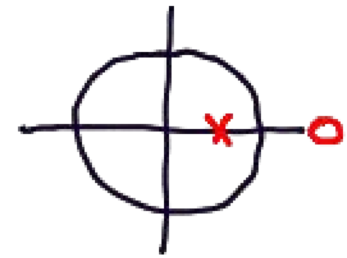
All-Pass Systems

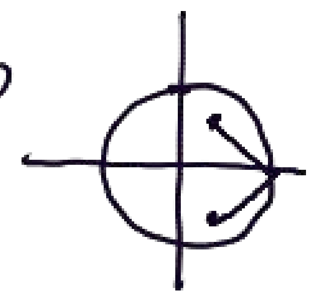
what is the magnitude response of

$$H(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$$



$$|H(e^{j\omega})| = \frac{|e^{-j\omega} - a^*|}{|1 - a e^{-j\omega}|} = \frac{|e^{-j\omega}(1 - a^* e^{j\omega})|}{|1 - a e^{-j\omega}|} =$$

$$= \frac{|1 - a^* e^{j\omega}|}{|1 - (a^* e^{j\omega})^*|} = 1 \quad \forall \omega$$



A generall all-pass system: ③

$$H_{ap}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \cdot \prod_{k=1}^{M_c} \frac{z^{-1} - e_k^*}{1 - e_k z^{-1}} \cdot \frac{z^{-1} - e_k}{1 - e_k^* z^{-1}}$$

Real

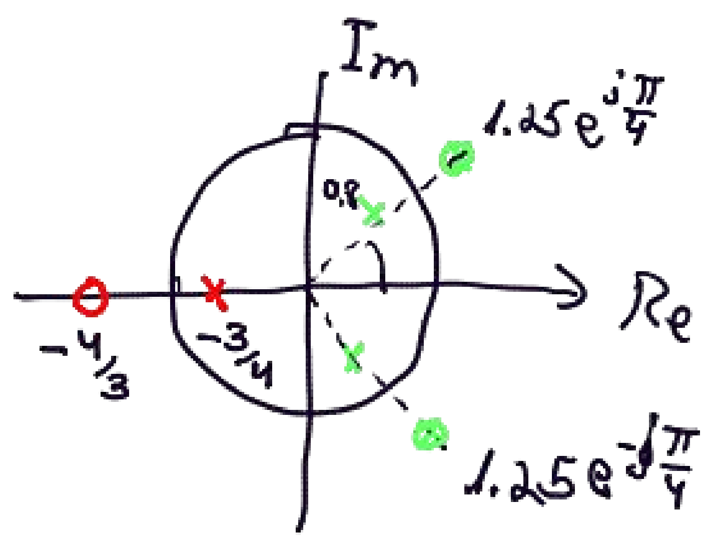
complex

d_k: real Poles

e_k: complex poles paired w/ conjugate e_k^*

$$|H_{ap}(e^{j\omega})| \equiv 1$$

Example



Phase response of an all-pass:

(4)

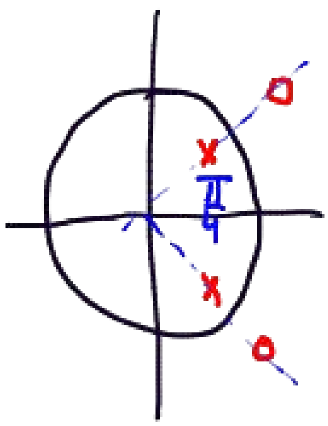
$$\arg\left[\frac{e^{-j\omega}-\overbrace{re^{-j\Theta}}^{a^*}}{1-\underbrace{re^{j\Theta}}_{a}e^{-j\omega}}\right]=\arg\left[\frac{e^{-j\omega}(1-re^{-j\Theta}e^{j\omega})}{1-re^{j\Theta}e^{-j\omega}}\right]=$$

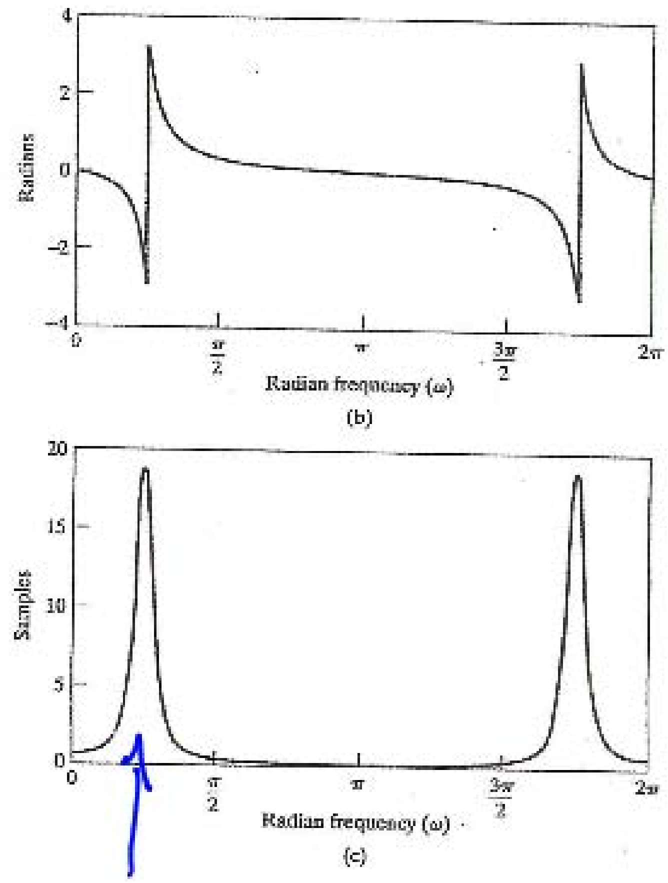
$$=\underbrace{\arg\left[e^{-j\omega}\right]}_{-\omega}-2\arg\left[1-re^{j\Theta}e^{-j\omega}\right]$$

$$\operatorname{grd}\left[\frac{e^{-j\omega}-re^{-j\Theta}}{1-re^{j\Theta}e^{-j\omega}}\right]=1-2\operatorname{grd}\left[1-re^{j\Theta}e^{-j\omega}\right]$$

< figure 5.20 >

Example:





(b)

(c)

can be used to compensate phase distortion.

Claim: for a stable ap system $H_{ap}(z)$:

(i) $\mathrm{grd}\left[H_{ap}(e^{j\omega})\right] > 0$

(ii) $\arg\left[H_{ap}(e^{j\omega})\right] \leq 0$

Delay positive → causal

phase negative → phase lag.

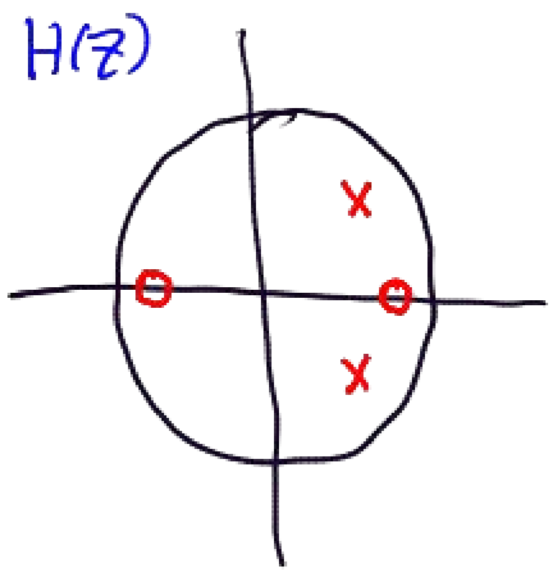
proof in book.

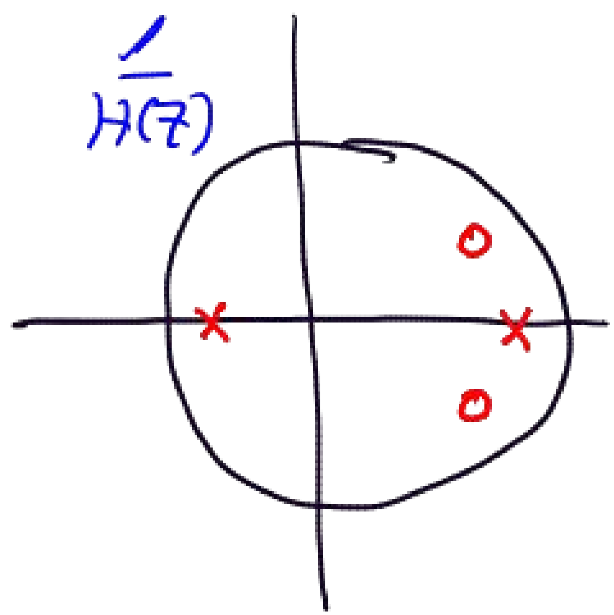
Minimum-Phase Systems

Definition: a $\underbrace{\text{stable and causal}}_{\text{poles insid unit circle}}$ system $H(z)$

who's inverse $\dfrac{1}{H(z)}$ is also stable & causal

zeros are insid unit circle.





AP-Min-Phase decomposition:

stable, causal system can be decomposed to:

$$H(z) = \underbrace{H_{min}(z)}_{\text{min phase}} \cdot \underbrace{H_{ap}(z)}_{\text{all pass}}$$

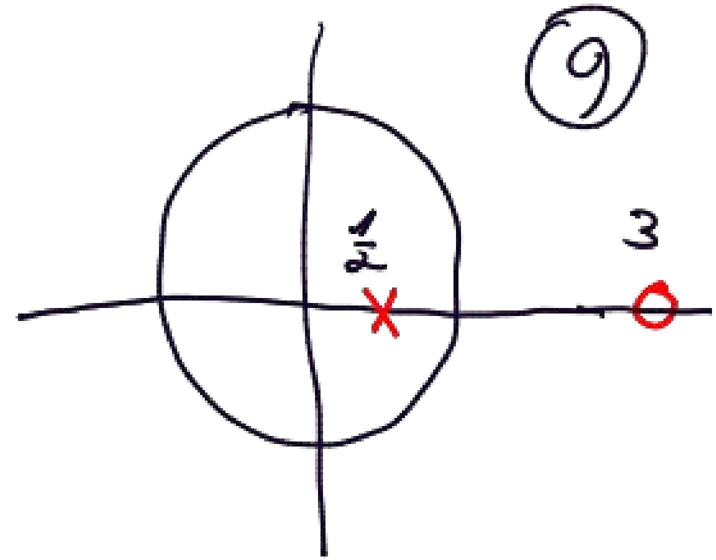
Approach: ① first construct H_{ap} with all zeros outside unit circle

② compute

$$H_{min}(z) = \frac{H(z)}{H_{ap}(z)}$$

Example

$$H(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}}$$



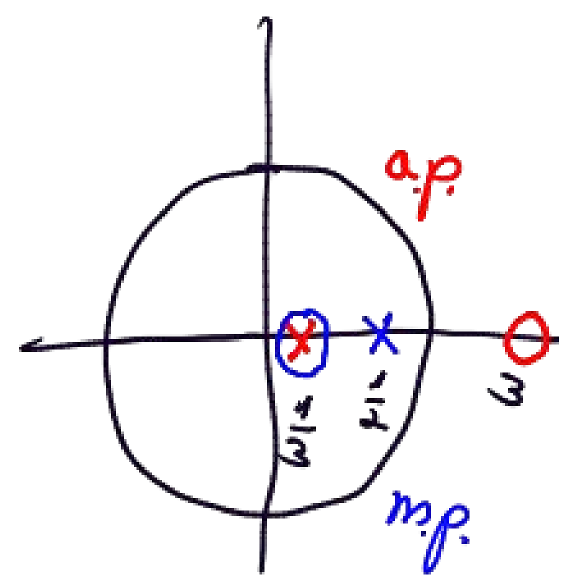
Set:

$$H_{ap} = \frac{z^{-1}-\frac{1}{3}}{1-\frac{1}{3}z^{-1}}$$

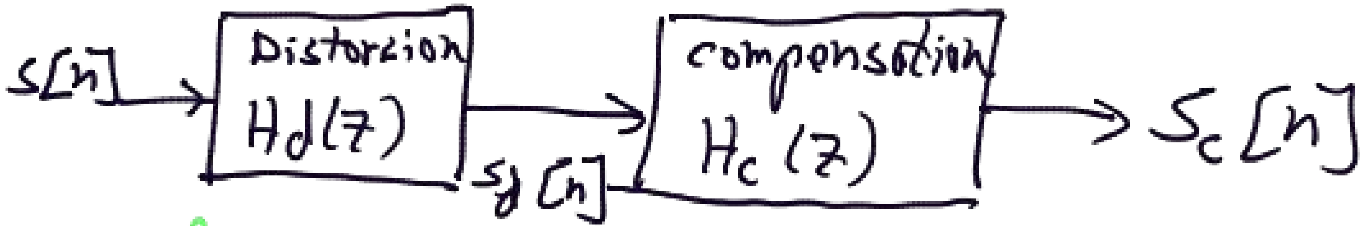
$-3 =$

$$H_{min}(z) = \frac{1-3z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1-\frac{1}{3}z^{-1}}{z^{-1}-\frac{1}{3}} =$$

$$= -3\,\frac{1-\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$



why m.p. property important?



for ex. communication chan.

If $H_d(z)$ is minimum phase, design

$$H_c(z) = \frac{1}{H_d(z)} \quad \text{(stable!)}$$

If not m.p., decompose: $H_d(z) = H_{d,mp}(z) \cdot H_{d,ap}(z)$

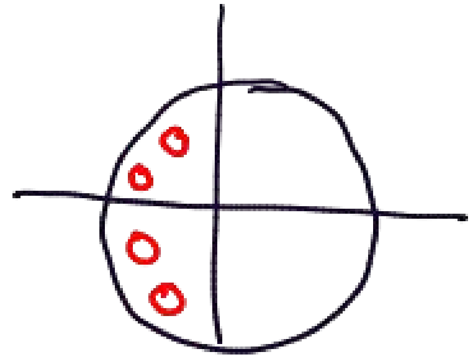
$$H_c(z) = \frac{1}{H_{d,min}(z)} \Rightarrow H_d H_c = H_{d,ap}(z)$$

only compensate for mag.

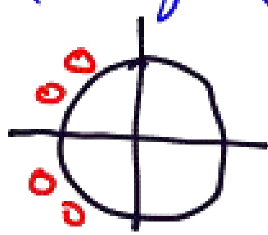
Why called "minimum phase"? (11)

Different systems can have same mag. response.

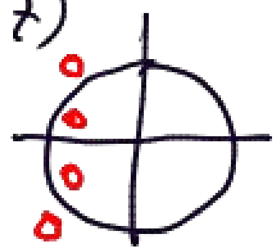
$H_1(z)$ min phase:



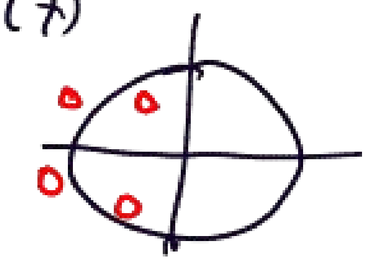
$H_2(z)$ (max phase)



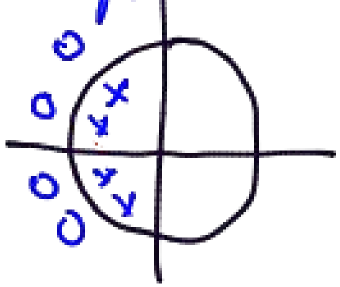
$H_3(z)$



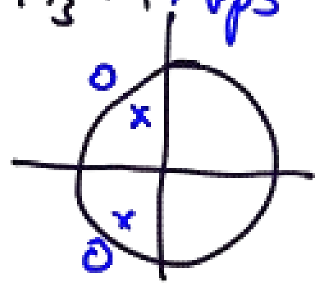
$H_4(z)$



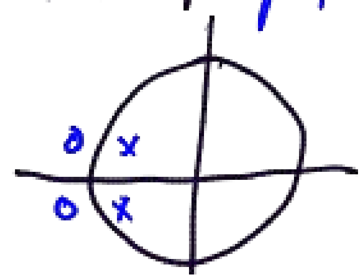
$H_2 = H_1 H_{op,2}$



$H_3 = H_1 H_{op3}$



$H_4 = H_1 \cdot H_{op4}$



of all, $H_1(z)$ has minimum phase by (12)

because:

$$\arg[H_i(e^{j\omega})] = \arg[H_1(e^{j\omega})] + \arg[H_{ap_i}]$$

$$\geq 0$$

other properties:

<u>minimum group delay</u>:

$$\mathrm{grd}[H(e^{j\omega})] = \mathrm{grd}[H_{min}] + \mathrm{grd}[H_{ap}]$$

$$\geq 0$$

<u>minimum energy delay</u>:

Problem 5.72