EE123
Digital Signal Processing

Lecture 6

based on slides by J.M. Kahn

Announcements

- HW1 solutions posted -- grading due tonight
- HW2 due Friday
- SDR give after GSI Wednesday
- Finish Ch. 8, start Ch. 9

Satellite

- Saudisat 1c has an FM repeater

Last Time

- Discrete Fourier Transform
  - Similar to DFS
  - Sampling of the DTFT (subtitles....more later)
  - Properties of the DFT
- Today
  - Linear convolution with DFT
  - Fast Fourier Transform

Properties of DFT

- Inherited from DFS (EE120/20) so no need to be proved

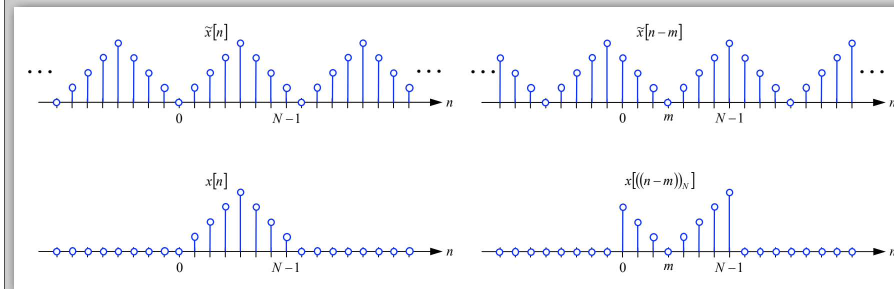
- Linearity

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \leftrightarrow \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

- Circular Time Shift

$$x[((n-m))_N] \leftrightarrow X[k]e^{-j(2\pi/N)km} = X[k]W_N^{km}$$

Circular shift



Properties of DFT

- Circular frequency shift

$$x[n]e^{j(2\pi/N)nl} = x[n]W_N^{-nl} \leftrightarrow X[((k-l))_N]$$

- Complex Conjugation

$$x^*[n] \leftrightarrow X^*[((-k))_N]$$

- Conjugate Symmetry for Real Signals

$$x[n] = x^*[n] \leftrightarrow X[k] = X^*[((-k))_N]$$

Show....

Properties of DFT

- Parseval's Identity

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

- Proof (in matrix notation)

$$\mathbf{x}^*\mathbf{x} = \left(\frac{1}{N}\mathbf{W}_N^*\mathbf{X}\right)^* \left(\frac{1}{N}\mathbf{W}_N^*\mathbf{X}\right) = \frac{1}{N^2}\mathbf{X}^*\underbrace{\mathbf{W}_N\mathbf{W}_N^*}_{N\cdot\mathbf{I}}\mathbf{X} = \frac{1}{N}\mathbf{X}^*\mathbf{X}$$

Circular Convolution Sum

- Circular Convolution:

$$x_1[n] \circledN x_2[n] \triangleq \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$$

for two signals of length N

- Note: Circular convolution is commutative

$$x_2[n] \circledN x_1[n] = x_1[n] \circledN x_2[n]$$

Properties of DFT

- Circular Convolution: Let x1[n], x2[n] be length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

Very useful!!! (for linear convolutions with DFT)

- Multiplication: Let x1[n], x2[n] be length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$

Linear Convolution

- Next....
  - Using DFT, circular convolution is easy
  - But, **linear** convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Used DFT to do linear convolution

Circular Convolution Sum

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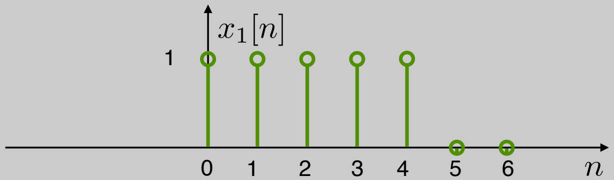
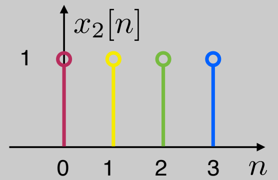
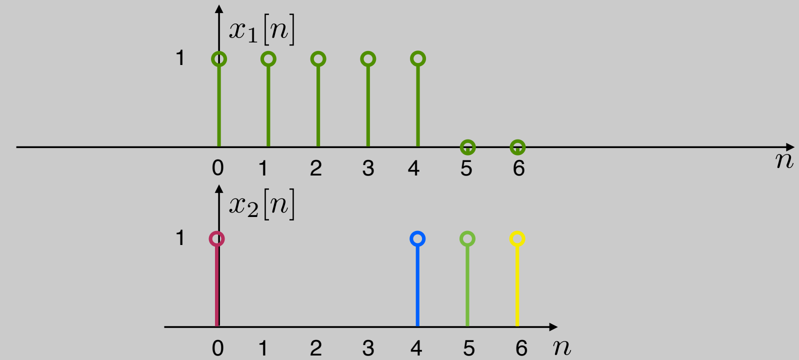
Compute Circular Convolution Sum





$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$

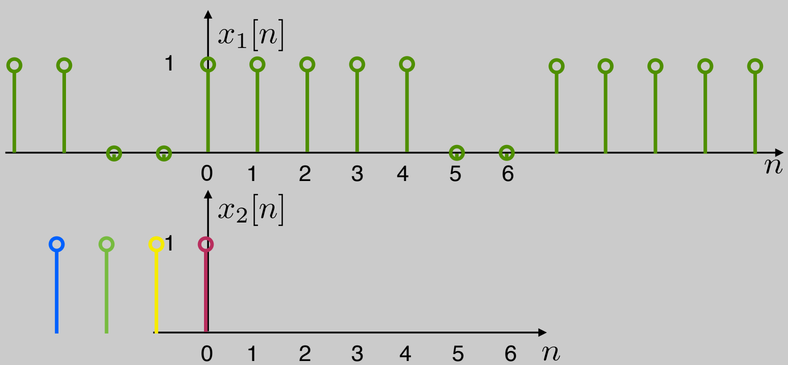
Compute Circular Convolution Sum



Circular 'flip'
multiply and add
Here: y[0]

$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$

Compute Circular Convolution Sum

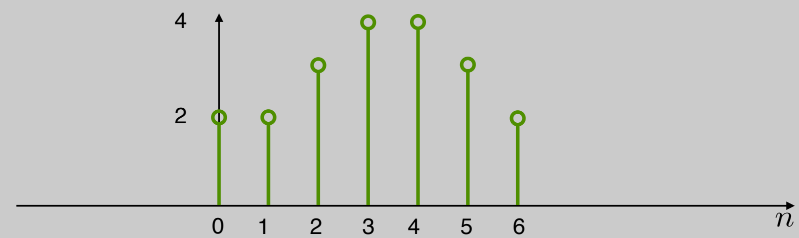


Equivalent periodic convolution over a period

$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$

Result

$$y[n] = x_1[n] \; \textcircled{7} \; x_2[n] = ?$$



Properties of DFT

- **Circular Convolution:** Let x1[n], x2[n] be length N

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

Very useful!!! (for linear convolutions with DFT)

- **Multiplication:** Let x1[n], x2[n] be length N

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$

Linear Convolution

- Next....
  - Using DFT, circular convolution is easy
  - But, **linear** convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Used DFT to do linear convolution

Linear Convolution using the DFT

We start with two nonperiodic sequences:

$$x[n] \quad 0 \le n \le L-1$$
$$h[n] \quad 0 \le n \le P-1$$

We can think of $x[n]$ as a signal, and $h[n]$ as a filter inpulse response.
We want to compute the linear convolution:

$$y[n] = x[n] * h[n] = \sum_{m=0}^{L-1} x[m] * h[n-m] = \sum_{m=0}^{P-1} x[n-m]h[m]$$

$y[n] = x[n] * h[n]$ is nonzero only for $0 \le n \le L+P-2$, and is of length $L+P-1 = M$.

Linear Convolution using the DFT

We will look at two approaches for computing $y[n]$:

(1) Direct Convolution

- Evaluate the convolution sum directly.

- This requires $L \cdot P$ multiplications

(2) Using Circular Convolution

Linear Convolution using the DFT

(2) Using Circular Convolution

- Zero-pad $x[n]$ by $P-1$ zeros:

$$x_{zp}[n] = \begin{cases} x[n] & 0 \le n \le L-1 \\ 0 & L \le n \le L+P-2 \end{cases}$$

- Zero-pad $h[n]$ by $L-1$ zeros:

$$h_{zp}[n] = \begin{cases} h[n] & 0 \le n \le P-1 \\ 0 & P \le n \le L+P-2 \end{cases}$$

- Both zero-padded sequences $x_{zp}[n]$ and $h_{zp}[n]$ are of length $M = L+P-1$

Linear Convolution using the DFT

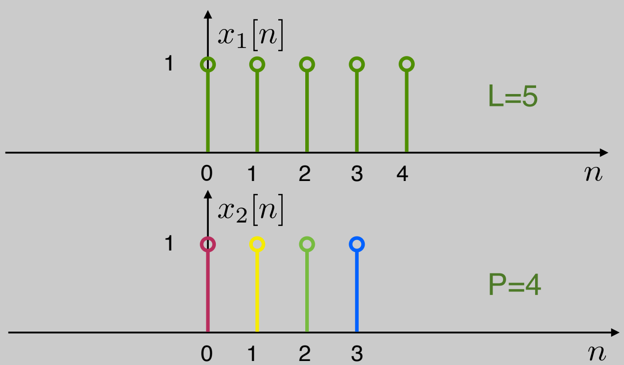
- Both zero-padded sequences $x_{zp}[n]$ and $h_{zp}[n]$ are of length $M = L+P-1$

- We can compute the linear convolution $x[n] * h[n] = y[n]$ by computing circular convolution $x_{zp}[n] \circledM h_{zp}[n]$:

Linear convolution via circular

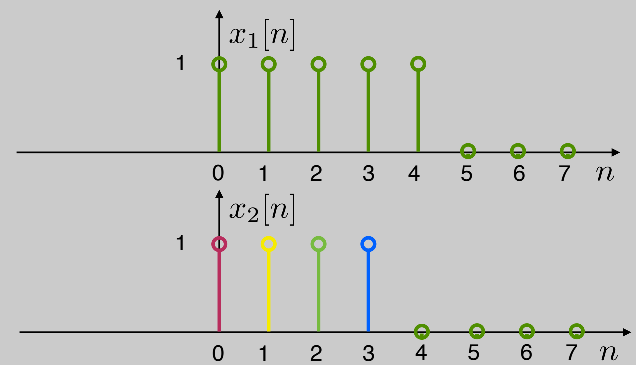
$$y[n] = x[n] * y[n] = \begin{cases} x_{zp}[n] \circledM h_{zp}[n] & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

Example



M = L + P - 1 = 8

Example



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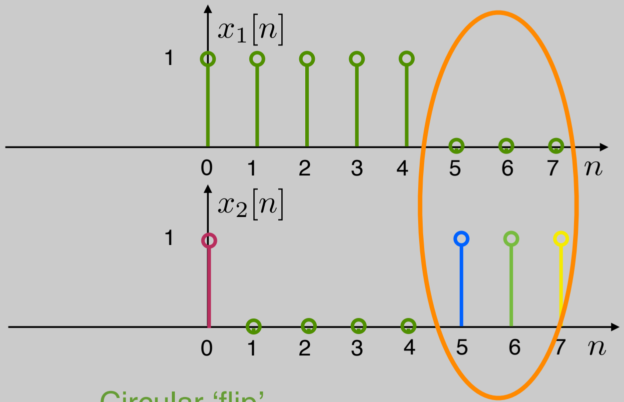

M = L + P - 1 = 8

$$y[n] = x_1[n] \circledast_8 x_2[n] = x_1[n] * x_2[n]$$

Linear Convolution using the DFT

- In practice, the circular convolution is implemented using the DFT circular convolution property:

$$\begin{aligned} x[n] * h[n] &= x_{zp}[n] \circledast_M h_{zp}[n] \\ &= \mathcal{DFT}^{-1}\{DFT x_{zp}[n] \cdot \mathcal{DFT}\{h_{zp}[n]\}\} \end{aligned}$$

  for $0 \leq n \leq M-1$, $M = L + P - 1$.
- Advantage: This can be more efficient than direct linear convolution because the FFT and inverse FFT are $O(M \cdot \log_2 M)$.
- Drawback: We must wait until we have all of the input data. This introduces a large delay which is incompatible with real-time applications like communications.

Linear Convolution using the DFT

- In practice, the circular convolution is implemented using the DFT circular convolution property:

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- Advantage: This can be more efficient than direct linear convolution because the FFT and inverse FFT are $O(M \cdot \log_2 M)$.
- Drawback: We must wait until we have all of the input data. This introduces a large delay which is incompatible with real-time applications like communications.
- Approach: Break input into smaller blocks. Combine the results using 1. *overlap and save* or 2. *overlap and add* .

Block Convolution

Problem

An input signal $x[n]$ has very long length, which can be considered infinite.
An impulse response $h[n]$ has length P.
We want to compute the linear convolution

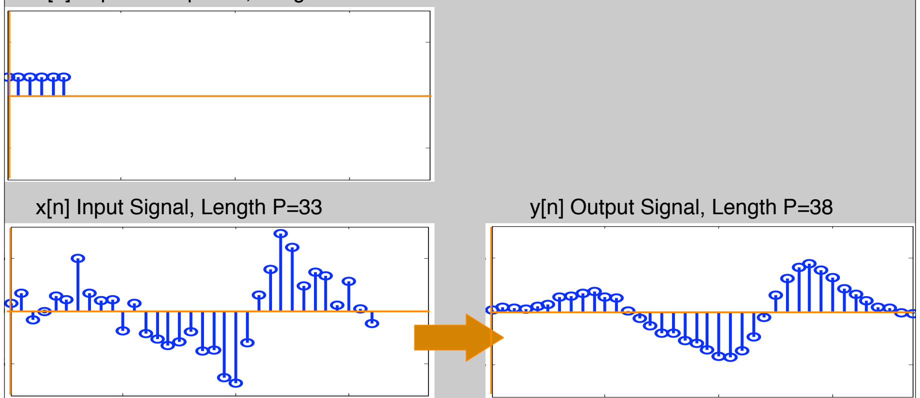
$$y[n] = x[n] * h[n]$$

using block lengths shorter than the input signal length.

Block Convolution

Example:

h[n] Impulse response, Length P=6



Overlap-Add Method

We decompose the input signal $x[n]$ into non-overlapping segments $x_r[n]$ of length L:

$$x_r[n] = \begin{cases} x[n] & rL \leq n \leq (r+1)L - 1 \\ 0 & \text{otherwise} \end{cases}$$

The input signal is the sum of these input segments:

$$x[n] = \sum_{r=0}^{\infty} x_r[n]$$

The output signal is the sum of the output segments $x_r[n] * h[n]$:

$$y[n] = x[n] * h[n] = \sum_{r=0}^{\infty} x_r[n] * h[n] \qquad (1)$$

Each of the output segments $x_r[n] * h[n]$ is of length $N = L + P - 1$.

Overlap-Add Method

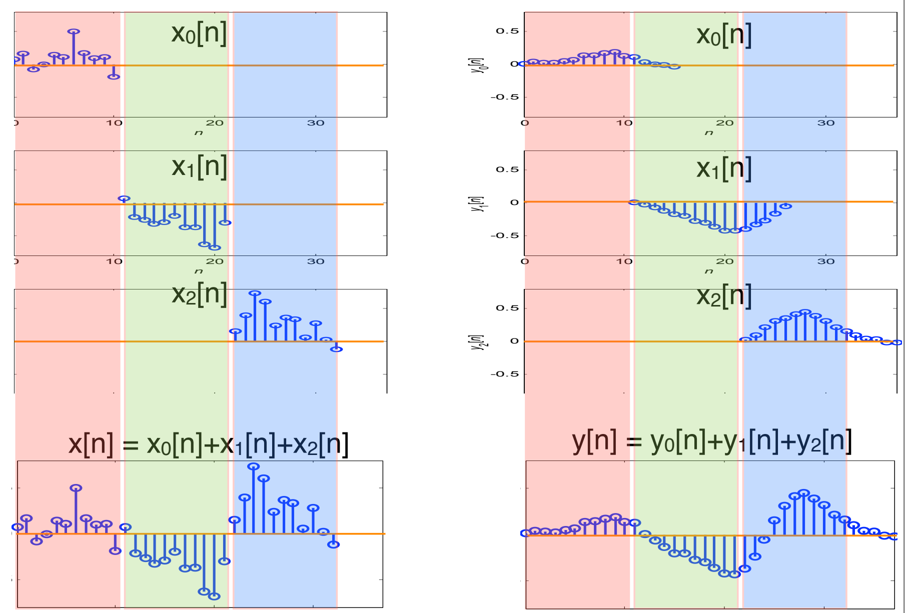
We can compute each output segment $x_r[n] * h[n]$ with linear convolution.
DFT-based circular convolution is usually more efficient:

- Zero-pad input segment $x_r[n]$ to obtain $x_{r,zp}[n]$, of length N.
- Zero-pad the impulse response $h[n]$ to obtain $h_{zp}[n]$, of length N (this needs to be done only once).
- Compute each output segment using:

$$x_r[n] * h[n] = \mathcal{DFT}^{-1}\{\mathcal{DFT}\{x_{r,zp}[n]\} \cdot \mathcal{DFT}\{h_{zp}[n]\}\}$$

Since output segment $x_r[n] * h[n]$ starts offset from its neighbor $x_{r-1}[n] * h[n]$ by L, neighboring output segments overlap at $P - 1$ points.
Finally, we just add up the output segments using (1) to obtain the output.

Example of overlap and add:



Overlap-Save Method

Basic Idea

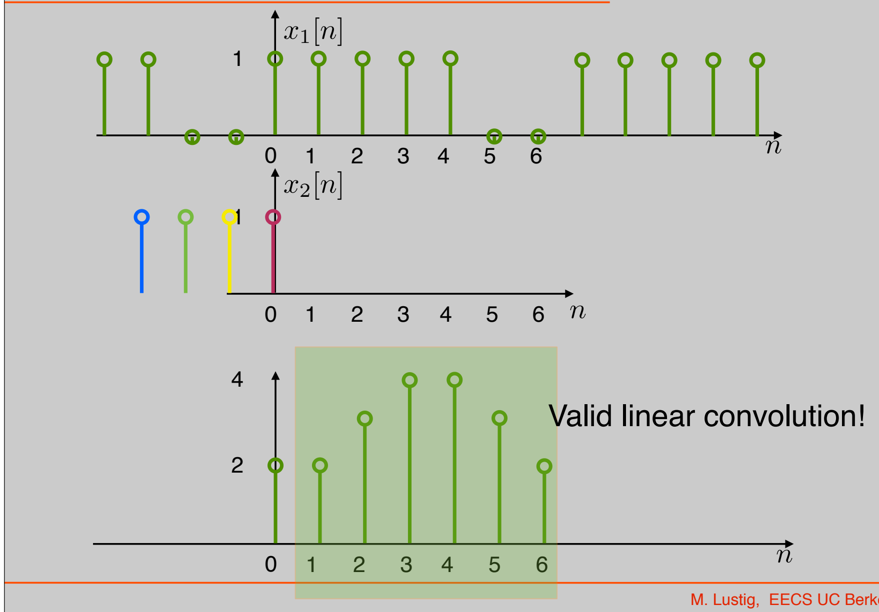
We split the input signal $x[n]$ into overlapping segments $x_r[n]$ of length $L+P-1$.

Perform a circular convolution of each input segment $x_r[n]$ with the impulse response $h[n]$, which is of length P using the DFT.

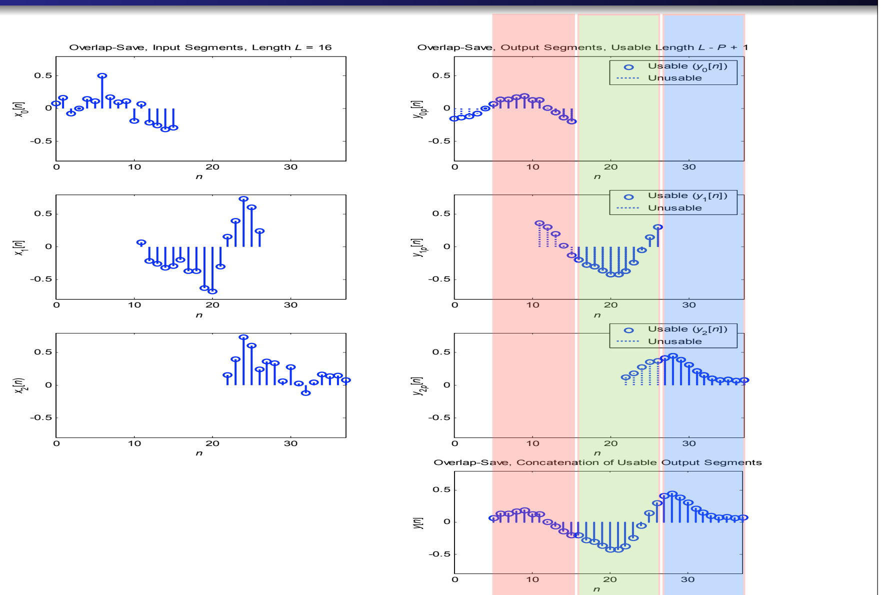
Identify the L-sample portion of each circular convolution that corresponds to a linear convolution, and save it.

This is illustrated below where we have a block of L samples circularly convolved with a P sample filter.

Recall:



e Method



DFT vs DTFT (revisit)

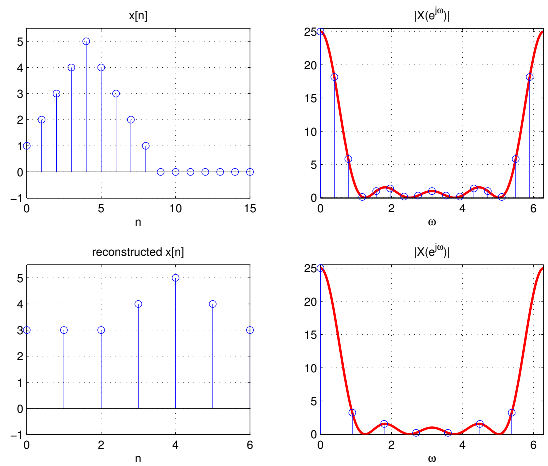
- Back to moving average example:

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n}$$

$$= e^{-j2\omega}\frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})}$$

DFT and Sampling the DTFT

$$X(e^{j\omega}) = e^{-j4\omega}\frac{\sin^2(5\omega/2)}{\sin^2(\omega/2)}$$



Circular Convolution as Matrix Operation

Circular convolution:

$$h[n] \circledN x[n] = \begin{bmatrix} h[0] & h[N-1] & \cdots & h[1] \\ h[1] & h[0] & & h[2] \\ & & \vdots & \\ h[N-1] & h[N-2] & & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$
$$= H_c x$$

- H_c is a circulant matrix
- The columns of the DFT matrix are Eigen vectors of circulant matrices.
- Eigen vectors are DFT coefficients. How can you show?

Circular Convolution as Matrix Operation

- Diagonalize:

$$W_N H_c W_n^{-1} = \begin{bmatrix} H[0] & 0\cdots & 0 \\ 0 & H[1]\cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix}$$

- Right-multiply by W_N

$$W_N H_c = \begin{bmatrix} H[0] & 0\cdots & 0 \\ 0 & H[1]\cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix} W_N$$

- Multiply both sides by x

$$W_N H_c x = \begin{bmatrix} H[0] & 0\cdots & 0 \\ 0 & H[1]\cdots & 0 \\ \vdots & 0 & H[N-1] \end{bmatrix} W_N x$$