EE123 Homework 8 Solutions

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- **Problem 8.43**

(a) For overlap-add, $L+P-1=N$. Therefore, each segment of the input is $L=N-P+1$ long. Hence, number of iterations is given by $\lceil \frac{10000}{N-P-1} \rceil = 64$. Therefore, we need to do 64 N point DFTs for the input sections, one N point DFT for the filter and 64 N point IDFTs after the product. Then, we will need to do $P-1$ additions 63 times to piece together the individual signals in the time domain. Note: in doing the complexity analysis, the number of multiplications corresponding to multiplying the DFTs must be factored in (i.e. 64 multiplies of 256).

(b) For overlap-save, $L=N$. The easiest way to look at the number of iterations is to look at the resulting signal. We know that the output is built in blocks of length $N-P+1=157$. The overall length of the output will be $10000+100-1=10099$, so the overall number of input blocks will be $\lceil \frac{10099}{157} \rceil = 65$. Hence, we need to do 65 N point DFTs for the input sections, one N point DFT for the filter and 65 N point IDFT after the product.

(c)

Method	Number of multiplications	Number of additions
overlap-add	148480	270429
overlap-save	150784	268288
Direct method	1000000	989901

The direct method numbers come from the fact that there are 100 points moving along a 10000 point sequence. Each point in the 100 point sequence will overlap with each point in the 10000 point sequence exactly once. Thus, there will be $100 * 10000 = 1000000$ overall multiplications. For additions, one fewer addition at each point in the resulting sequence than multiplications. So, we need only subtract the length of the sequence from the number of multiplications to get the number of additions.

So, we are lower in complexity by an order of magnitude.

- **Problem 8.44**

$$\begin{aligned} X_1(z) &= \sum_n x_1[n] z^{-n} \\ &= \sum_{n=0,\pm4,\pm8,\ldots} x\left[\frac{n}{4}\right] z^{-n} \\ &= \sum_n x[n] z^{-4n} \\ X_1(z) &= X(z^4) \end{aligned}$$

From the tables, $X(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$, ROC:$|z| > \frac{1}{4}$. Hence,

$$Q(0) = X_1(1) = \frac{4}{3} = X_1(-1) = Q(3)$$

Just set the other vales of Q to be zero. The corresponding sequence would be

$$q[n] = \begin{cases} \frac{4}{9} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

. This signal only holds for n within 0 to 5. Otherwise, the signal is zero.

- **Problem 8.47 (a)**

$$\begin{aligned} \left[\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi nk}{8}}\right]\Bigg|_{n=9} &= \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi n9}{8}} \\ &= \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi n}{8}} \\ &= x[((9))_8] \\ &= x[1] \end{aligned}$$

(b) For $k = 0, \ldots, 7$,

$$\begin{aligned} V[k] &= X(z)|_{z=2e^{j\frac{2\pi k+\pi}{8}}} \\ &= \sum_{n=0}^{7} x[n]z^{-n}|_{z=2e^{j\frac{2\pi k+\pi}{8}}} \\ &= \sum_{n=0}^{7} x[n]\left[2e^{j\frac{2\pi k+\pi}{8}}\right]^{-n} \\ &= \sum_{n=0}^{7} \left[2^{-n}e^{-j\frac{\pi n}{8}}x[n]\right] e^{-j\frac{2\pi nk}{8}} \end{aligned}$$

Hence, $v[n] = 2^{-n}e^{-j\frac{\pi n}{8}}x[n]$ for $n = 0, \ldots, 7$.

(c) For $n = 0, \ldots, 3$

$$\begin{aligned} w[n] &= \sum_{k=0}^{3} W[k]e^{j\frac{2\pi nk}{4}} \\ &= \frac{1}{4}(\sum_{k=0}^{3} X[k]e^{j\frac{2\pi nk}{4}} + \sum_{k=0}^{3} X[k+4]e^{j\frac{2\pi nk}{4}}) \\ &= \frac{1}{4}(\sum_{k=0}^{3} X[k]e^{j\frac{2\pi nk}{4}} + \sum_{k=4}^{7} X[k]e^{j\frac{2\pi n(k-4)}{4}}) \\ &= 2\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi(2n)k}{8}} \end{aligned}$$

Hence, $w[n] = 2x[2n]$ for $n = 0, \ldots, 3$.

(d) The solution to this can be arrived either using duality to the previous part and repeating the DFT from $4, \ldots, 7$ or by deriving the formula as in the problem 8.26(c) and 8.32.

$$y[n] = x[n] + x[((n+4))_8] \text{ for } n = 0, \ldots, 7$$

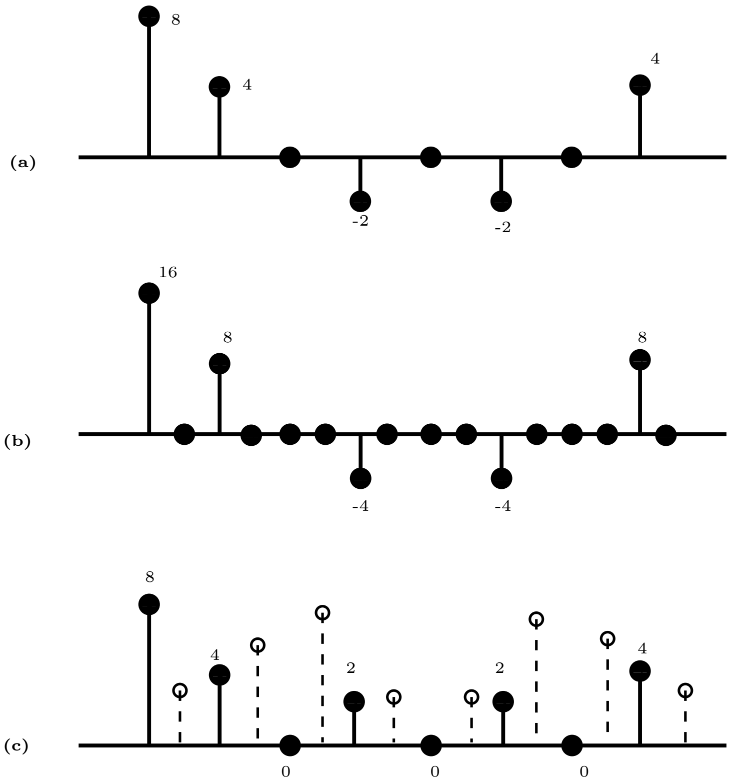

Figure 1: Problem 8.50

- **Problem 8.50 (a)** The two cosines in the problem are at $\omega_1 = \frac{\pi}{4}$ and $\omega_2 = \frac{3\pi}{4}$ and are powers of W_8.
$$x[n] = 1 + \frac{1}{2}e^{j\frac{\pi n}{4}} + \frac{1}{2}e^{-j\frac{\pi n}{4}} - \frac{1}{4}e^{-j\frac{3\pi n}{4}} - \frac{1}{4}e^{-j\frac{3\pi n}{4}}$$
From the properties of DFT, the DFT $X[k]$ is shown in Figure 1a.

  (b) Following similar steps of (a), we have the DFT shown in Figure 1b.

  (c) The async pulse which is now spanning over only 8 sample points still has a peak at $k = 0, 2, 6, 10, 14$. The zero-crossings still remain the same as (a), except wider apart as in (b). Hence, $k = 0, 2, 6, 10, 14$ are the same as part (b). All the four aliasing asyncs have zero-crossing at $k = 4, 8, 12$ (easily verified by a quick plot!). Hence, we know $|X_{16}|[k]$ for $k = 0, 2, 4, 6, 10, 12, 14$, which is shown in Figure 1c. Also note the symmetry which follows from the fact that $x[n[$ is real. Note that the values don't look just like interpolated values. This is because the frequency signal we're looking at is the convolution of a sinc (a rectangular window in time) with a bunch of deltas.
- **Problem 8.59**

$$\begin{aligned} r[n] &= x[n] \star y[n] \\ R(\omega) &= X(\omega)Y(\omega) \\ R_s[k] &= R(\omega)|_{\omega=\frac{2\pi k}{128}} \\ &= X(\omega)Y(\omega)|_{\omega=\frac{2\pi k}{128}} \end{aligned}$$

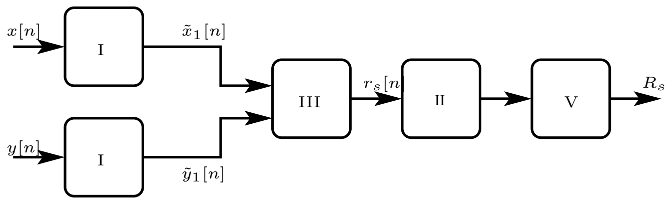

Figure 2: Problem 8.59

Since both $x[n]$ and $y[n]$ are finite length sequences on the right side of origin, their ROC includes the unit circle (and there is no cancellation of poles/zeros in $R(\omega)$,

$$R_s[k] = X(\omega)|_{\omega=\frac{2\pi k}{128}} \, Y(\omega)|_{\omega=\frac{2\pi k}{128}} = X_1[k]Y_1[k]$$

where $X_1[k] = X(\omega)|_{\omega=\frac{2\pi k}{128}}$ and $Y_1[k] = Y(\omega)|_{\omega=\frac{2\pi k}{128}}$ for $k = 0, \dots, 128$. If $x_1[n]$ and $y_1[n]$ denote the corresponding 128 point IDFT,

$$r_s[n] = x_1[n] \otimes_{128} y_1[n] = \sum_{n=0}^{127} \tilde{x}_1[m]\tilde{y}_1[n-m], \text{ for } n = 0, \dots, 128$$

$$\tilde{x}_1[n] = \sum_{l_1=-\infty}^{\infty} x[n+128l_1] \quad \tilde{y}_1[n] = \sum_{l_2=-\infty}^{\infty} y[n+128l_2]$$

Procedure

- given $x[n]$ and $y[n]$, use module I to get $\tilde{x}_1[n]$ and $\tilde{y}_1[n]$
- Use module III to get $r_s[n]$ from $\tilde{x}_1[n]$ and $\tilde{y}_1[n]$
- Extract the first 128 sample points using module II
- Use module V to get $R_s[k]$.

Total Cost = 110 as shown in Figure 2. Note that technically, module II in the figure is not needed because module V only cares about the signal over the range 0 to 127. But, because the block doesn't cost us anything and because it would require me to remake the figure, the solution will be left as is.

- **Problem 8.60**

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z - bz^{-1}}$$

Therefore, $x[n] = y[n+1] - by[n-1]$.Now,

$$V[k] = \left[W_N^{-k} - bW_N^k\right] Y[k]$$

Therefore, $v[n] = y[((n+1))_N] - by[((n-1))_N]$. If $n = 1, \dots, N-2$, then, $((n+1))_N = n+1$ and $((n-1))_N = n-1$. Hence,

$$v[n] = x[n] \text{ for } n = 1, \dots, N-2$$

For $n = 0$ and $n = N-1$, we cannot determine $x[n]$ if $y[n]$ is given only for $n = 0, \ldots, N-1$. In general (even if $x[n]$ could be determined), the first and last values are not necessarily equal.

- **Problem 8.63**

This is a modified version of the overlap-save. We have $P = 50$, $N = 128$, and $L = 100$. If we did linear convolution of the blocks, we would get a length of $L + P - 1 = 149$ samples. But, since N is less than this number, we get aliasing in the time domain. The problem is, however, that we are using the overlap-save method. If we overlap the samples by anything larger than $V = 49$, we see that the resulting shift in input would require adding of the resulting output (i.e. not overlap save). This number comes straight from the overlap save equations. With $V = 49$, we find that $M = 51$ in order to advance the input correctly. The correct samples are the samples in the range 49 to 99 inclusively.