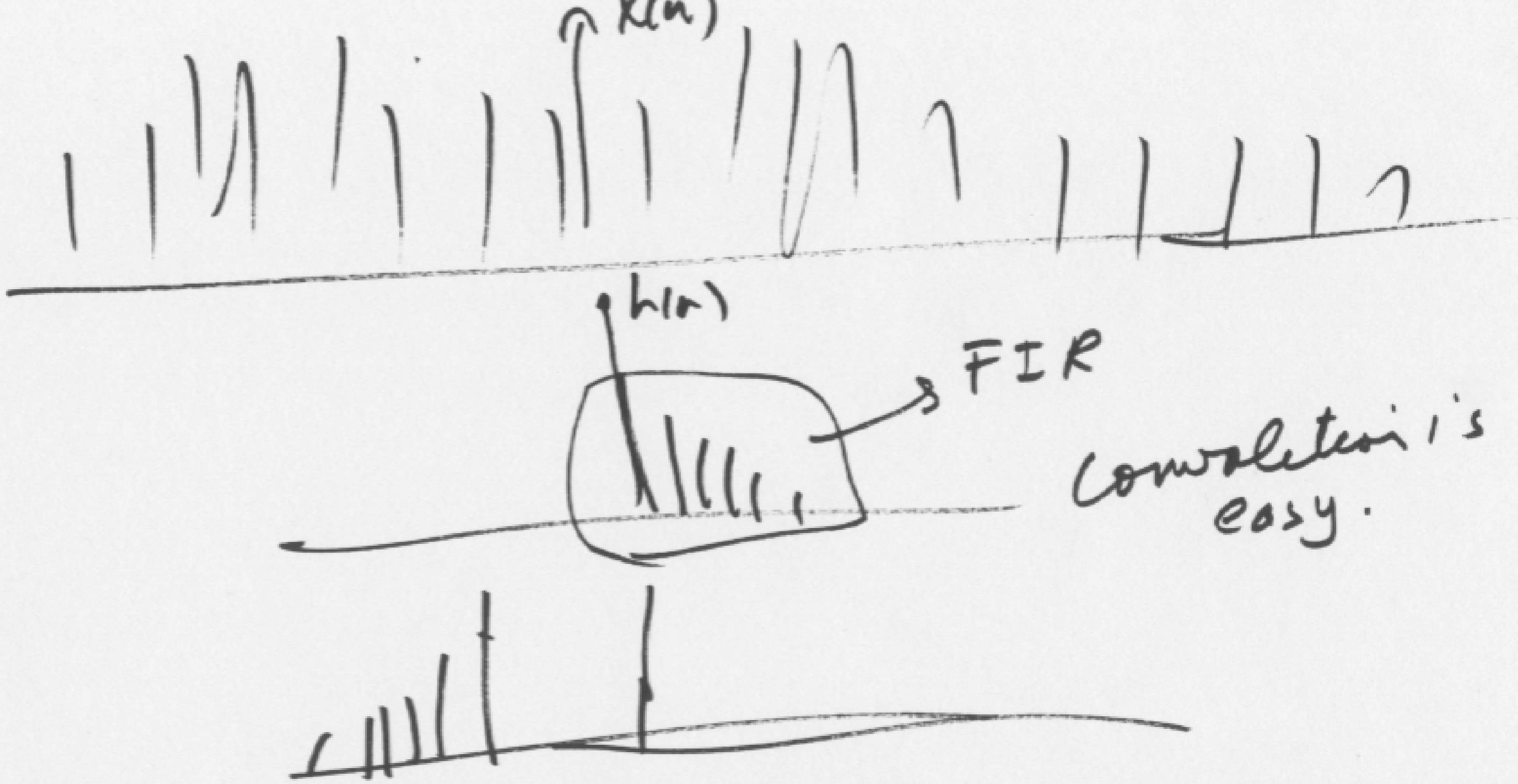
$\frac{P(z)}{Q(z)} = H(z)$:

polynomial

$H(z) = \frac{1}{1-az^{-1}}$ $|z| > |a|$

Do not have rational Transfer fn.

$h(n)$: sinc.



Convolution is easy.

IIR filters with Rational Transfer fn
We use Difference Equation. D.E.

Ex $h(n) = a^n u(n) \longrightarrow H(z) = \dfrac{1}{1 - a z^{-1}}$

$$x \rightarrow \boxed{\begin{matrix} H(z) \\ h(n) \end{matrix}} \rightarrow y(n)$$

$$Y(z) = X(z)\, H(z)$$

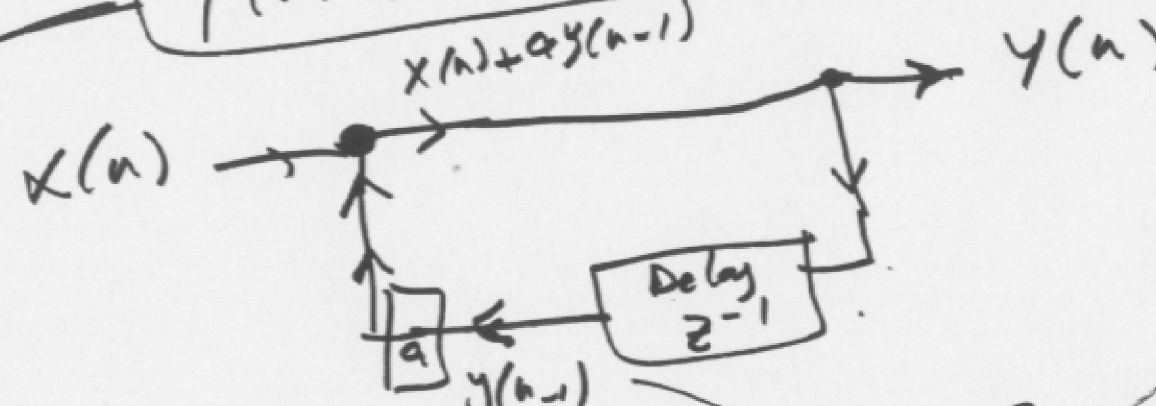
$$Y(z) = X(z) \cdot \frac{1}{1-az^{-1}}$$

$$Y(z)\left[1 - a z^{-1}\right] = X(z)$$

$$Y(z) - a z^{-1} Y(z) = X(z)$$

I.Z.T.

$$\boxed{Y(n) - a\, Y(n-1) = x(n)} \leftarrow \text{D.E.}$$



→ Linear : No $x^2(n)$, $y^2(n)$, $x^2(n-1)$, $x(n)y(n)$, $x(n)y^2(n)$

→ coefficient are constant (a)

Not constant coeffi: $ny(n) - an^2y(n-1)$

Linear Constant Coefficient
Difference Equation
L.C.C.D.E.

Used for implementing IIR
filters, with Rational Transfer fns.

<u>Q</u>: Under what Conditions does
L.C.C.D.E correspond to LTI
system?

or Causal LTI system?

<u>Ex</u> $y(n) = a\,y(n-1) + b\,x(n)$

does not correspond To a <u>system</u>

<u>System</u>: unique input correspond To a unique output.

<u>proof</u>. Suppose $y_1(n)$ is a soln.

Then $y_2(n) = y_1(n) + K a^n$ is also a solution:

Assume y_1 is a soln $\Rightarrow$ $y_1(n) = a\,y_1(n-1) + b\,x(n)$

<u>Q</u>: Does y_2 also satisfy Eqn ?

$$y_2(n) \overset{??}{=} a\,y_2(n-1) + b\,x(n)$$

$$y_1(n) + K a^n \overset{??}{=} a\left[y_1(n-1) + K a^{n-1}\right] + b\,x(n)$$

$\Longrightarrow$ True ?? yes

Fix: Add. I.C. = Initial Condition.

Claim Then with I.C. $\longrightarrow$ system.

$$y(n) = a\,y(n-1) + x(n)$$

$$x(n) = b\,\delta(n)$$

I.C. $y(-1) = y_0$

1 $n \geq 0$

$$y(0) = a\,y(-1) + b\,\delta(0)$$
$$= a\,y_0 + b$$
$$y(1) = a\,y(0) + b\,\delta(1)$$
$$= a(a y_0 + b)$$
$$= a^2 y_0 + ab$$

$y(2)$ ...

$n \geq 0 \longrightarrow \boxed{y(n) = a^n (a y_0 + b)}$

(2) $n \leq -2$

$$y(n) = a\, y(n-1) + x(n)$$

$$a\,y(n-1) = y(n) - b\,\delta(n)$$

$$y(n-1) = \frac{1}{a} y(n) - \frac{b}{a}\delta(n)$$

$n = -1$

$$y(-2) = \frac{1}{a} y(-1) - \frac{b}{a}\delta(-1)$$

$$= \frac{1}{a} y_0$$

$$y(-3) = \frac{1}{a^2} y_0$$

$$\vdots$$

$$y(n) = a^{n+1} y_0$$

$n \leq -2$

$$\boxed{n \leq -1 \longrightarrow y(n) = a^{n+1} y_0}$$

Combine: $y(n) = a^n (a y_0 + b) u(n) +$

$$a^{n+1} y_0 \, u(-n-1)$$

$\forall n$ $$\boxed{y(n) = a^{n+1} y_0 + a^n b \, u(n)}$$

<u>Aliter</u>:

(a) find homogeneous Soln. y_h

(b) " particular Soln y_p.

(c) $y(n) = y_h(n) + y_p(n)$

(d) Impose. I.C.

(a) homogen soln: To be gen. soln if input is zero: Let $x(n)=0$

$$y(n) = a\, y(n-1)$$

given: $y(n) = K\alpha^n$

$$K\alpha^n = a\, K\alpha^{n-1}$$

$$\Longrightarrow \boxed{y_h(n) = K a^n}$$

(b) particular soln: $y(n) = a\, y(n-1) + b\,\delta(n)$

$$Y(z) = a z^{-1} Y(z) + b$$

$$Y(z) = \frac{b}{1 - a z^{-1}}$$

$|z| > |a|$ b.1

$|z| < |a|$ b.2

b.1 $\rightarrow$ $y_{p_1}(n) = b\,a^n\,u(n)$

b.2 $y_{p_2}(n) = -b\,a^n\,u(-n-1)$

(c) Combine particular soln with homogeneous solution:

$$y_{tot} = y_h(n) + y_p(n)$$

(d) Impose I.C. $\rightarrow$ d.1, d.2

d.1 $y_{tot}(n) = K a^n + b a^n u(n)$

Apply I.C.: $n=-1$ $\quad y_{tot}(-1) = K a^{-1} + b a^{-1} u(-1) = y_0$

$\Rightarrow K = a y_0$

d.1 $y_{Tot}(n) = a^{n+1} y_0 + b a^n u(n)$

d.2 $y_{Tot}(n) = K a^n - b a^n u(-n-1)$

Apply I.C: $y_{Tot}(-1) = y_0 = K a^{-1} - b a^{-1} u(0)$

$\Longrightarrow K = a y_0 + b.$

$y_{Tot}(n) = a^n (a y_0 + b) - b a^n u(-n-1)$

d.2 $\boxed{y_{Tot}(n) = a^{n+1} y_0 + b a^n u(n)}$

$\Rightarrow$ Conclusion: Same answer

Does This define a linear system

$$y(n) = a\,y(n-1) + x(n)$$

$$x(n) = b\,\delta(n)$$

I.C. $y(-1) = y_0$

<u>No</u>: Not a linear system.
zero input i.e. $b=0$ does not result in zero output

$$y_{tot}(n) = a^{n+1}\,\underline{y_0}$$

However $y_0 = 0 \longrightarrow$ Does correspond to a linear system

$\Rightarrow$. For LCCDE to correspond to linear system I.C. $= 0$

Q: Does this system correspond to T.I. system?

Let $x(n) = b\,\delta(n)$

output $y_{TOT}(n) = a^{n+1} y_0 + b\,a^{n}\,u(n)$

Let's shift input $w(n) = b\,\delta(n-n_0)$

output due to $w(n)$ ..... $\rightarrow a^{n+1} y_0 + \underbrace{b\,a^{n-n_0}\,u(n-n_0)}_{\text{output due to } w}$

Is this same as $y_{TOT}(n-n_0)$?

$$y_{TOT}(n-n_0) = \underbrace{a^{(n-n_0)+1} y_0 + b\,a^{n-n_0}\,u(n-n_0)}$$

Not T.I. , let $y_0 = 0 \Longrightarrow$ T.I.

Proposition for LCCDE to correspond to LTI system, must have $y_0 = 0$

i.e. zero I.C.

If also want causality, $\Longrightarrow$ Initial Rest Condition. I.R.C.

LCCDE + I.RC $\longleftrightarrow$ LTI + causality.

Def I.R.C.

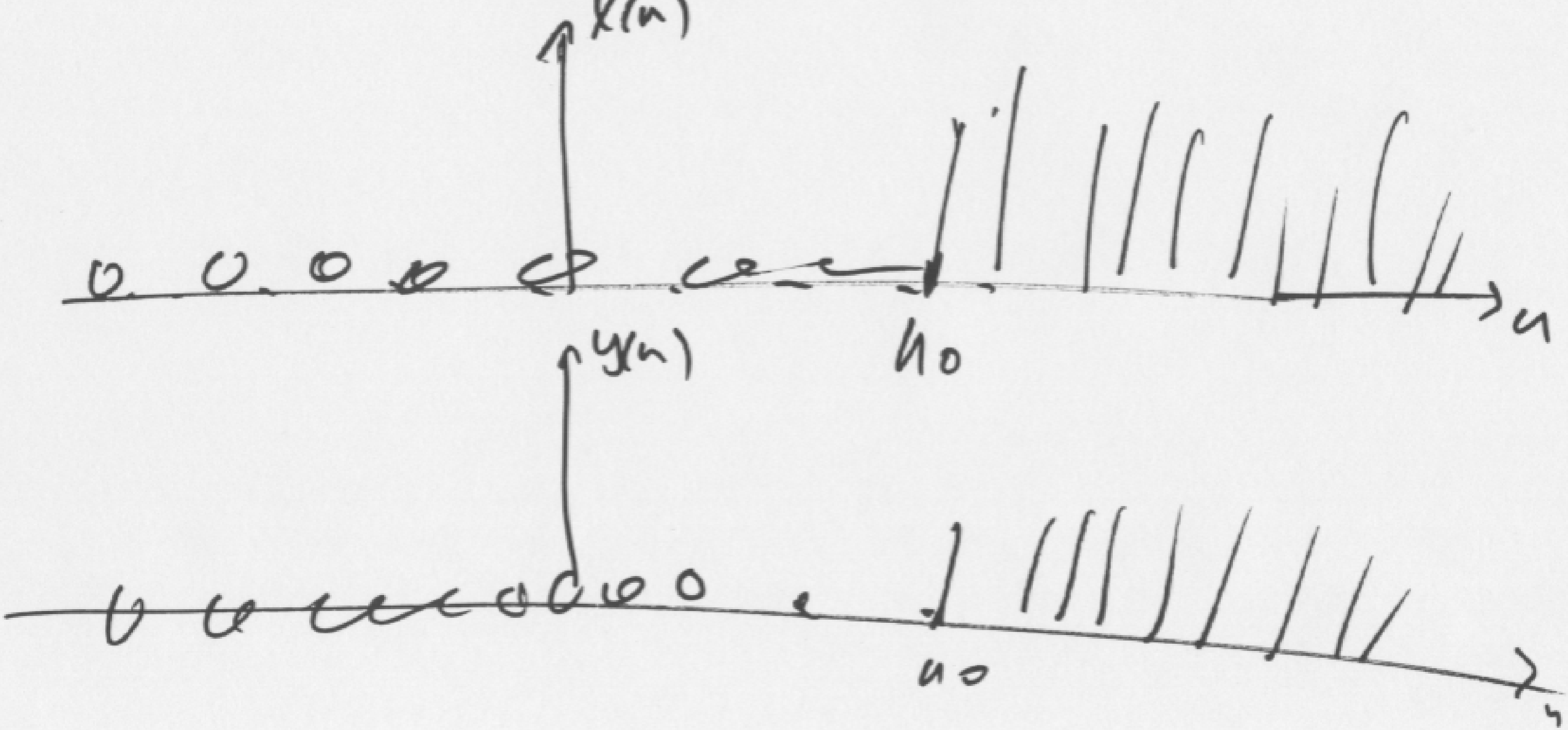
I.R.C. sys has I.R.C.

iff

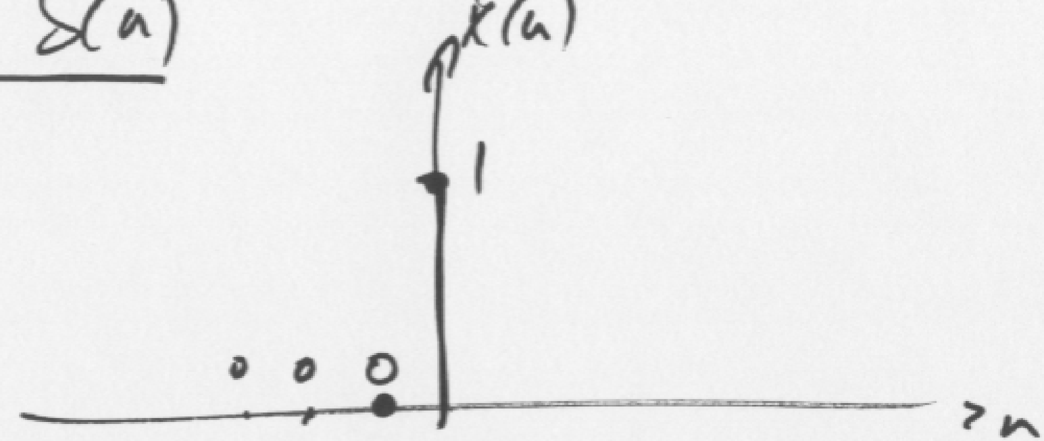
for $x(n)=0$ $n<n_0$

results in

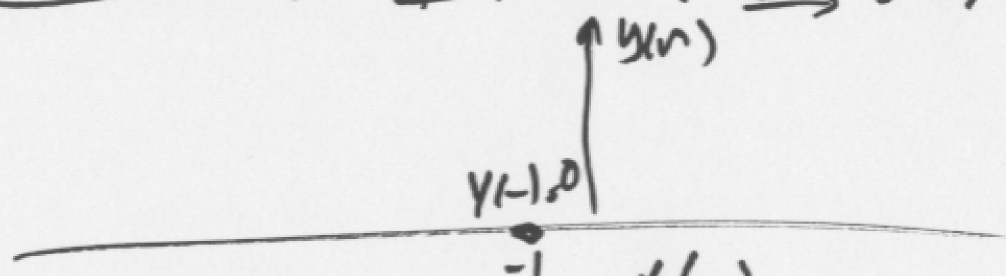
$y(n)=0$ $n<n_0$



$\underline{EX}$ $\underline{x(n) = \delta(n)}$

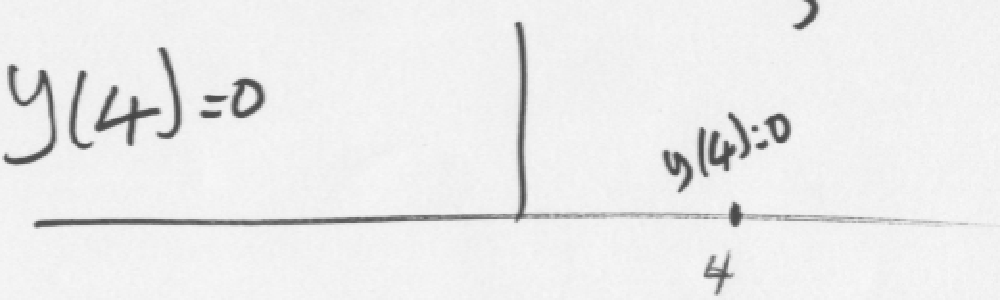


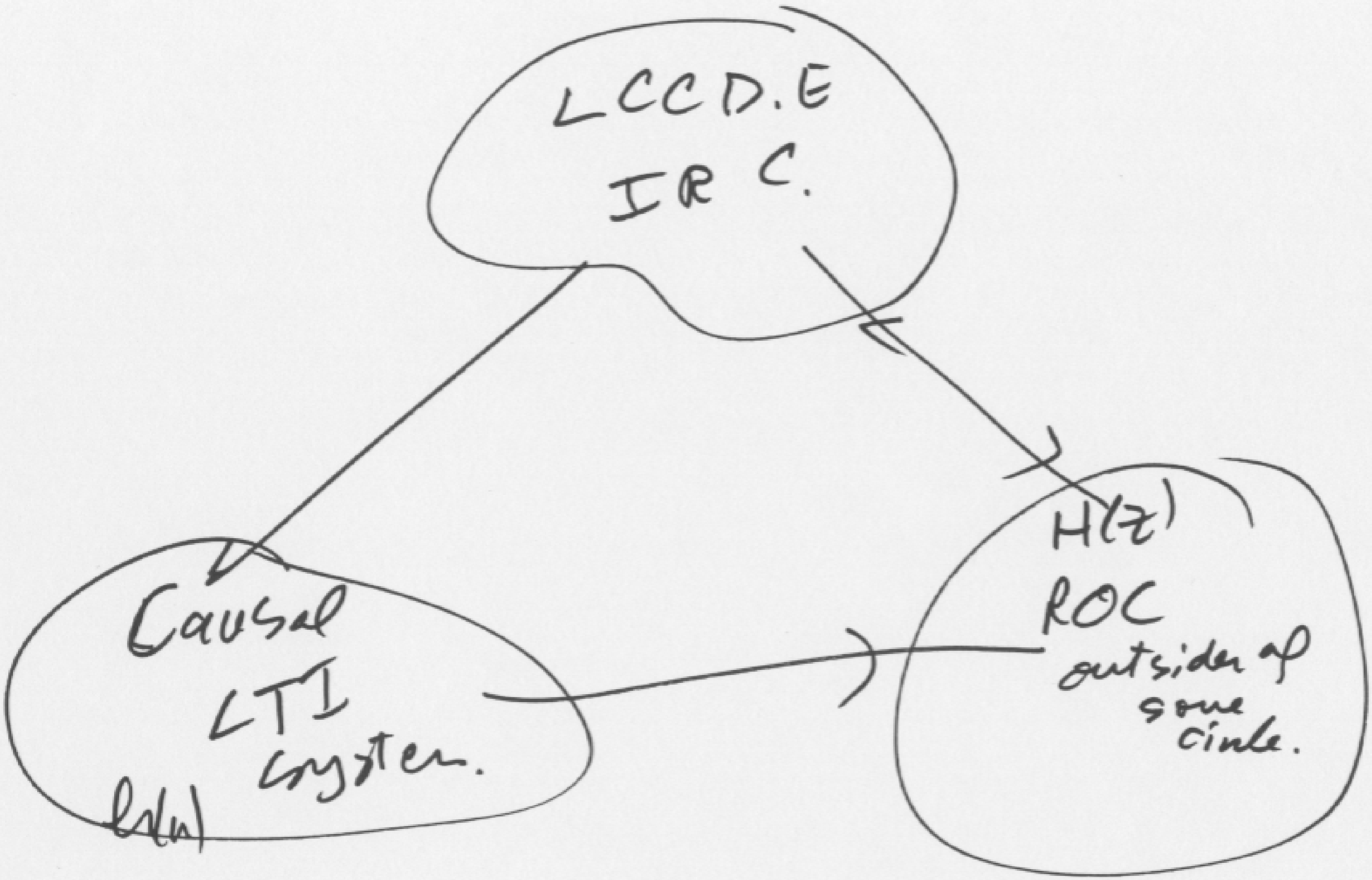
First order D.E $\Longrightarrow$ I.R.C $\Rightarrow$ output$(-1)=0$



$\underline{Ex2}$ $x(n) = \delta(n-5)$

~~output~~ I.R.C. $y(4)=0$




LCCD.E
IR C.
Causal
LTI
system.
h(n)
H(z)
ROC
outside of
some
circle.

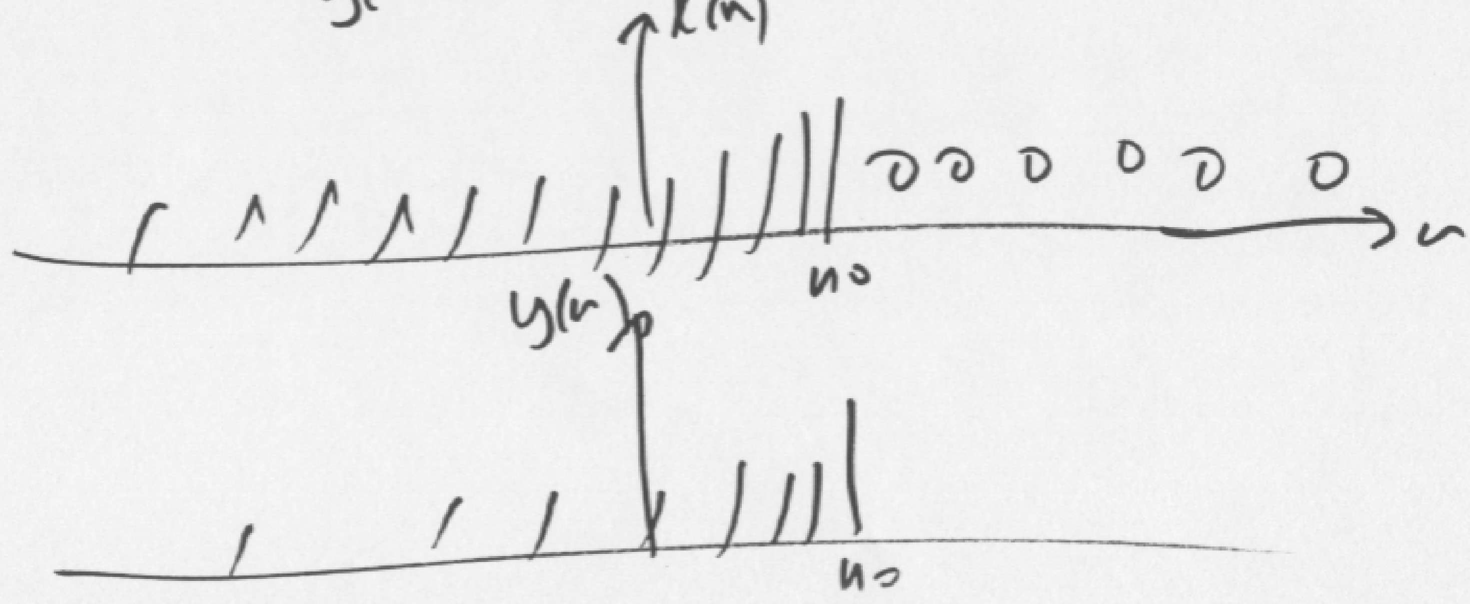
Final Rest Condition F.R.C.

LCCDE ~~+ anti~~ + F.RC. $\longleftrightarrow$ LTI anticausal.

F.RC sys has FRC iff.

for $x(n) = 0$ for $n > n_0$

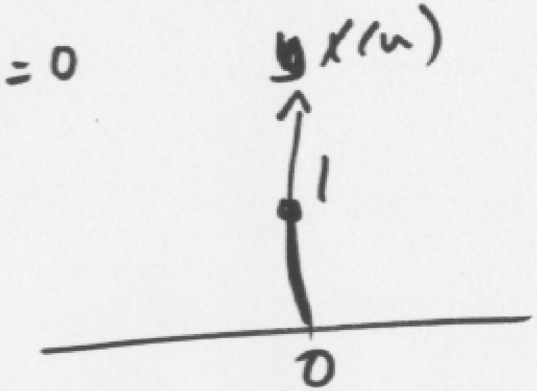
results in $y(n) = 0$ for $n > n_0$



EX $y(n) = a\,y(n-1) + \underbrace{b\,\delta(n)}_{x(n)}$

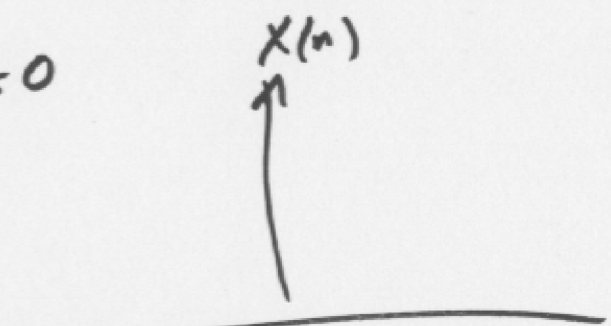
(1) I.R.C. $\longrightarrow$ $y(-1) = 0$

causal.



(2) F.R.C. $\longrightarrow$ $y(1) = 0$

anti causal.



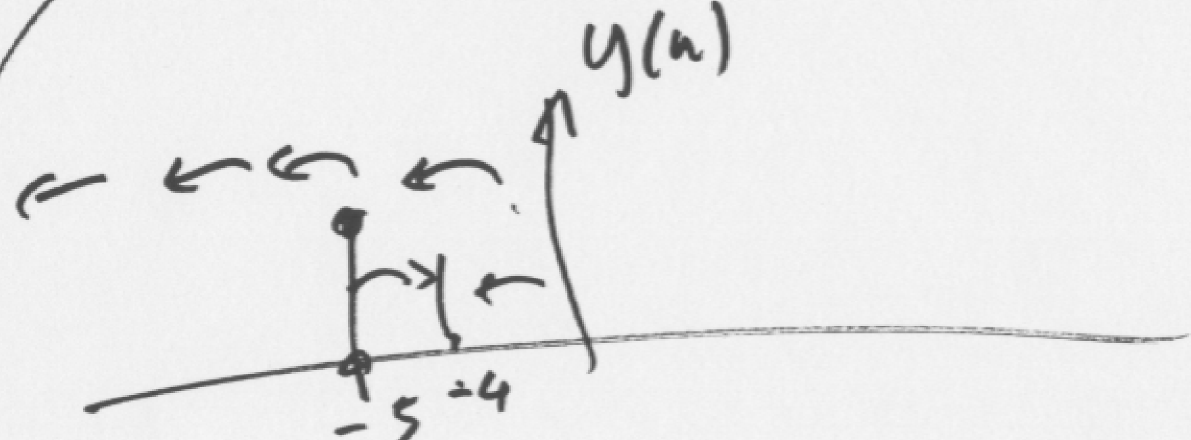
D.E corresponds to $\underline{\underline{2}}$ different systems.

Causal implementation

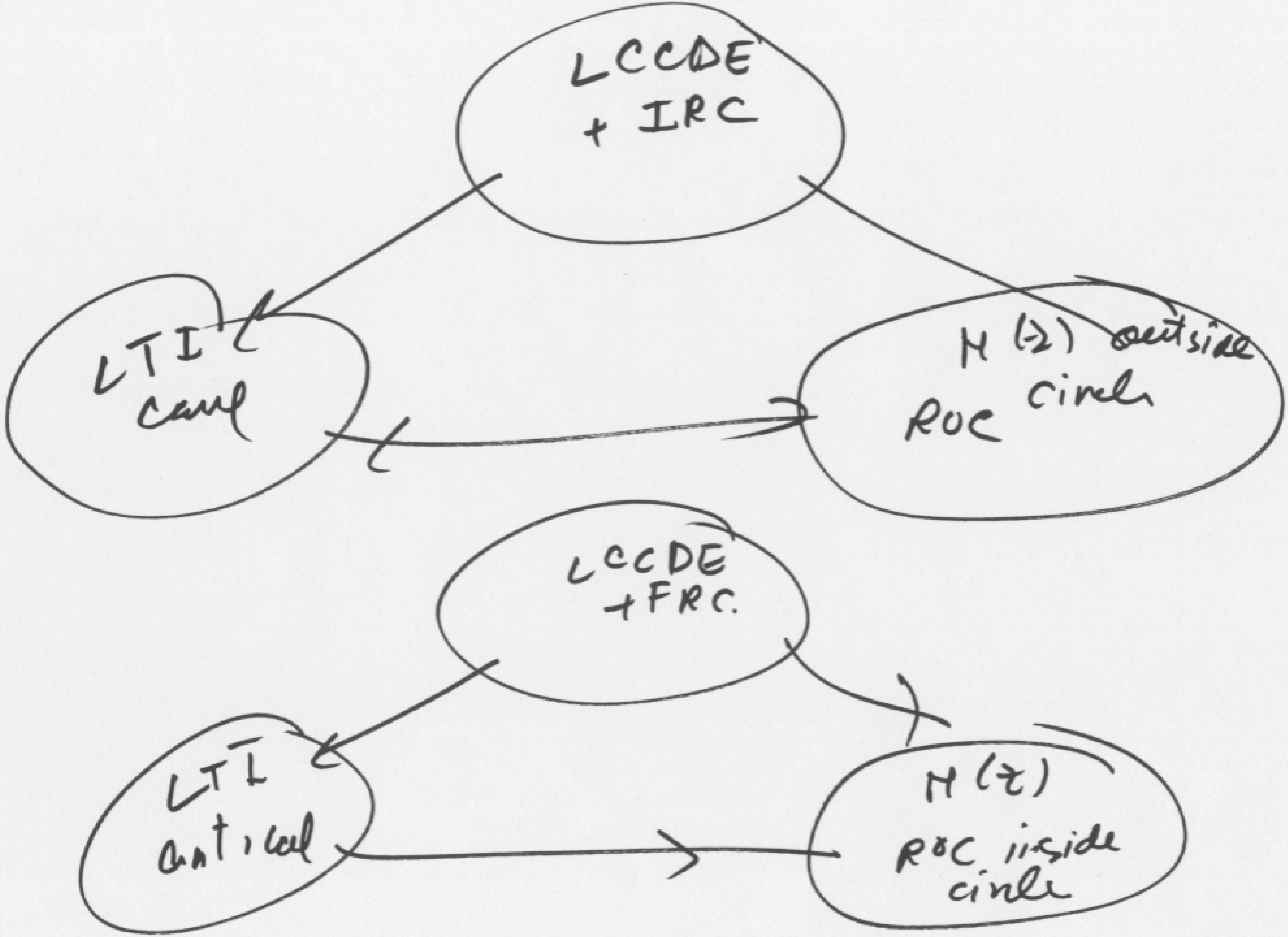
$$y(n) \longleftarrow a\,y(n-1) + x(n)$$

Anticausal implementation

$$y(n-1) \longleftarrow \frac{1}{a}y(n) - \frac{1}{a}x(n)$$



$$y(n) \longleftarrow \frac{1}{a}y(n+1) - \frac{1}{a}x(n+1)$$


LCCDE + IRC
LTI causal
H(z) outside circle ROC
LCCDE + FRC.
LTI anticausal
H(z) ROC inside circle

$$y(n) = \alpha y(n-1) + \alpha y(n-2) + x(n)$$

$$y(n) \longleftarrow y(n-1), \ldots y(n-2)$$

$$y(n) \longleftarrow y(n-1), y(n+1)$$

$$y(n) \longleftarrow y(n+1), y(n+2)$$