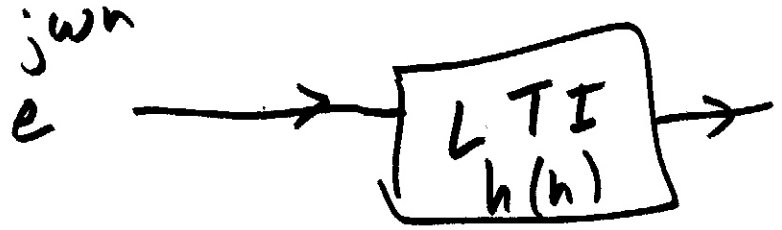
oct 14, 2005

Linear Phase Filtering

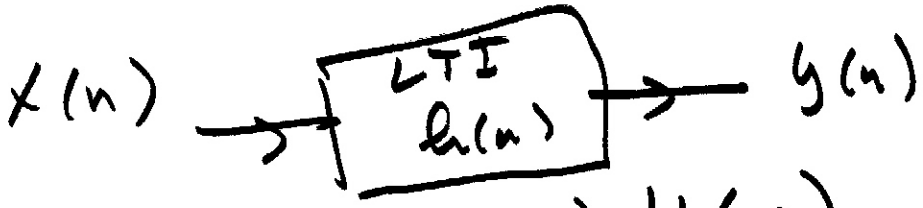


$H(\omega)$

↓

DTFT $\{h(n)\}$

= Freq. Response filter at ω.



$$Y(\omega) = X(\omega)\, H(\omega)$$

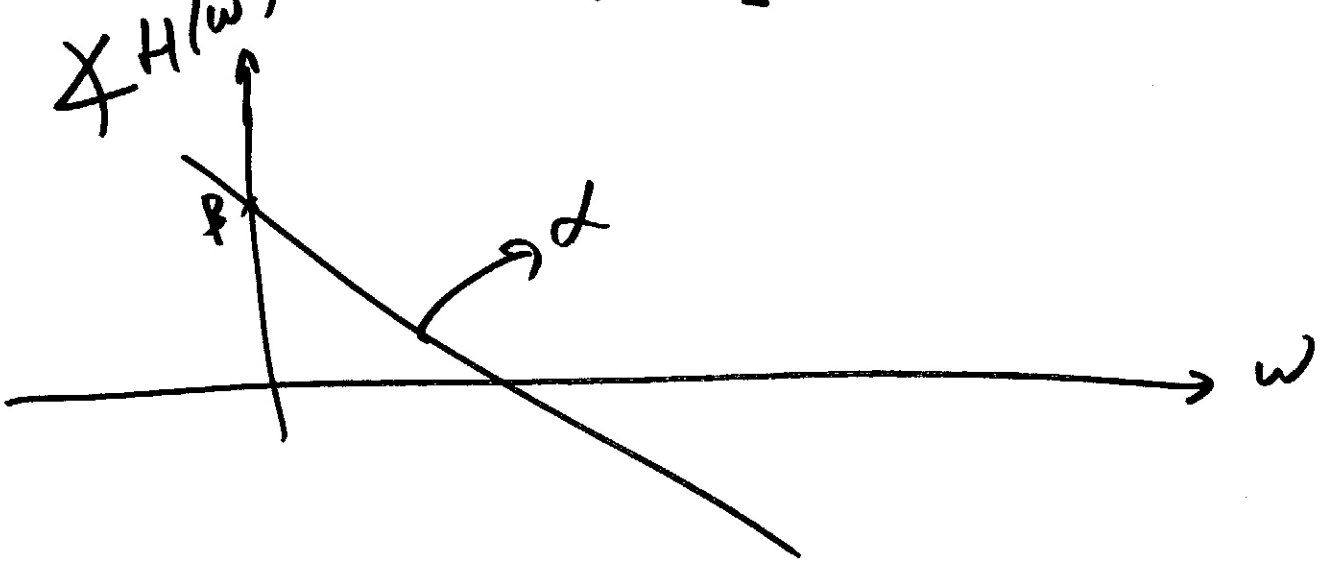
$$|Y(\omega)| = |H(\omega)|\, |X(\omega)|$$

$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

Def Linear phase LTI system : (LTI: filter.)

if $\angle H(\omega)$ is linear in ω

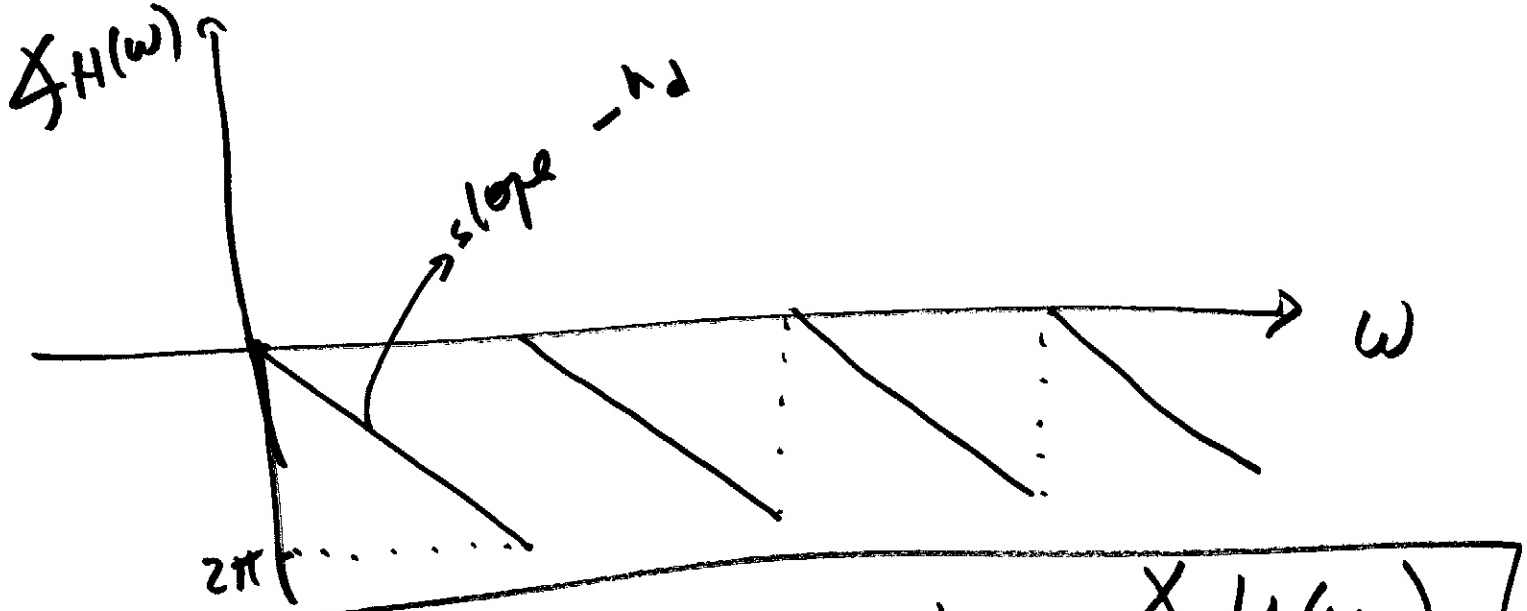
it is of the form $\alpha\omega + \beta$



Consider a pure delay LTI system

$$h(n) = \delta(n - n_d)$$

$|H(\omega)| = 1$ $\quad\quad$ $\angle H(\omega) = -\omega n_d$

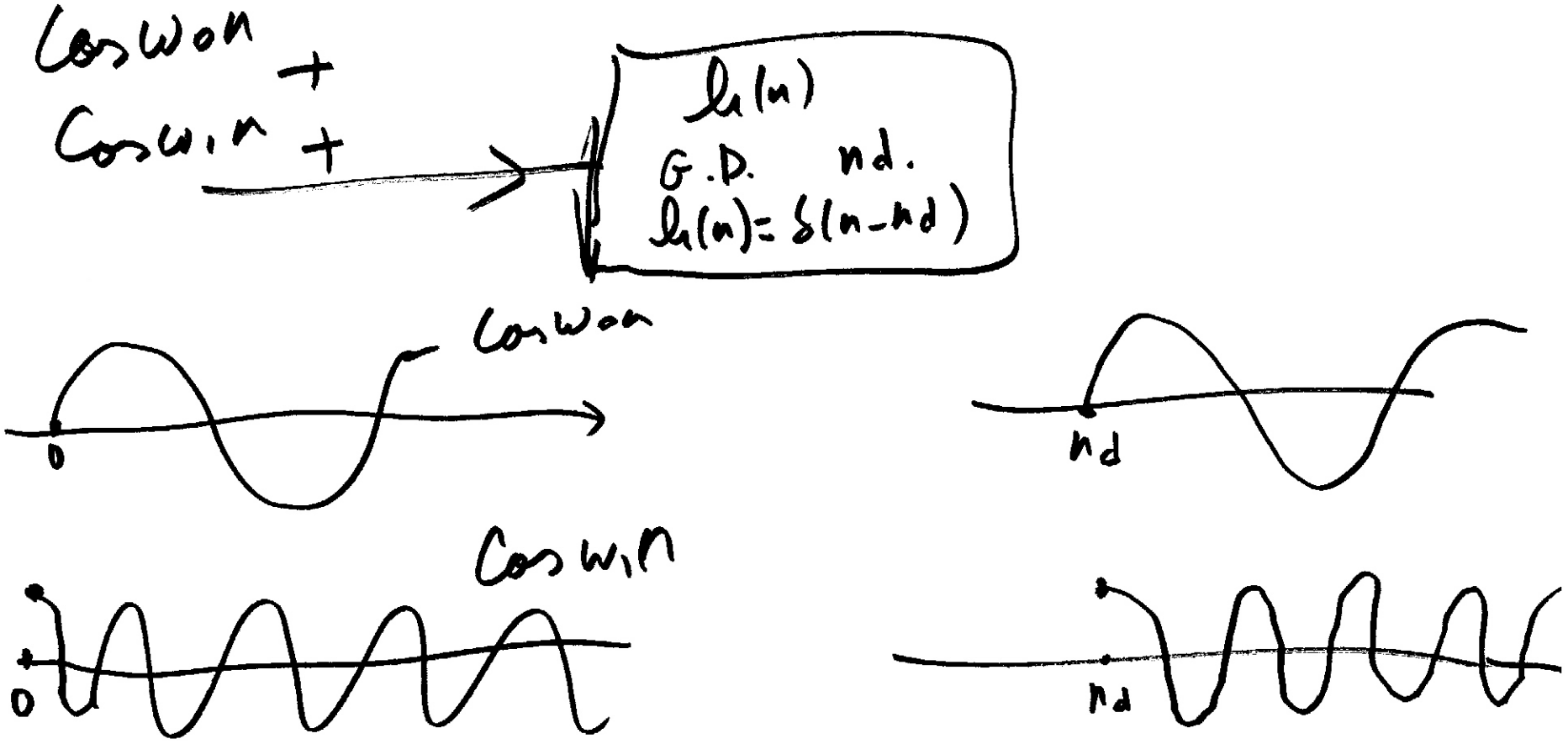


Def. → Group delay $\triangleq -\dfrac{d}{d\omega} \angle H(\omega)$

$\angle H(\omega) = -\omega n_d \Rightarrow$ G.D. $= -\dfrac{d}{d\omega}(-\omega n_d) = n_d$

intuitively. n_d is the amount of delay for ALL Freq. is indpendant of ω

All frequencies are delayed by the same amount as they pass through the system.



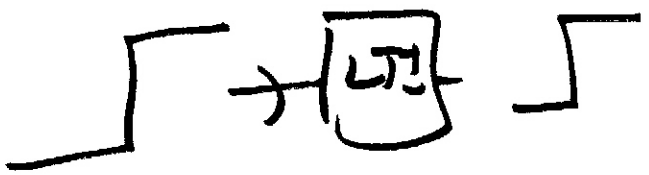
observation if Linear phase ⟶ group delay is constant.

⟹ relative phase between sinusoidal components are same as they go through the system

⟹ no distortion added to the signal

Q why processing relative phase is important?

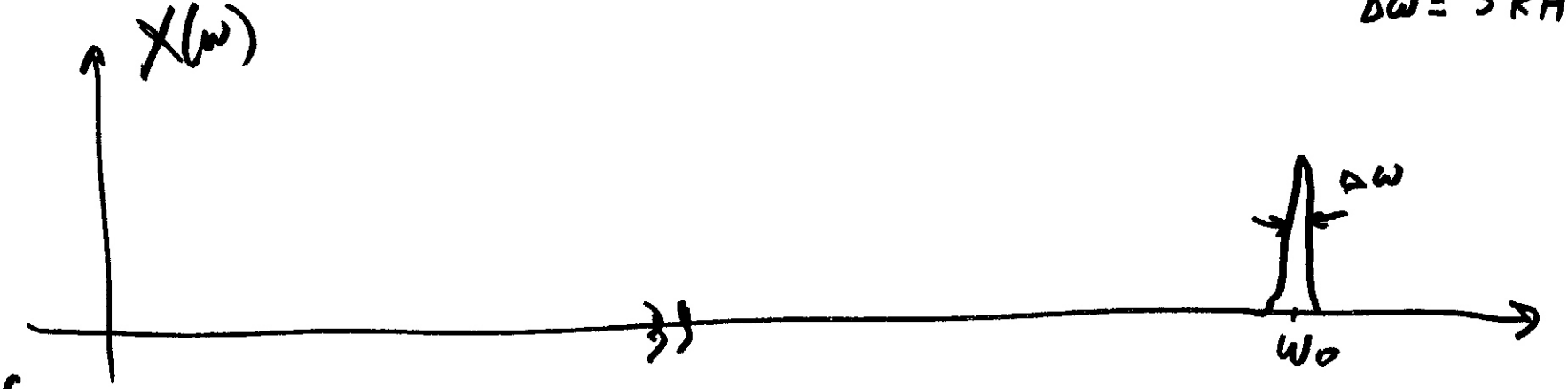
image processing.



To preserve edges in image processing, it is good to have linear phase

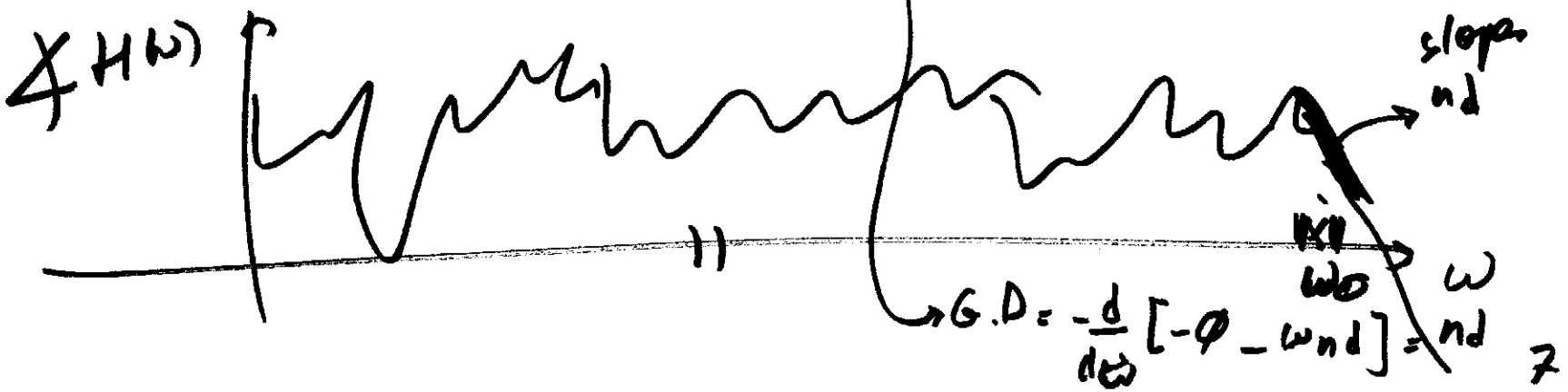
Consider a narrow band signal. cellphone $\omega_0 = 900\,MHz$.

$\Delta\omega = 5\,KHz$



If $x(n) = s(n)\cos\omega_0 n$

- pass $x(n)$ through LTI filter $H(\omega)$. s.Th. $|H(\omega)| = 1$, can approximate $\angle H(\omega)$ at $\omega = \omega_0$ with a linear Term $\left[\angle H(\omega)\right]_{\omega=\omega_0} = -\phi - \omega n_d$



Then

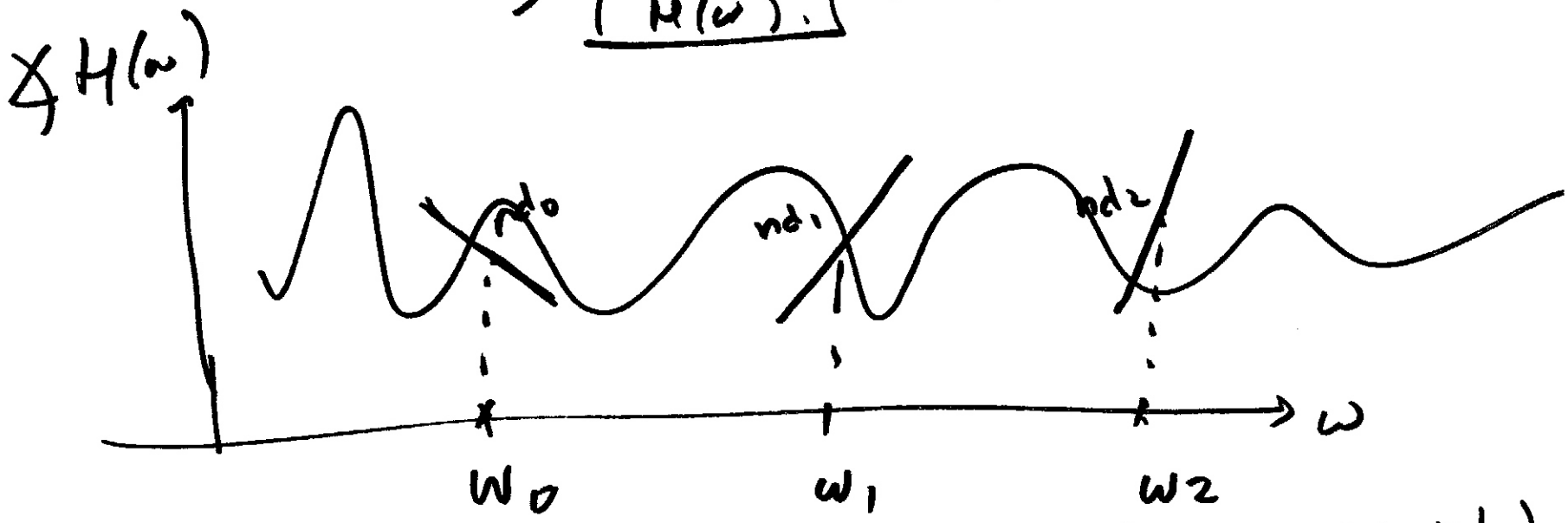
Can show: $y(n) = s(n - n_d)\cos(\omega_0 n - \omega_0 n_d - \phi)$

Conclusion: For a narrowband signal centered around ω_0 delay is proportional To

$$GD = \left[-\frac{d}{d\omega} \not\angle H(\omega) \right]_{\omega=\omega_0}$$

ω_0 = is center freq. of narrowband signal.

Consider $x(n) = a\cos\omega_0 n + b\cos\omega_1 n + c\cos\omega_2 n$.

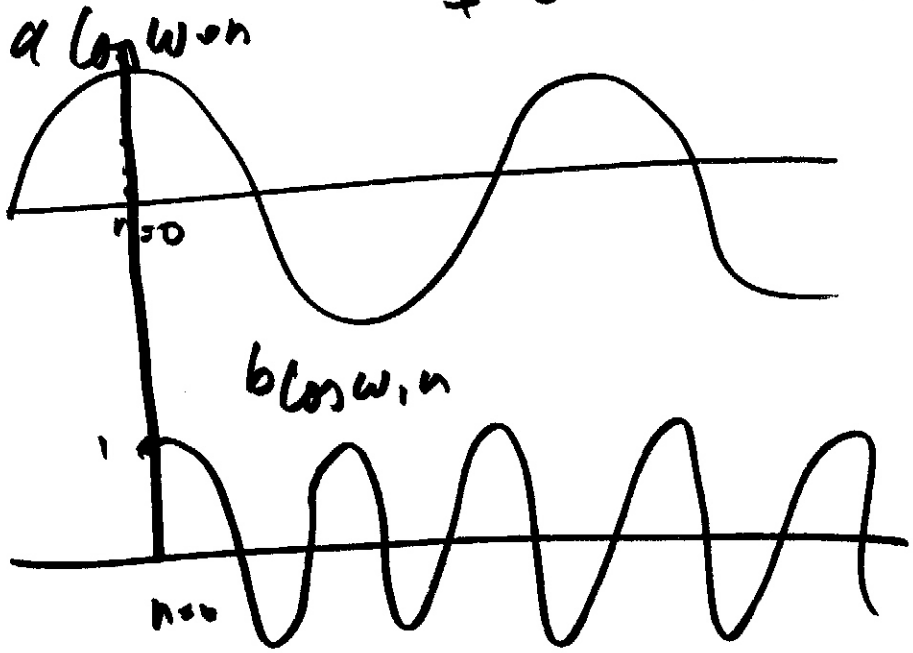


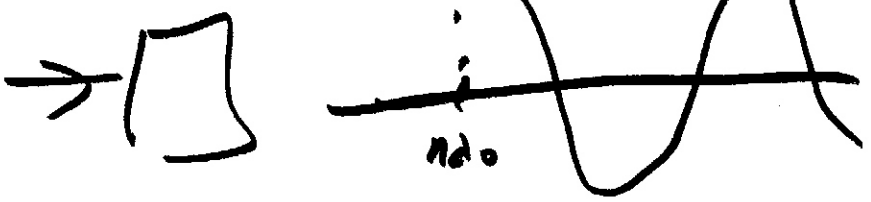
$a\cos\omega_0 n \longrightarrow a\cos(\omega_0 n - \omega_0 nd_0)$

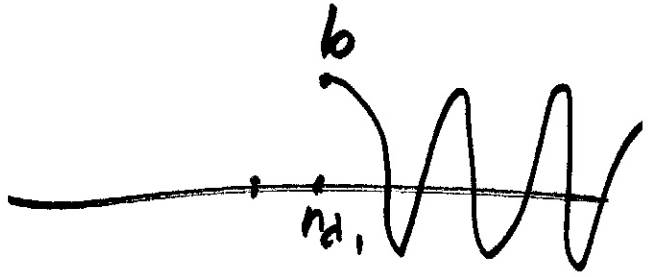
$b\cos\omega_1 n \longrightarrow b\cos(\omega_0 n - \omega_1 nd_1)$

$c\cos\omega_2 n \longrightarrow c\cos(\omega_2 n - \omega_2 nd_2)$

$$y(n) = a\cos(\omega_0 n - \omega_0 n d_0) + b\cos(\omega_1 n - \omega_1 n d_1) + c\cos(\omega_2 n - \omega_2 n d_2)$$







$n = n_0$

$a + b$ = amplitude

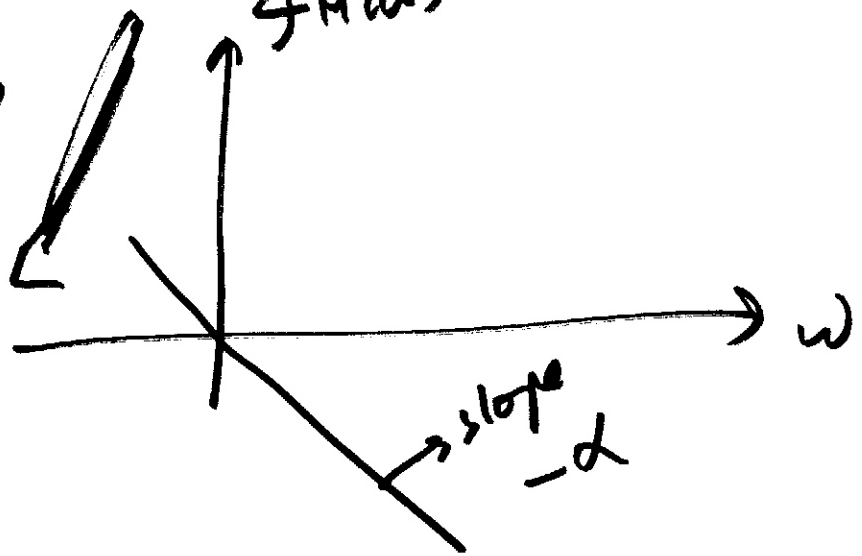
Unless $n d_0 = n d_1$

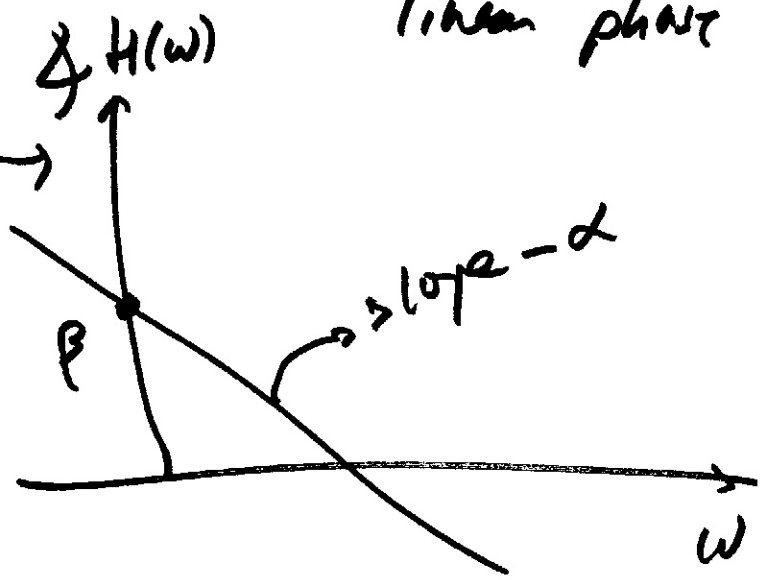
$\Rightarrow$ introduced distortion

Linear Phase filtering

Def $H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}}\, e^{-j\alpha\omega}$ linear phase

$H(\omega) = \underbrace{H_m(\omega)}_{\text{Real}}\, e^{j(\beta-\alpha\omega)}$ Generalized linear phase



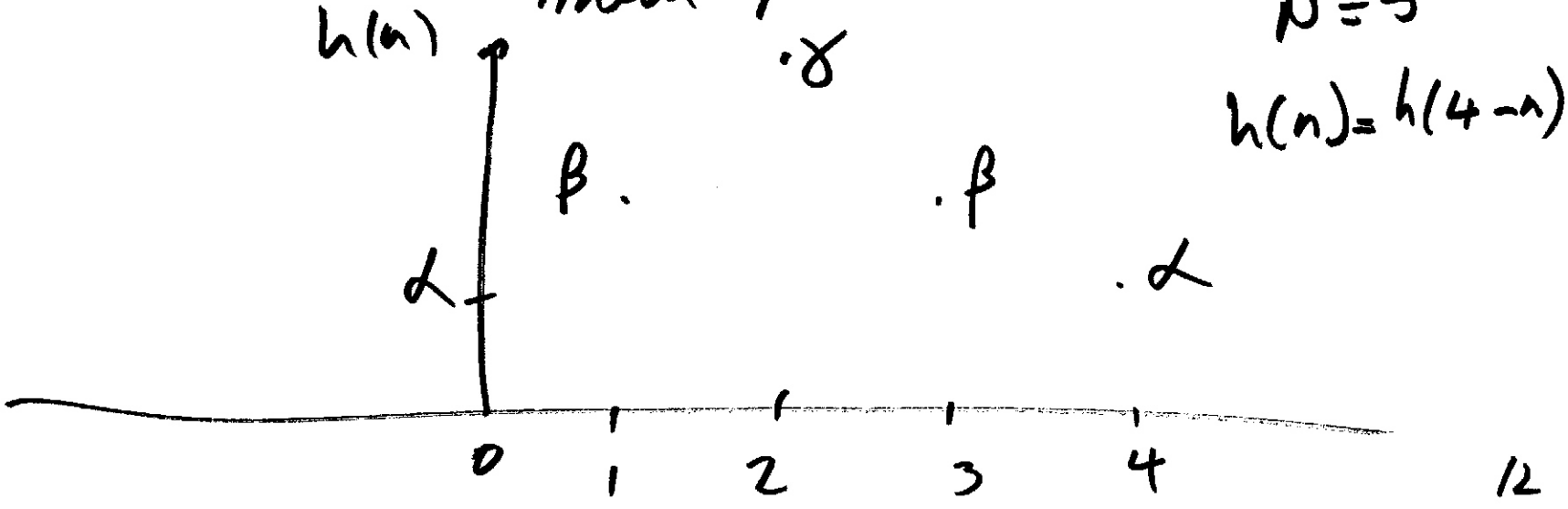


Theme How to get filters that are design
Linear phase?

Answer Impose symmetry or anti symmetry on filter coeff.

Show: If $h(n) = h(N-1-n)$ for FIR filter with real coeff $\Rightarrow$ Then linear phase.

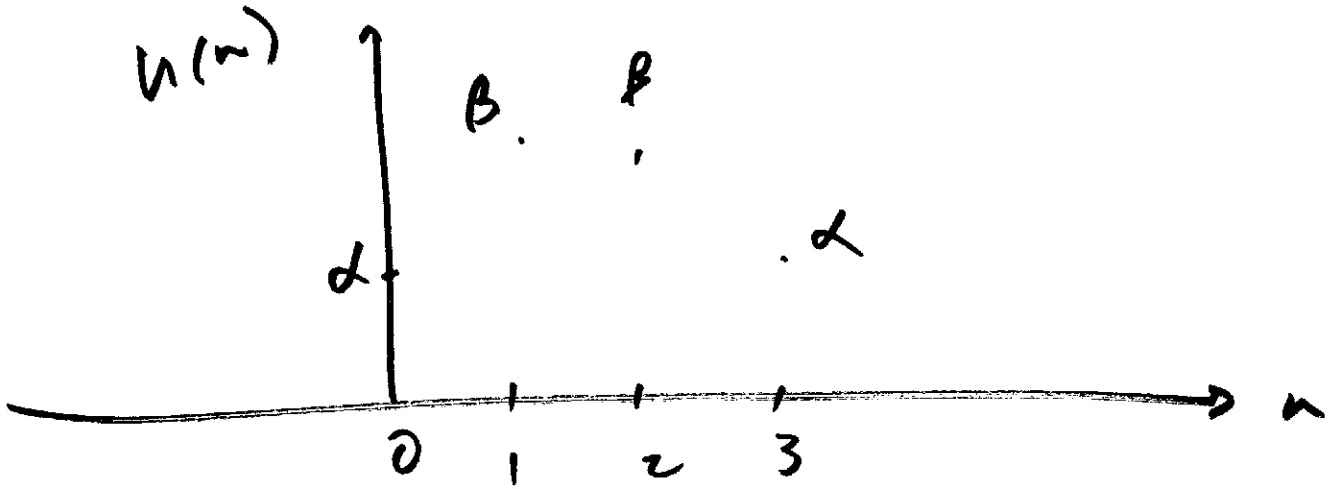
EX



$N = 5$

$h(n) = h(4-n)$

Ex $N=4$ $\quad h(n) = h(3-n)$



Prof: $H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

↳ change of variable $m = N-1-n$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \sum_{m=0}^{N/2-1} h(N-1-m)\, e^{-j\omega(N-1-m)}$$

↓ impose assumption $h(n) = h(N-1-n)$

$$H(\omega) = \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n + j\omega\left(\frac{N-1}{2}\right)} + e^{j\omega n - j\omega\left(\frac{N-1}{2}\right)} \right]$$

$$H(\omega) =$$

$$H(\omega) = \underbrace{e^{-j\omega\left(\frac{N-1}{2}\right)}}_{e^{-j\alpha\omega}} \underbrace{\sum_{n=0}^{N/2-1} h(n)\left[2\cos\left(\omega n - \frac{\omega}{2}(N-1)\right)\right]}_{\text{Real } H_m(\omega)}$$

with $\alpha = \frac{N-1}{2}$

$$H(\omega) = H_m(\omega)\, e^{-j\alpha\omega}$$

$\Longrightarrow$ linear phase.

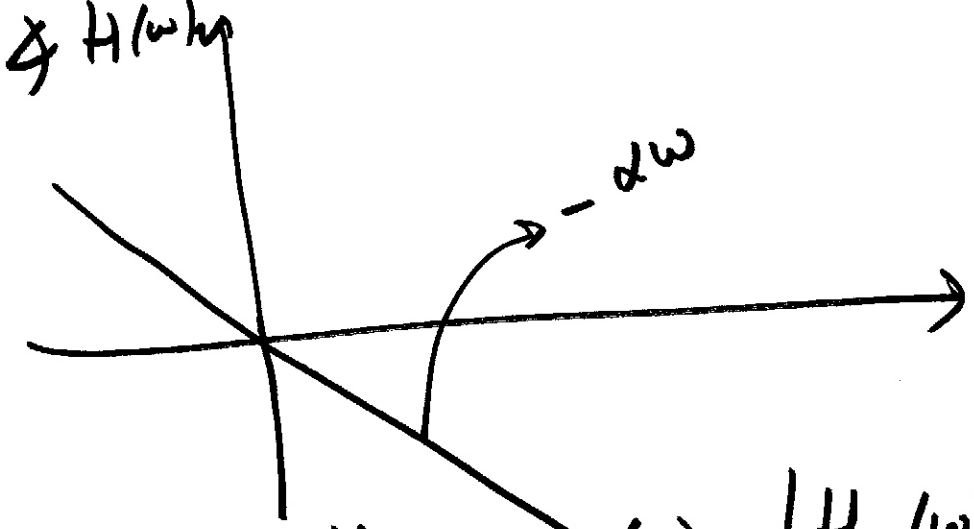
Q: what is $\angle H(\omega)$

$$H(\omega) = |H(\omega)|\, e^{j \angle H(\omega)} = H_m(\omega)\, e^{-j\alpha\omega}$$

Consider 2 case

① $H_m(\omega) > 0$ positive $\Rightarrow \angle H(\omega) = e^{-j\alpha\omega}$

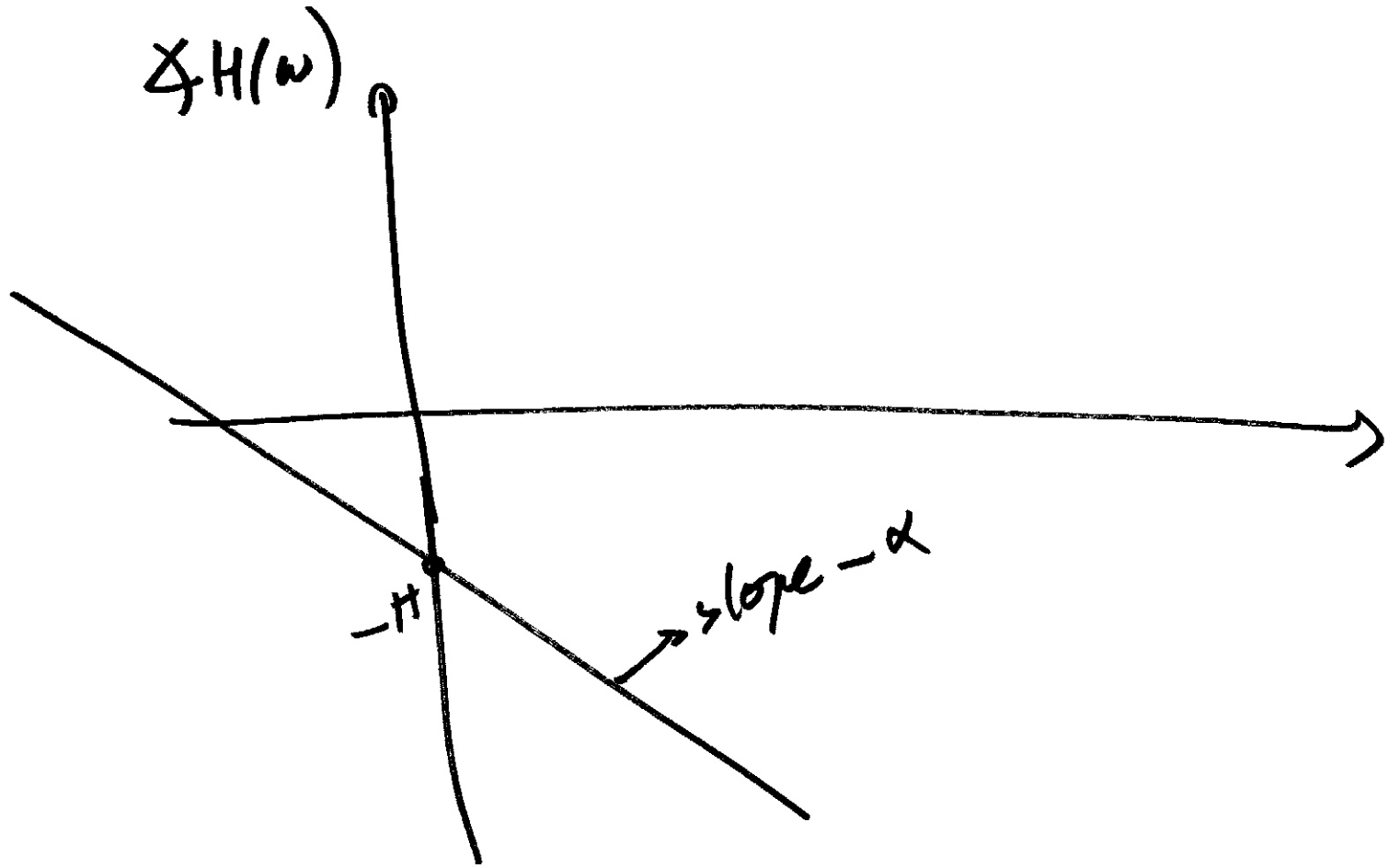
$\Rightarrow |H_m(\omega)| = |H(\omega)|$



② $H_m(\omega) < 0$ negative $\Rightarrow H(\omega) = |H_m(\omega)|(-1)\, e^{-j\alpha\omega}$

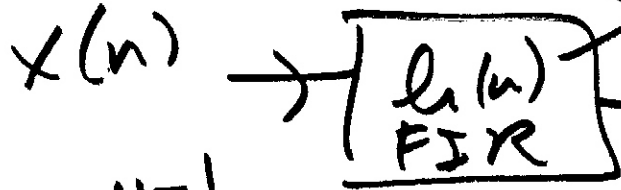
$= |H_m(\omega)|\, e^{-j\pi}\, e^{-j\alpha\omega}$

$= |H_m(\omega)|\, e^{-j(\alpha\omega+\pi)}$


∠H(ω)
-π
slope -α

Show Symmetry Condition that makes FIR filters linear phase, also reduces implementation complexity

$h(n) = h(N-1-n)$



$$y(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

$$= \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{k=N/2}^{N-1} h(k)\, x(n-k)$$

→ change of variable $k = N-1-m$

$$y(n) = \sum_{k=0}^{N/2-1} h(k)\, x(n-k) + \sum_{m=0}^{N/2-1} h(N-1-m)\, x(n-N+1+m)$$

$h(N-1-m) \rightarrow h(m)$

$$y(n) = \sum_{k=0}^{N/2-1} h(k) \underbrace{\left[x(n-k) + x(n-N+1+k) \right]}_{W(k)}$$

Each $y(n)$ $\quad N/2$ mults, N adds $\Big\}$ → linear phase

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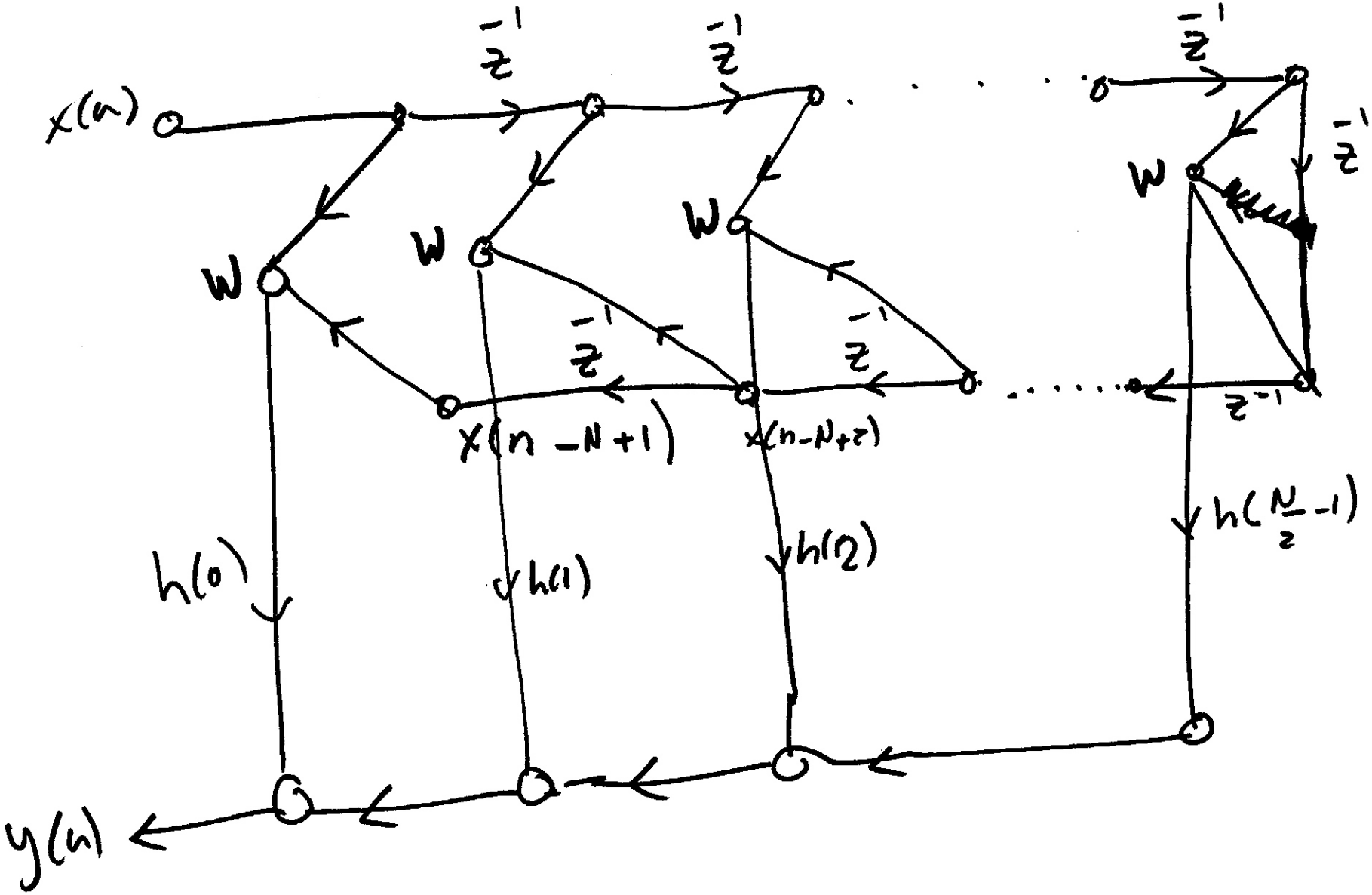
$$\sum_{k=0}^{N-1} h(k)\, x(n-k)$$

N mults

N adds.

---

linear phase $\quad 1/2$ as many mult.


x(n)
z^{-1}
z^{-1}
z^{-1}
W
W
W
W
z^{-1}
z^{-1}
z^{-1}
z^{-1}
x(n-N+1)
x(n-N+2)
h(0)
h(1)
h(2)
$h(\frac{N}{2}-1)$
y(n)