Nov 15, 03

How To use DFT To do Convolution

Convolution : $\longrightarrow$ linear Convolution

$$X_1 * X_2 = X_3$$

$$X_3(n) = \sum_k x_1(k)\, x_2(n-k)$$

LTI systems.

$$x(n) \rightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \rightarrow y(n) = \sum_k x(k) h(n-k)$$

- periodic convolution

$$\underset{N}{\tilde{X}_1} \circledast \underset{N}{\tilde{X}_2} = \underset{N}{\tilde{X}_3(n)}$$

1

- Circular Convolution :

2 finite length sequences.

x_1 N pt x_2 N pt

$$\underbrace{x_1}_{N\,pt} \underbrace{\circledast}_{\text{circularly convolution}} \underbrace{x_2}_{N\,pt} = \underbrace{x_4}_{N\,pt.}$$

Def of circular Convolution :

$$x_4(n) \triangleq x_1 \circledast x_2 \triangleq \left[\tilde{x}_1 \circledast \tilde{x}_2\right] \cdot R_N(n)$$

Show: Npt DFT $x_4(n) = X_1(k)\, X_2(k)$

$$\underbrace{X_4(k)}_{N\text{pt DFT of } x_4} = \underbrace{X_1(k)}_{N\text{pt DFT of } x_1}\ \underbrace{X_2(k)}_{N\text{pt DFT of } x_2}$$

We know from DFS properties:

$$\tilde{x}_1(n) \circledast \tilde{x}_2(n) = \tilde{x}_4(n)$$

$$\tilde{X}_1(k) = \text{DFS}\left\{\tilde{x}_1(n)\right\} \quad \ldots\ldots$$

$$\tilde{X}_1(k)\, \tilde{X}_2(k) = \tilde{X}_4(k)$$

$$R_N(k)\left\{\tilde{X}_1(k)\,\tilde{X}_2(k)\right\} = R_N(k)\,\tilde{X}_4(k)$$

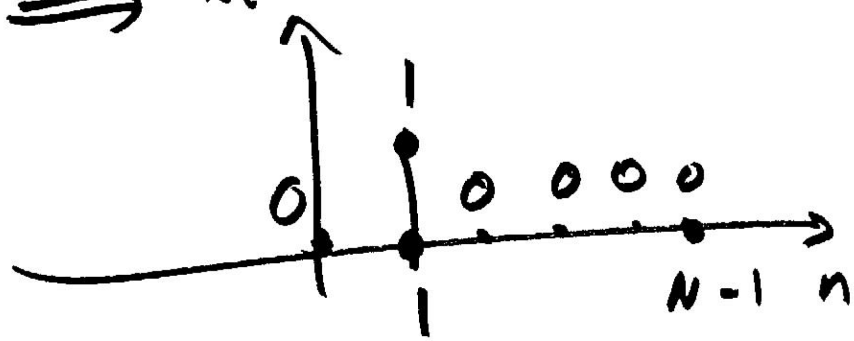
$$X_1(k)\,X_2(k) = X_4(k)$$

$\Longrightarrow$ If multiply N pt DFT of 2 N pt sequences, you get DFT of their circular convolution & not their linear convolution

Ex of circular Convolution:

$x_1(n) = \delta(n - n_0)$

$n_0 = 1 \Longrightarrow x_1(n) = \delta(n-1)$



$x_2(n)$ an N pt seq.

$$x_1(n) = \begin{cases} 0 & 0 \le n < 1 \\ 1 & n = 1 \\ 0 & 1 < n < N \end{cases}$$

Fig. 8.14 O&S.

$$x_3(n) = x_1(n) \circledast x_2(n)$$

$$= R_N(n)\left[\tilde{x}_1(n) \circledast \tilde{x}_2(n) \right]$$

$$= \sum_{m=0}^{N-1} x_2(m) \left[\tilde{x}_1(n-m) R_N(m) \right]$$

<u>Ex2</u> circular convolution of:

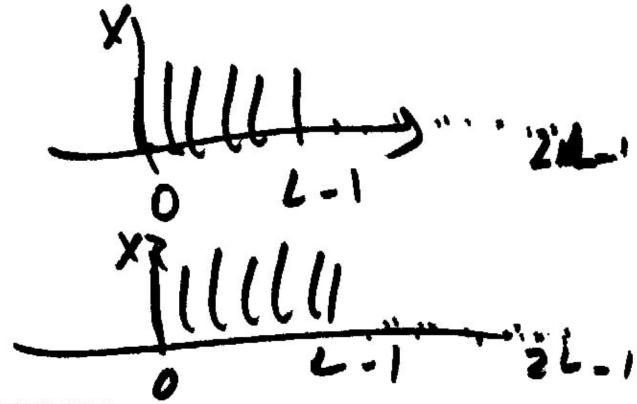
$$x_1(n) = x_2(n) = \begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

N pt circular convolution x_1 & x_2

2 Cases of N:

(1) $N = L$

(2) $N = 2L$



Case (1) $N = L$

Use DFT do our circ. conv..

Take $N_{pT} = L$ pt. DFT of x_1

$$X_{L}(k) = \sum_{n=0}^{L-1} x_1(n)\, e^{-j\frac{2\pi n k}{L}}$$

x_1 & x_2

$$X_L(k) = \begin{cases} L & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

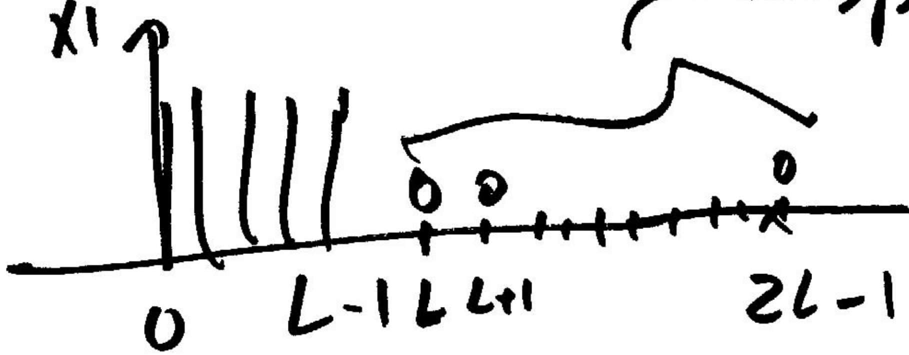
$X_3(k)$ = L pt. circ. Conv. of x_1 & x_2

$$= X_L(k)\, X_L(k) = \begin{cases} L^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{L pt IDFT}\{X_3(k)\} = \begin{cases} L & 0 \le n \le L-1 \\ 0 & \text{otherwise} \end{cases}$$

Case 2 $\quad N = 2L.$

- $2L$ pT. circ. con of x_1 & x_2.



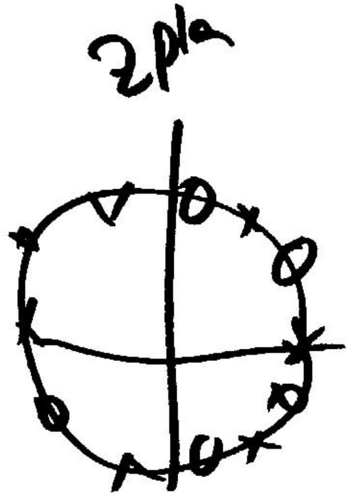
pad L extra zeros. & I get a $2L$ point seq.

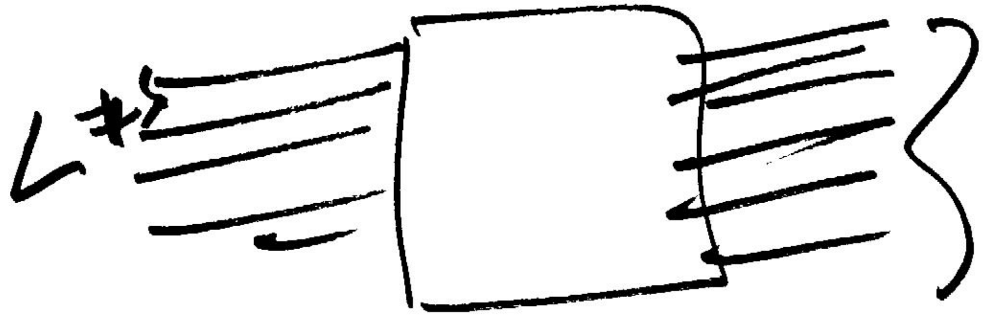
Compute $2L$ pt DFT of x_1

$$= \sum_{n=0}^{2L-1} x_1(n)\, e^{-j\frac{2\pi n k}{2L}}$$

$$= \sum_{n=0}^{L-1} x_1(n)\, e^{-j\frac{2\pi n k}{2L}}$$

$$= X_{2L}(k)$$





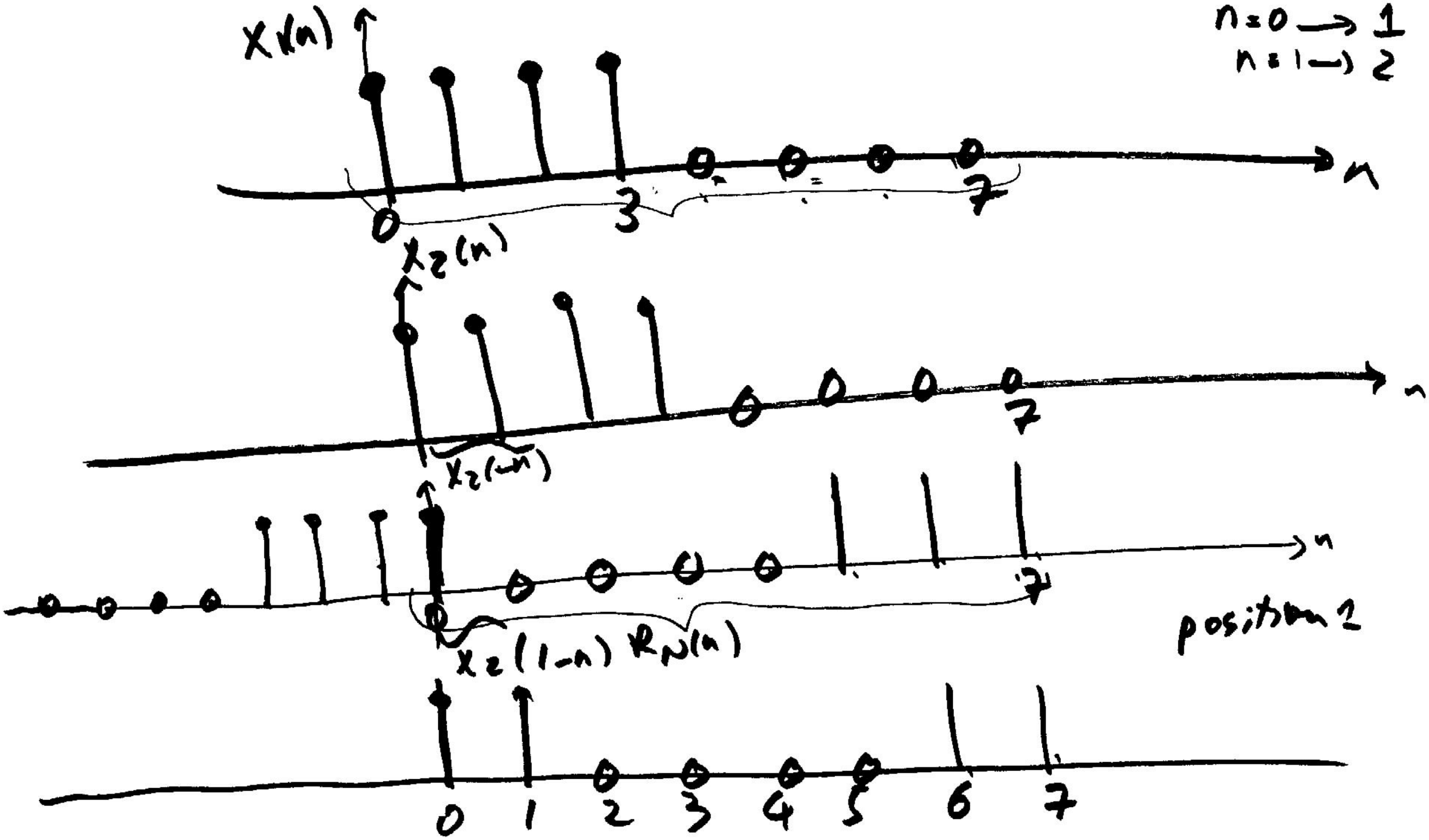
2L pT
DFT of x_1

2L pT circ. conv =

$$\mathrm{IDFT}_{2L}\left\{ X_{2L}(k)\, X_{2L}(k) \right\} = x_1 \circledast x_2$$

2L pT.

$n=0 \rightarrow 1$
$n=1 \rightarrow 2$



Goal: prove: padding with enough zeros can convert circ. to linear conv.

$x_1(n) \longrightarrow L$ pt. $N > L$

$x_2(n) \longrightarrow P$ pt. $N > P$

goal: $x_3(n) = x_1 * x_2$ = linear convolution.

To find linear convolution.

$$x_3(n) = \sum x_1(m)\, x_2(n-m)$$

$$\text{DTFT}\{x_3(n)\} = X_3(\omega) = \sum_n x_3(n)\, e^{-j\omega n}$$

$$X_3(\omega) = X_1(\omega)\, X_2(\omega)$$

DTFT of x_1 ; DTFT of x_2

Sample $X_3(\omega)$ at N equally spaced pt.

$$= Y(k)$$

$$Y(k) = \left[X_3(\omega)\right]_{\omega = \frac{2\pi k}{N}}$$

$$\mathrm{IDFT}\{Y(k)\} = \begin{cases} \sum_{r=-\infty}^{+\infty} X_3(n+rN) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

N pt.

Thought exp.

$$Y[k] = \left[X_1(\omega)\right]_{\omega = \frac{2\pi k}{N}} \left[X_2(\omega)\right]_{\omega = \frac{2\pi k}{N}}$$

= N pt DFT of x_1 · N pt DFT of x_2

$$\underset{N\,pt}{\mathrm{IDFT}}\left[Y(k)\right] = x_1 \circledast x_2$$

N pt circular convolution

$$x_1 \circledast x_2 = \begin{cases} \displaystyle\sum_{r=-\infty}^{+\infty} x_3(n + rN) & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

N pt circular

<u>Words</u>: N pT circ conv of x_1 & x_2 is same as then.

linear conv, x_3, periodicized with period N, Take one period.

$$x_3 \longrightarrow L+P-1$$

$$N \geq L+P-1$$

To ensure no aliasing.

x_3 [illegible] i.e. linear convolution can be computed just using circ. convol.

Know DFT can do circ. convolution

DFT can do linear convolution

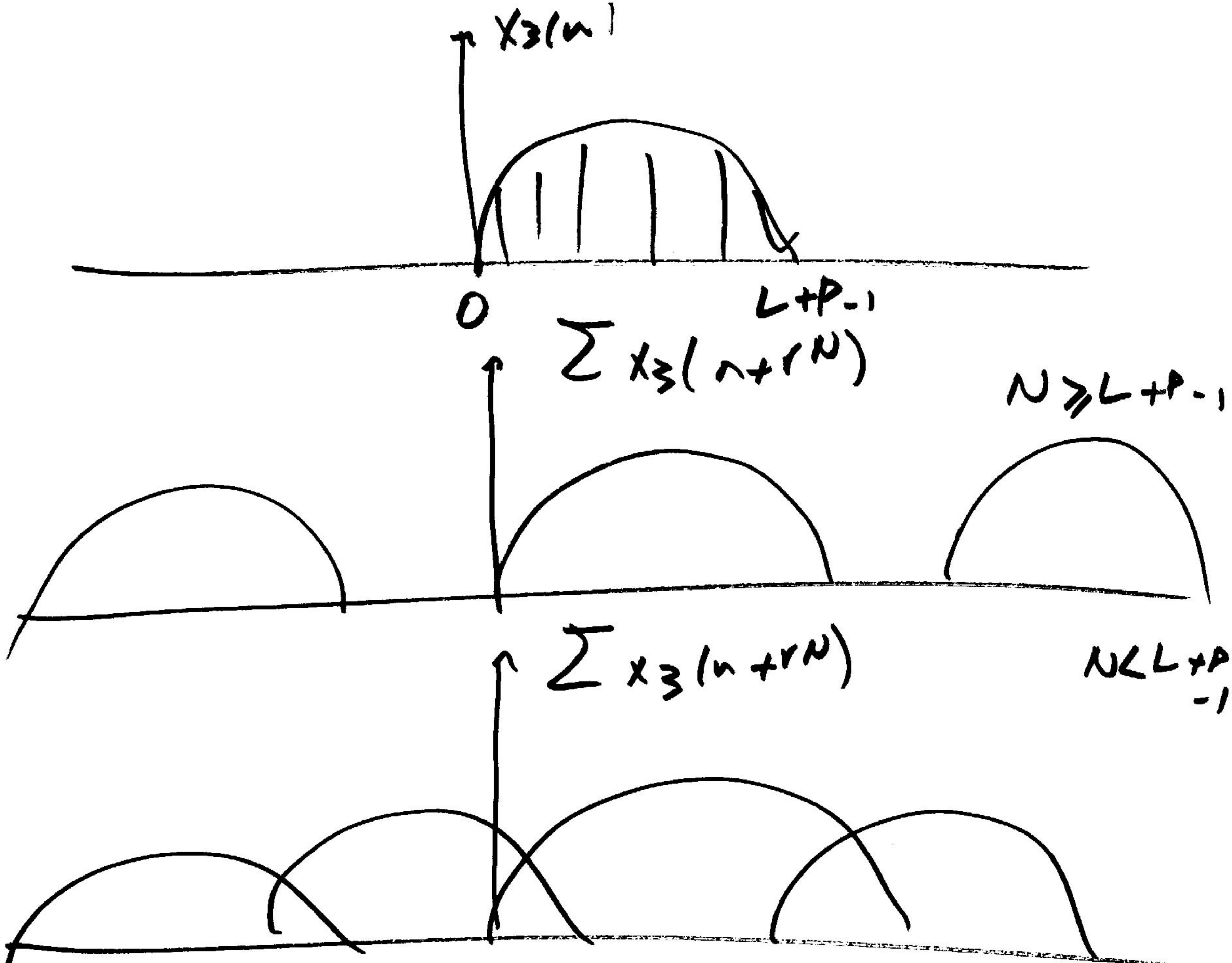
$$N \geq L + P - 1$$

$$x_1 \circledast x_2 = x_3(n)$$

Npt

linear conv

Npt circ conv of x_1 & x_2 same as linear convolution as long as N larger than the extent of linear convolution



$x_3(n)$
0
$L+P-1$
$\sum x_3(n+rN)$
$N \geqslant L+P-1$
$\sum x_3(n+rN)$
$N < L+P-1$

$L=10$
$P=5$
$N=20$

x_1 L pts.
x_2 P points.

① Pad x_1 $N-L$ zeros to get N pt.

② Pad x_2 with $N-P$ zeros to get N pt seq.

③ Take N pt DFT of the new seq we get in step 1 & 2.

③ Take IDFT N pt of the seq in step 3 $\longrightarrow$ linear conv $N \geq P+L-1$