EE123
Digital Signal Processing

Lecture 21
Tomography

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Impulse lines and line-integrals

In 1D

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

A sample @x=0

In 2D

$$\int_{-\infty}^{\infty} f(x,y)\delta(x)dx = f(0,y)$$

1D cross-section @x=0

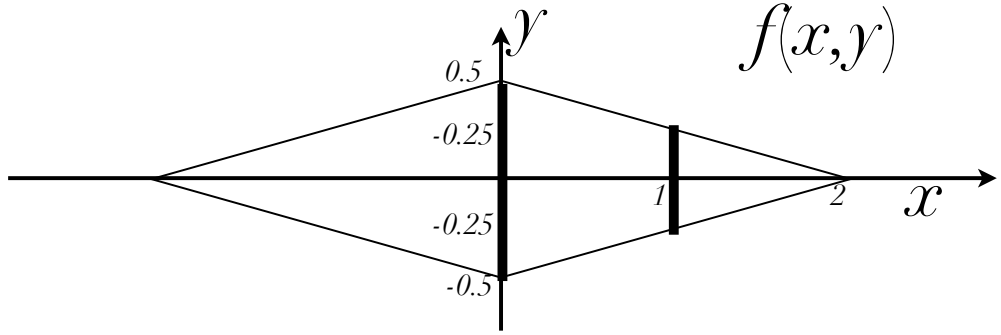
dx

Line Integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x)dy = \int_{-\infty}^{\infty} f(0,y)dy$$

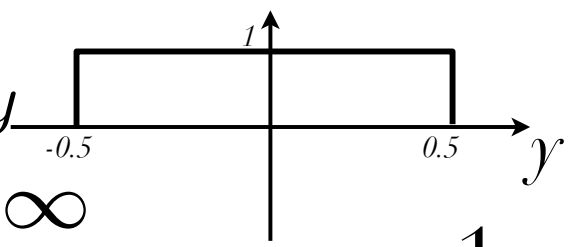
Integral of the 1D cross-section @x=0

Impulse lines and line-integrals



$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)\delta(x)dx = f(0,y) = \sqcap(y)$$

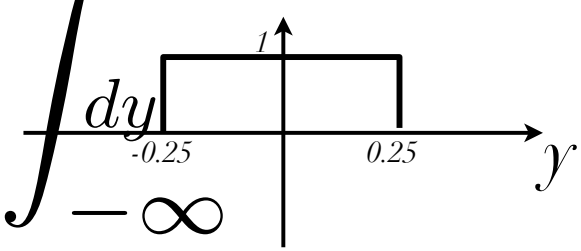
$$\int_{-\infty}^{\infty} dy$$



$$=1$$

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)\delta(x-1)dx = f(1,y) = \sqcap(2y)$$

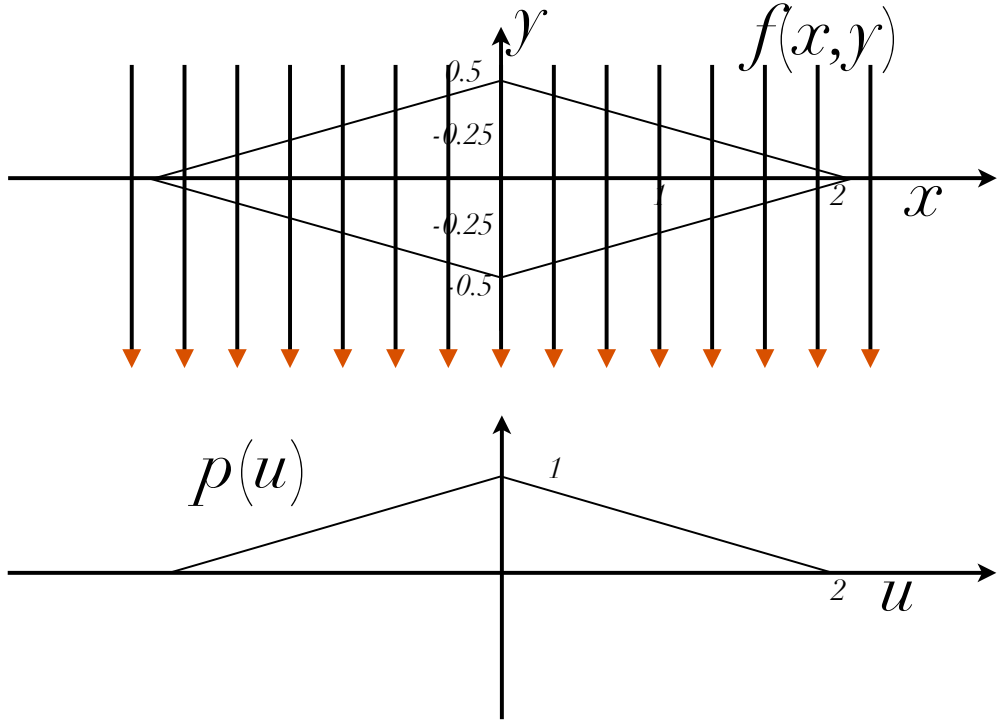
$$\int_{-\infty}^{\infty} dy$$



$$=0.5$$

what about line integral with $\delta(x\text{-}u)$?

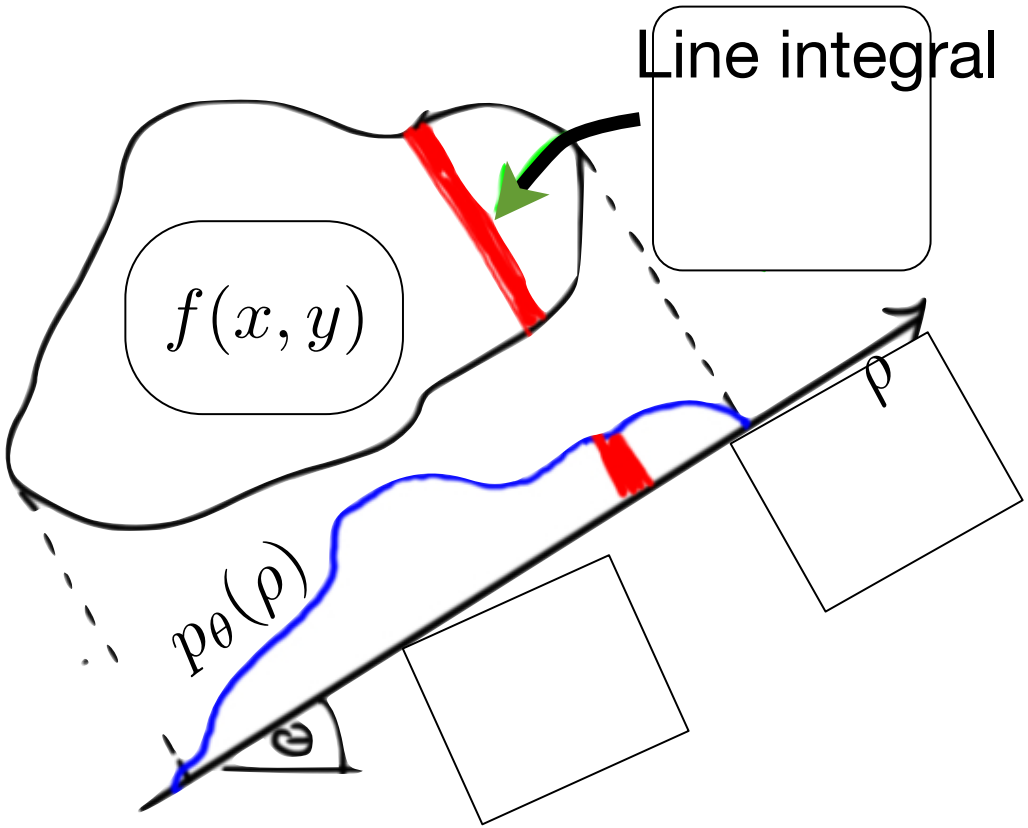
Line Integral and Projection



$$p(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-u)dxdy = \int_{-\infty}^{\infty} f(u,y)dy$$

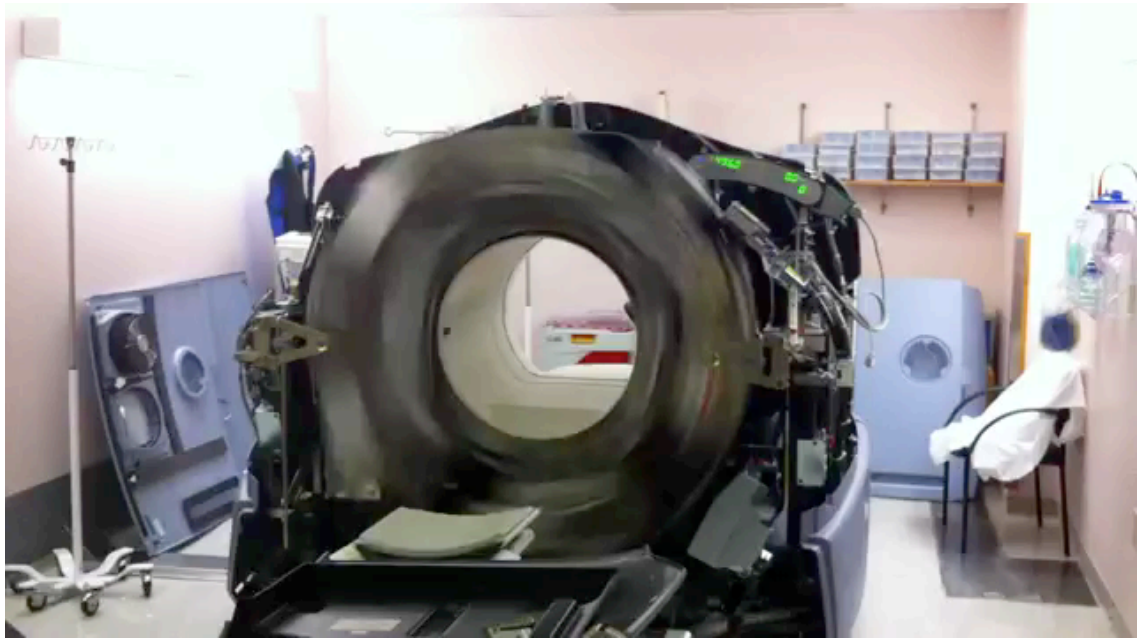
General Projections

$$p(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(\rho - x\cos\theta - y\sin\theta)dxdy$$

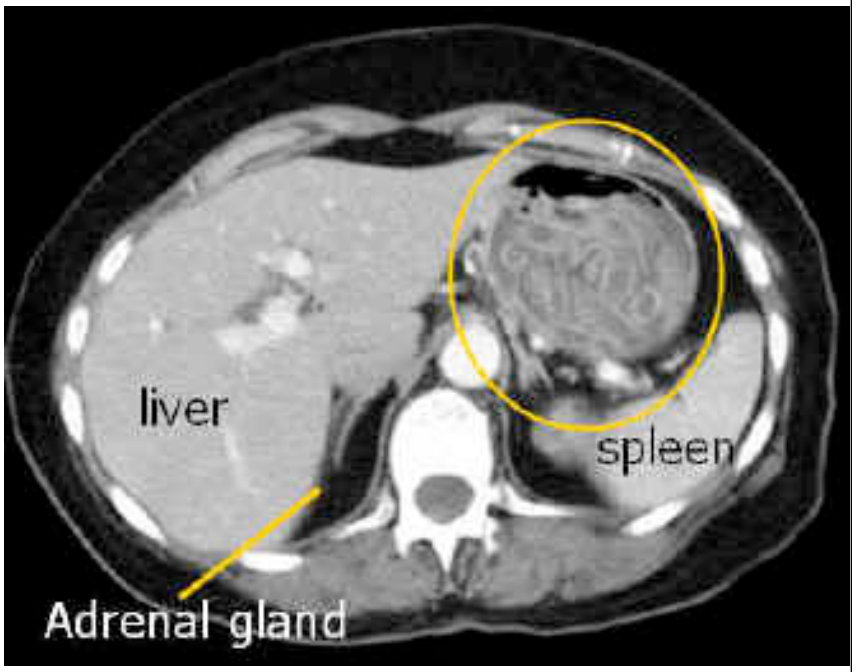


projection x-ray

Many Projections - Tomography

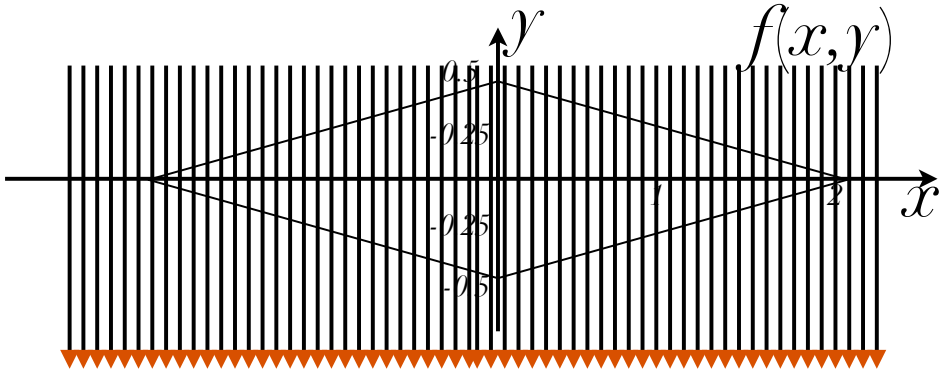


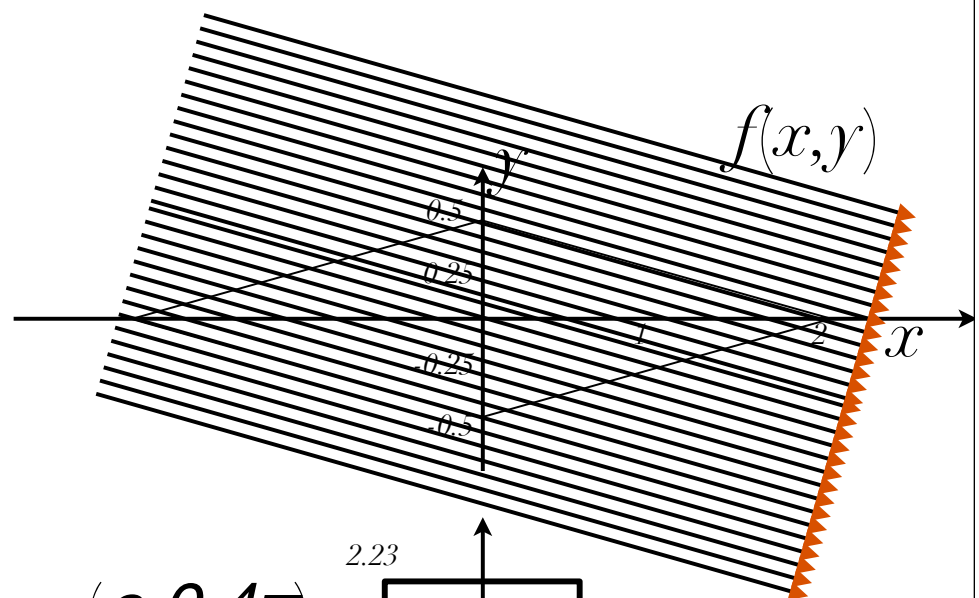
http://www.youtube.com/watch?v=4gklQHM19aY&feature=related

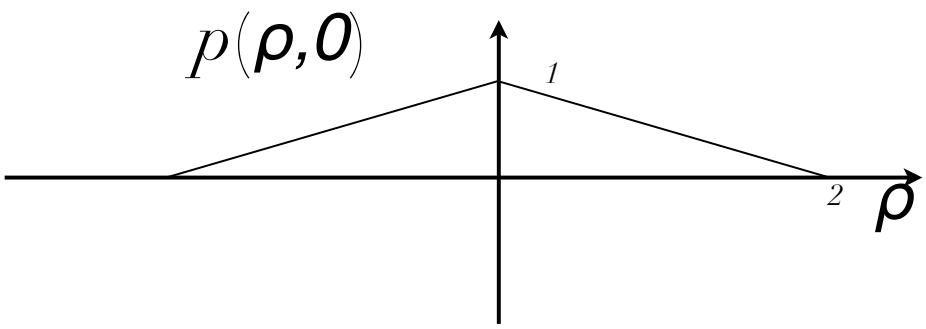


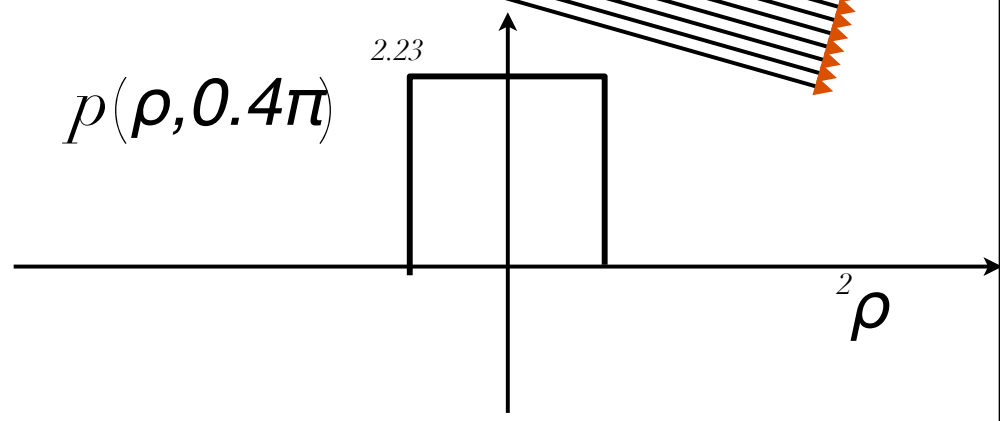
Radon Transform

$$p(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(\rho - x\cos\theta - y\sin\theta)dxdy$$



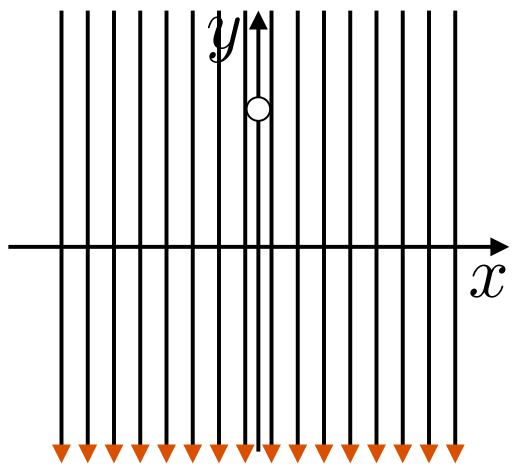


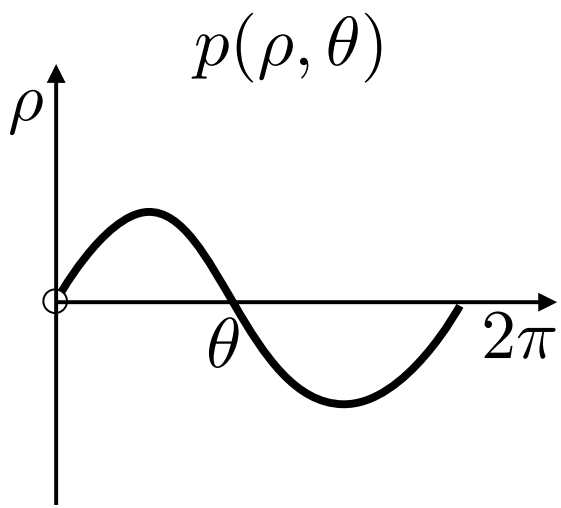




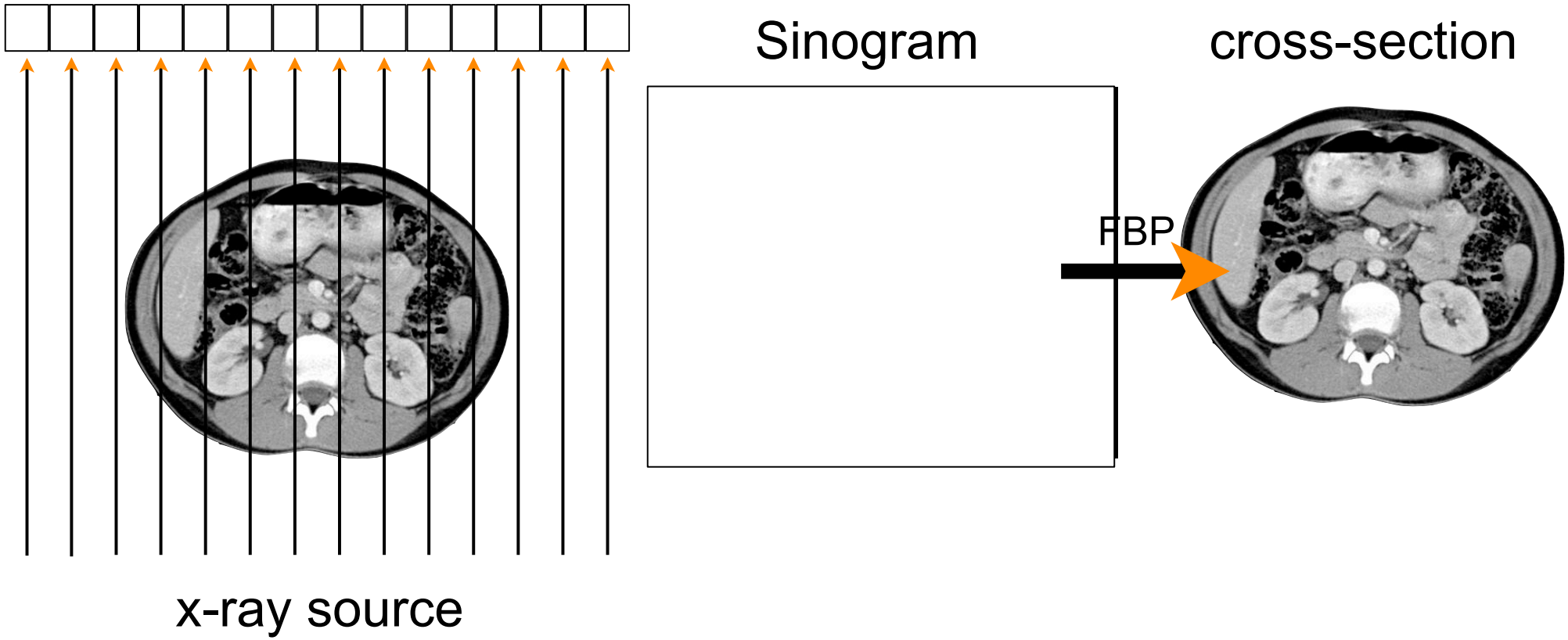
Radon Transform: Sinogram

- Also called Sinogram
- Impulse ⇒ Sinusoid



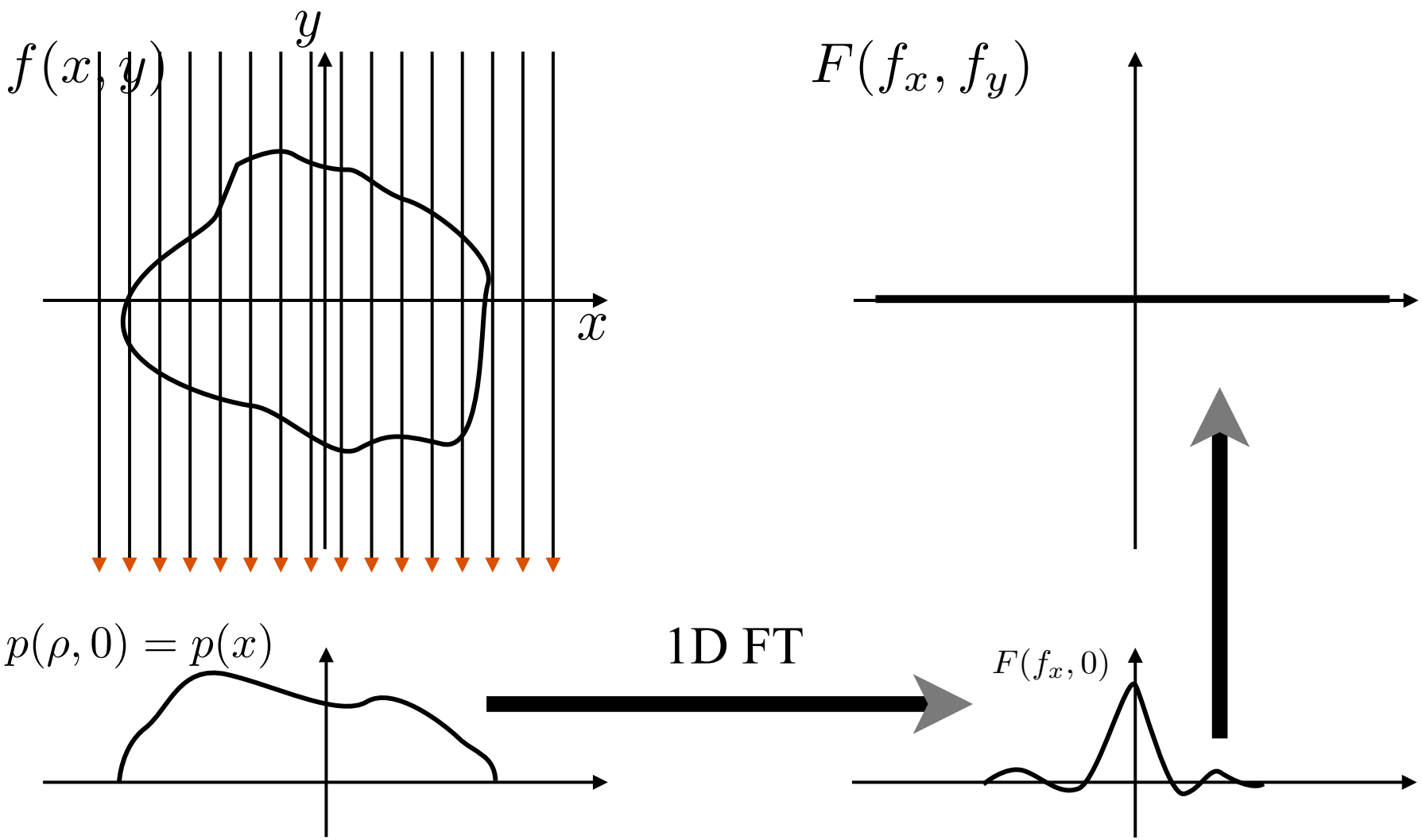


Computed Tomography



Projection Slice Theorem (Bracewell)

$$\mathcal{F}_{1\mathrm{D}}\{p(\rho,\theta)\} = F(\rho\cos\theta, \overset{\text{sine}}{\rho\cos\theta})$$



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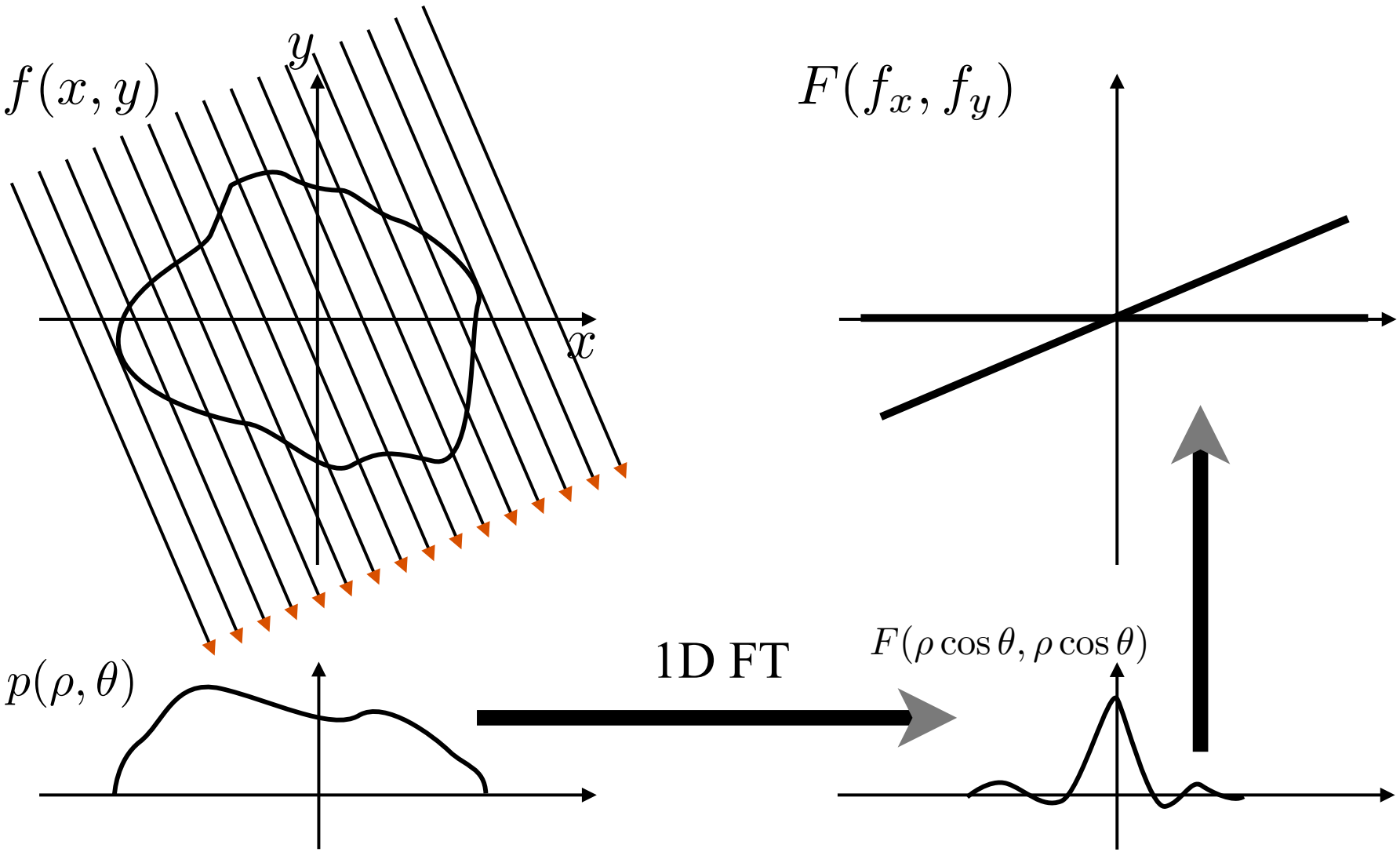

Projection Slice Theorem (Bracewell)

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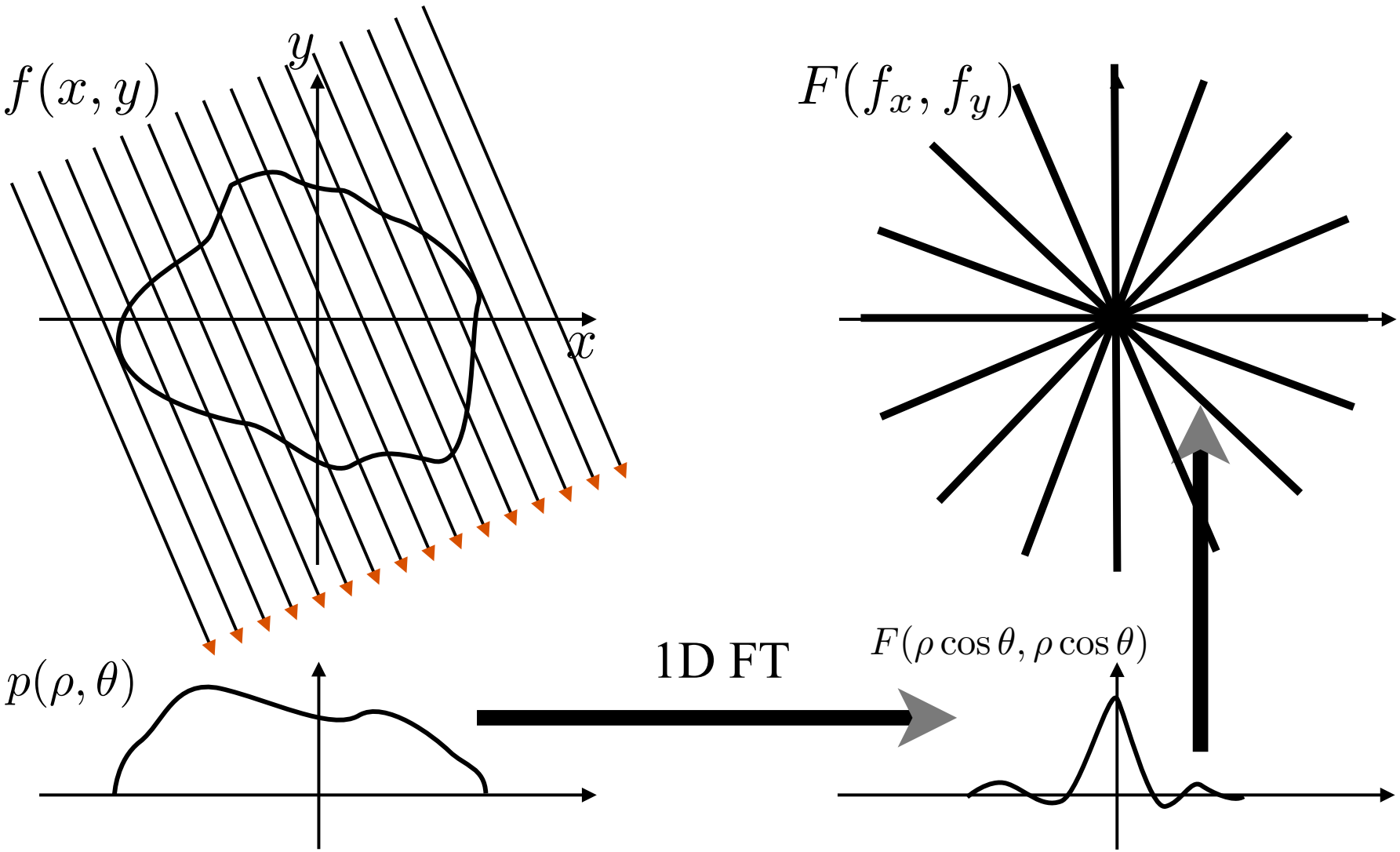


Projection Slice Theorem (Bracewell)

Proof (for $\theta = 0$)

$$p(x) = \int_{-\infty}^{\infty} m(x,y)\,dy$$

$$M(k_x, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x,y)\, e^{-i2\pi(k_x x + k_y y)}\,dx\,dy =$$

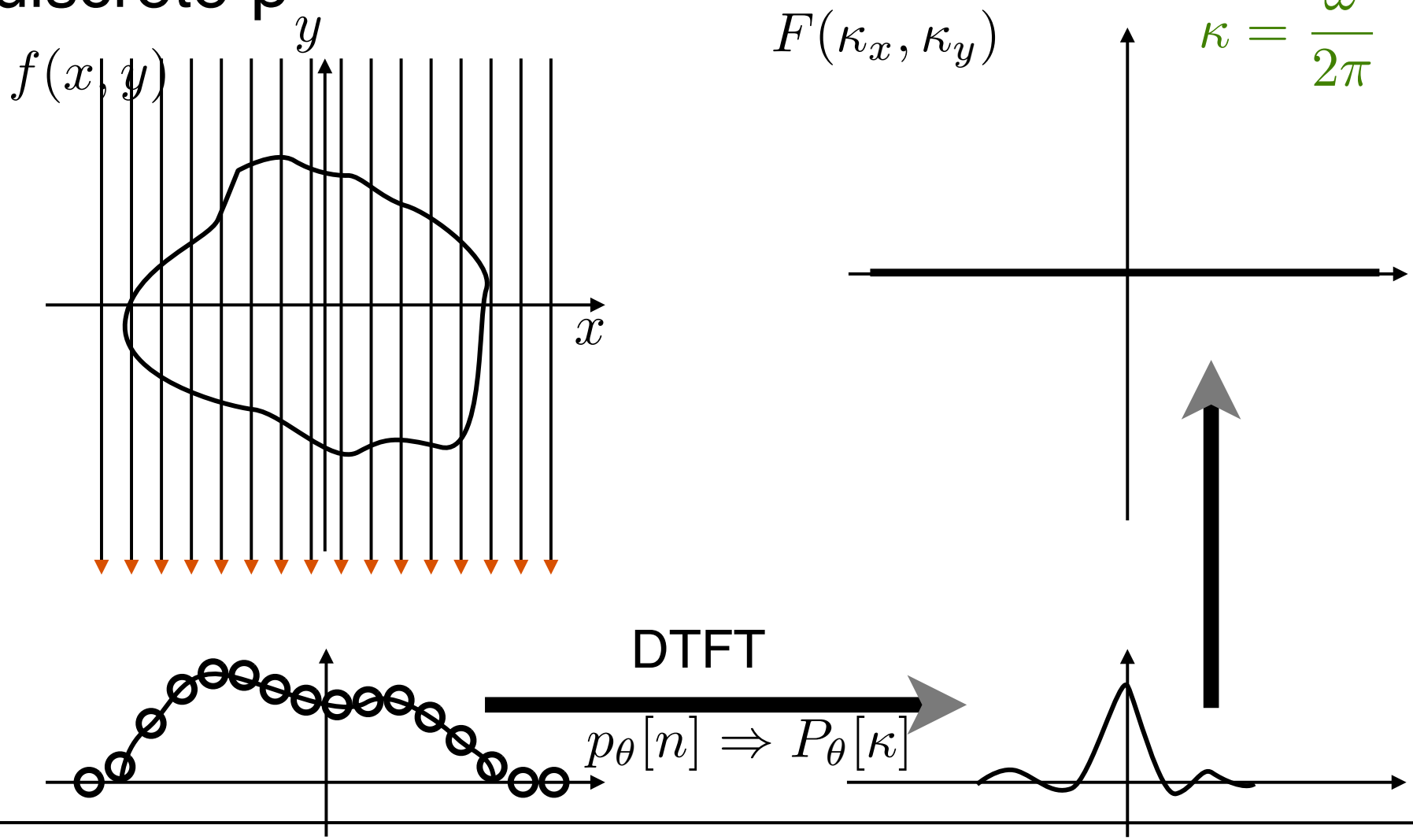
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(x,y)\, e^{-i2\pi k_x x}\,dx\,dy =$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} m(x,y)\,dy\right] e^{-i2\pi k_x x}\,dx =$$

$$= \int_{-\infty}^{\infty} p(x)\, e^{-i2\pi k_x x}\,dx = \mathcal{F}\{p(x)\}$$

Partly Discrete Reconstruction

- Let's assume continuous angle Θ, discrete ρ



Partly Discrete Reconstruction

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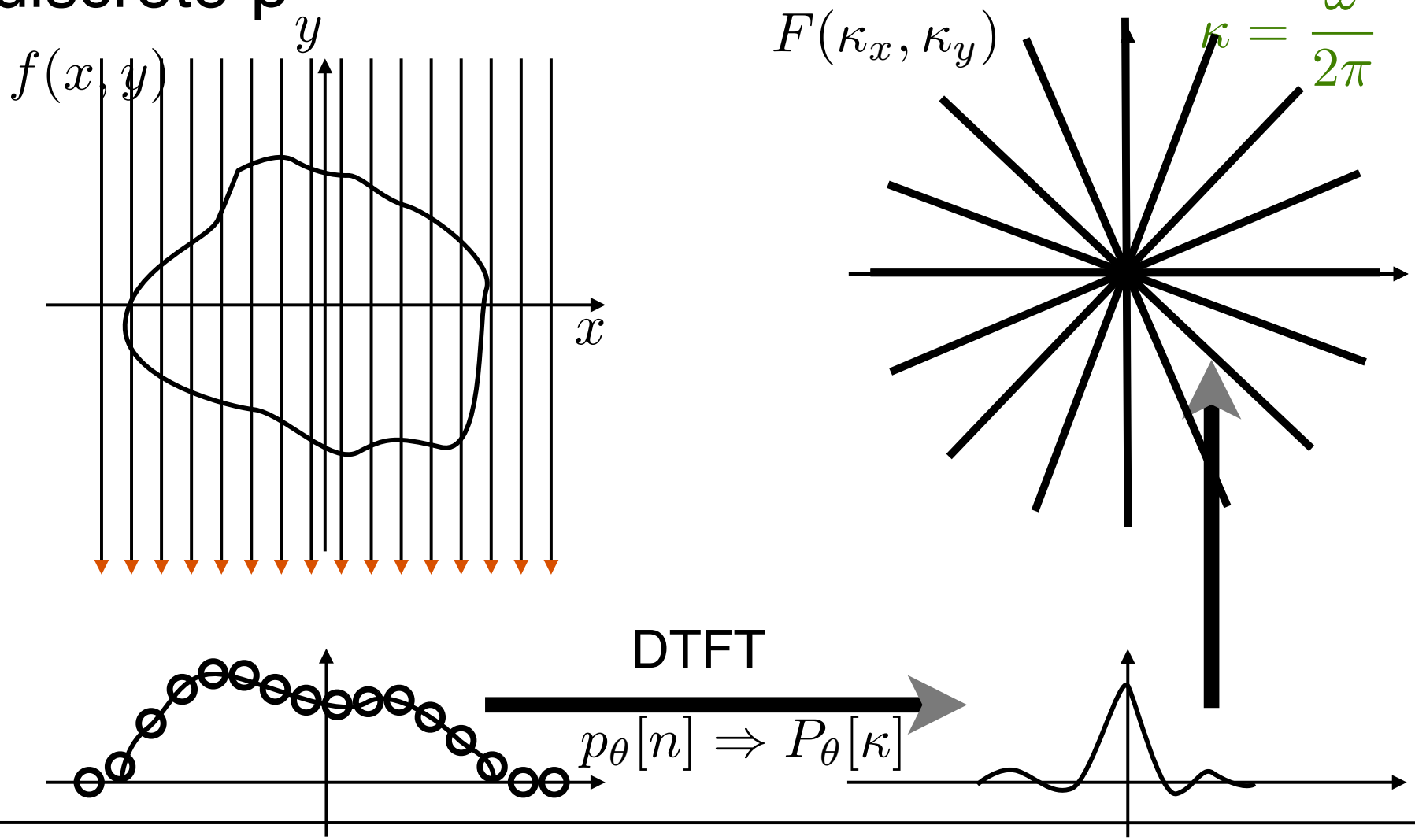


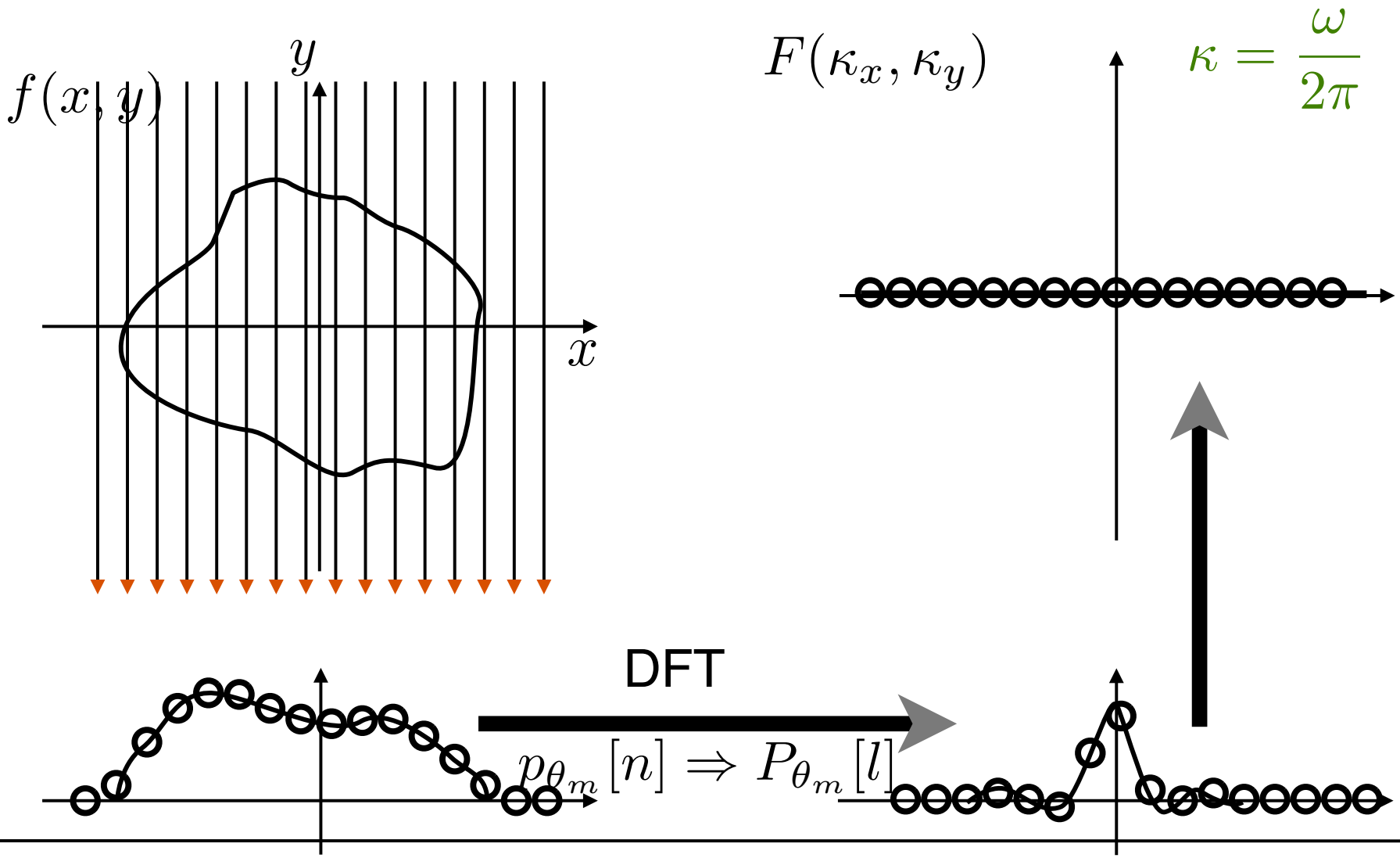
Reconstruction From Polar Coordinates

$$
\begin{aligned}
f[n,m] &= \int_{-0.5}^{0.5}\int_{-0.5}^{0.5} F(\kappa_x,\kappa_y)e^{2\pi j(\kappa_x n+\kappa_y m)}d\kappa_x d\kappa_y \\
&= \int_{0}^{\pi}\int_{\text{-0.5}}^{0.5} F(\rho,\theta)e^{2\pi j(\rho\cos(\theta)n+\rho\sin(\theta)m)}|\rho|d\rho d\theta
\end{aligned}
$$

- Polar frequency data must be multiplied by |ρ|
- Also called a rho filter

Discrete Reconstruction

- Let's assume discrete angle Θ_m, discrete ρ



Discrete Reconstruction

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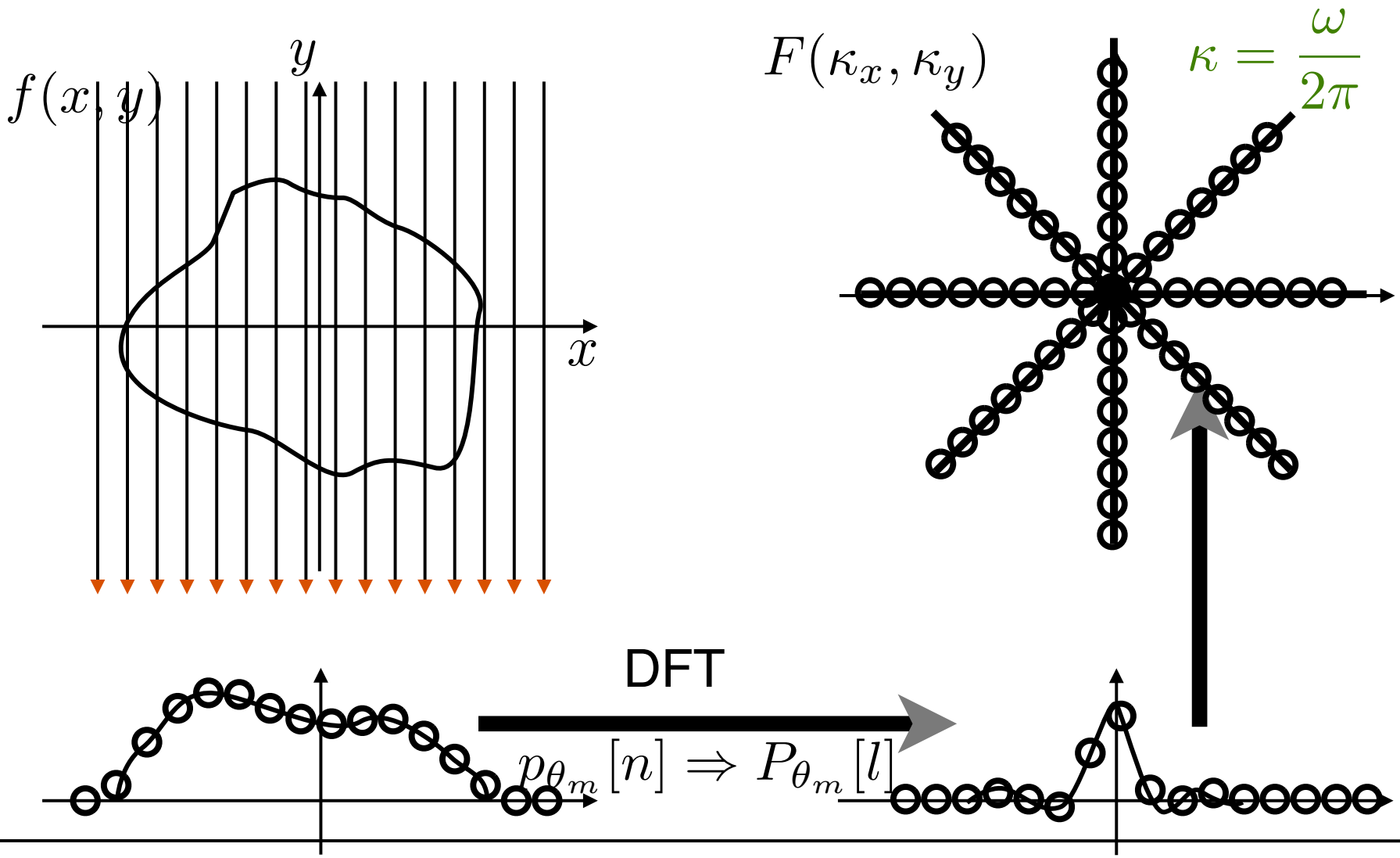


Filtered Back Projection

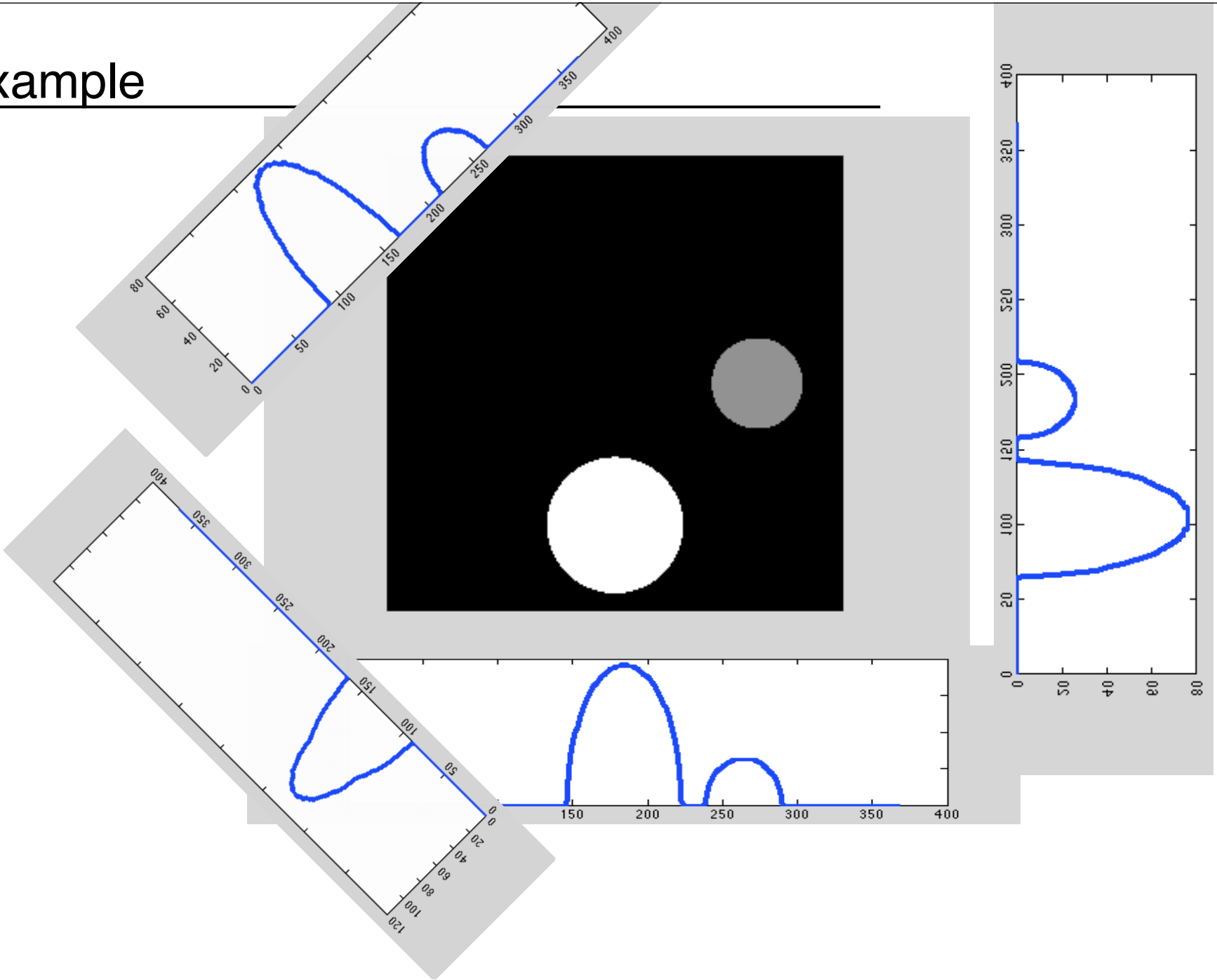
- Replace integrals with sums. Sum over radius and angle
- Define a (filtered) backprojection:

$$C_{\theta_m}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \theta_m] e^{2\pi j (l/N \cos(\theta_m) n_x + l/N \sin(\theta_m) n_y)} \underbrace{|l/N|}_{\rho}$$

So,

$$f[n_x, n_y] = \sum_m C_{\theta_m}[n_x, n_y]$$

Example


0
20
40
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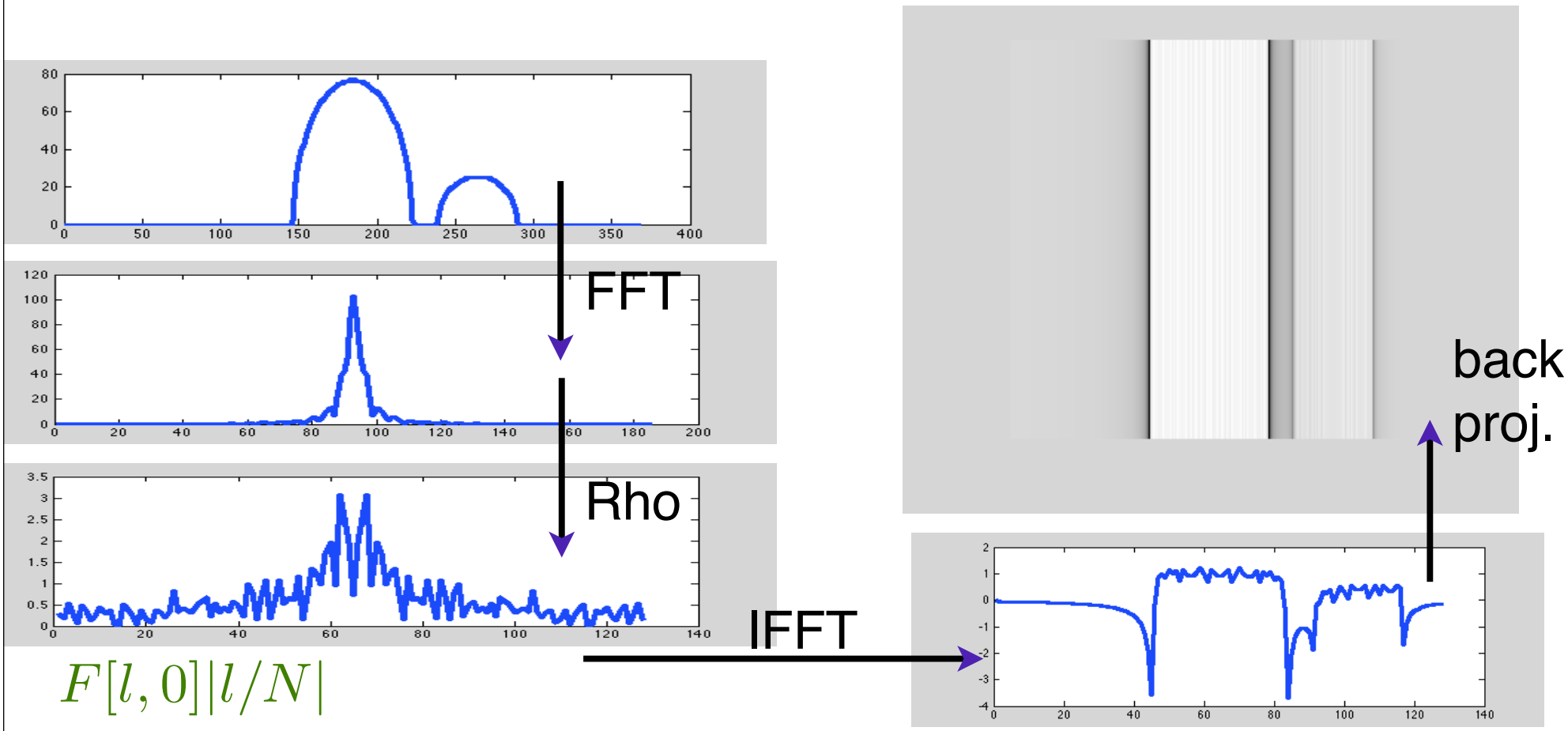
100
150
200
250
300
350
400
0
20
40
60
80
0
50
100
150
200
250
300
350
400

Example Convolution Back Projection

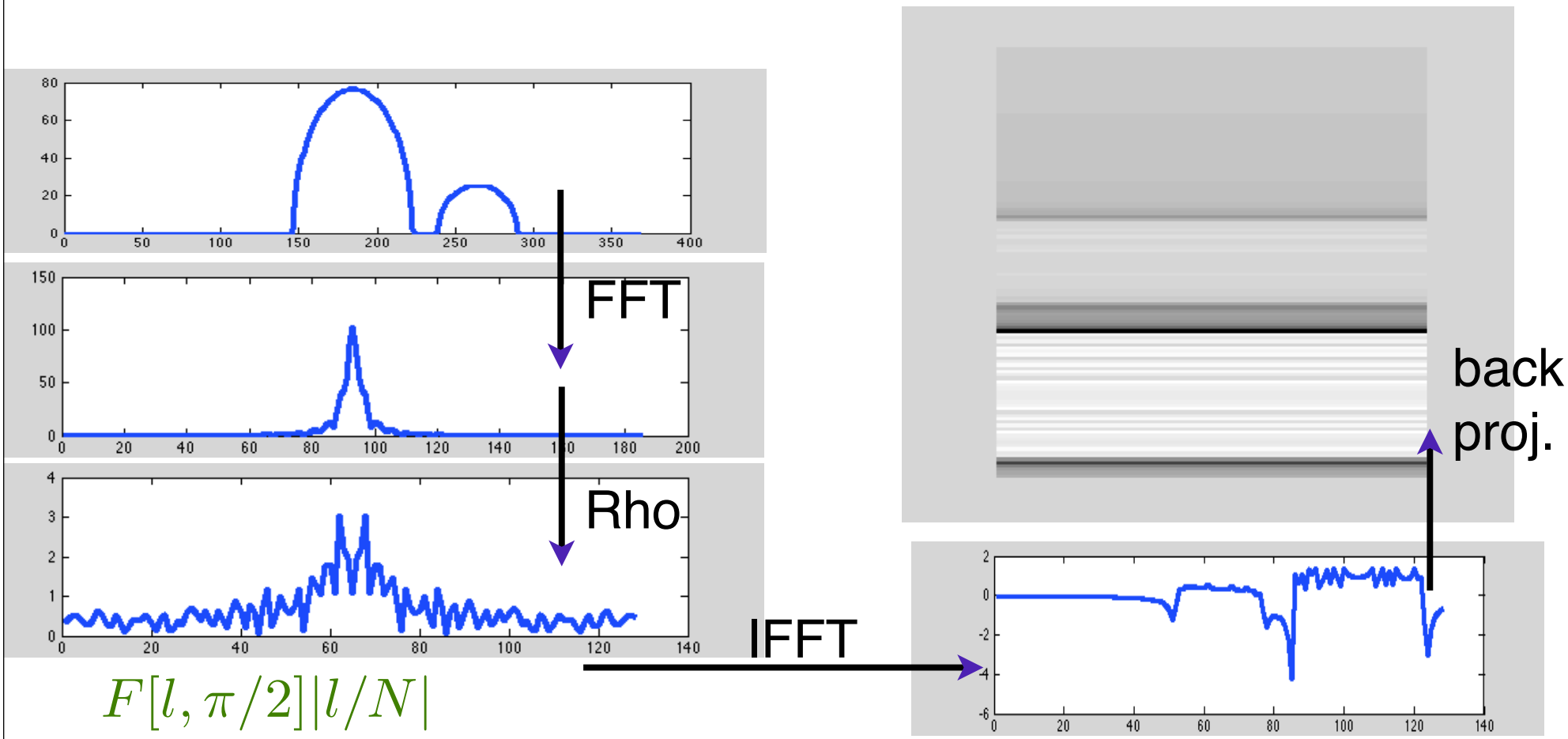
- For Θ=0

$$C_0[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l,0]|l/N|e^{2\pi j(l/N n_x)}$$

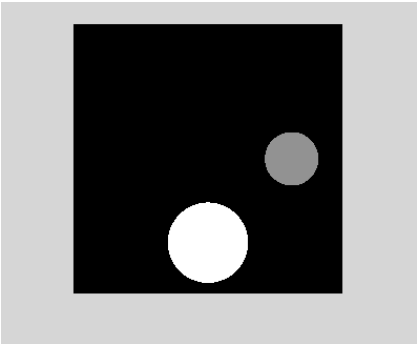


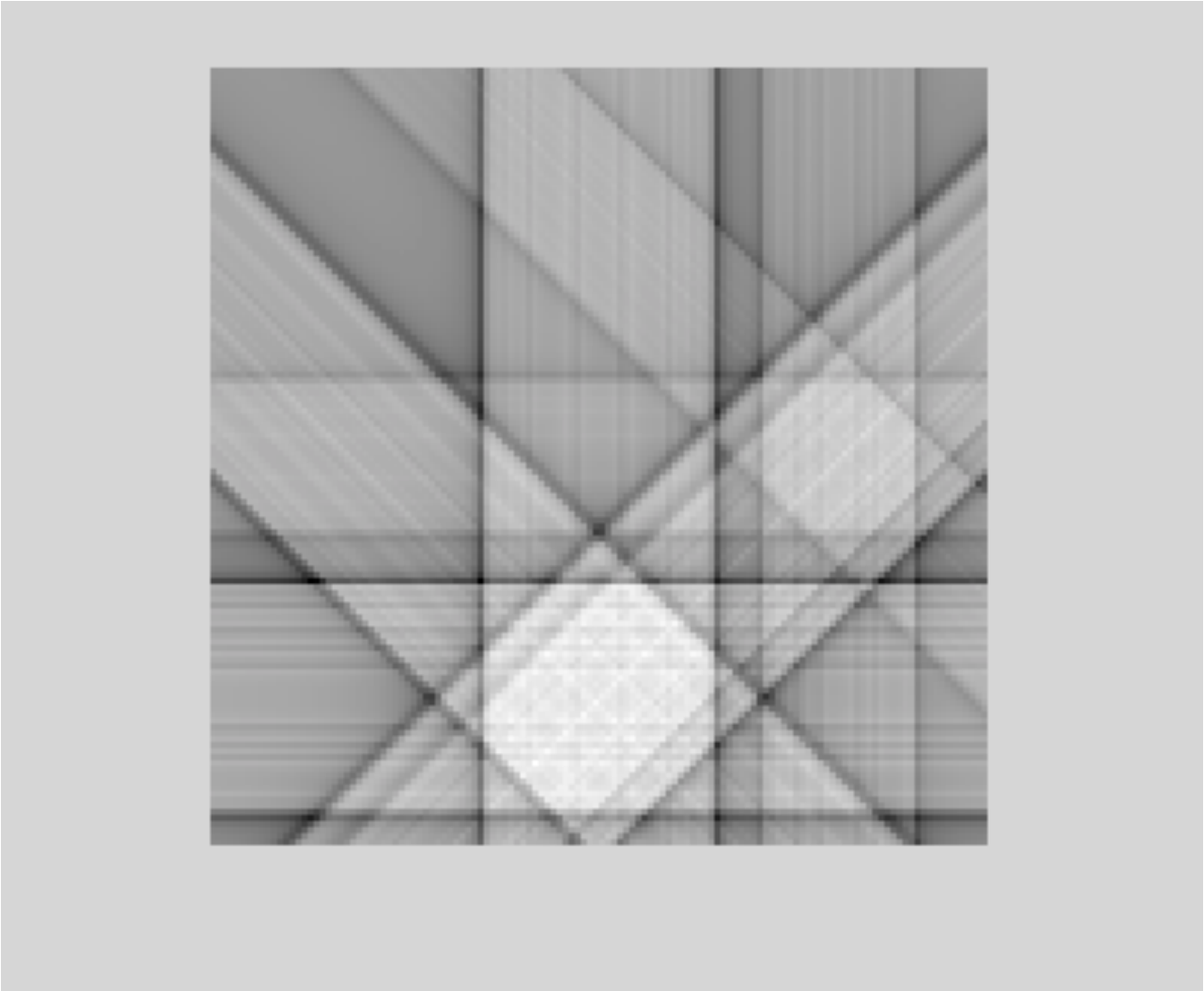
Example Convolution Back Projection

- For Θ=π/2 $$C_{\pi/2}[n_x, n_y] = \sum_{l=-N/2}^{(N/2)-1} F[l, \pi/2]|l/N|e^{2\pi j(l/N n_y)}$$

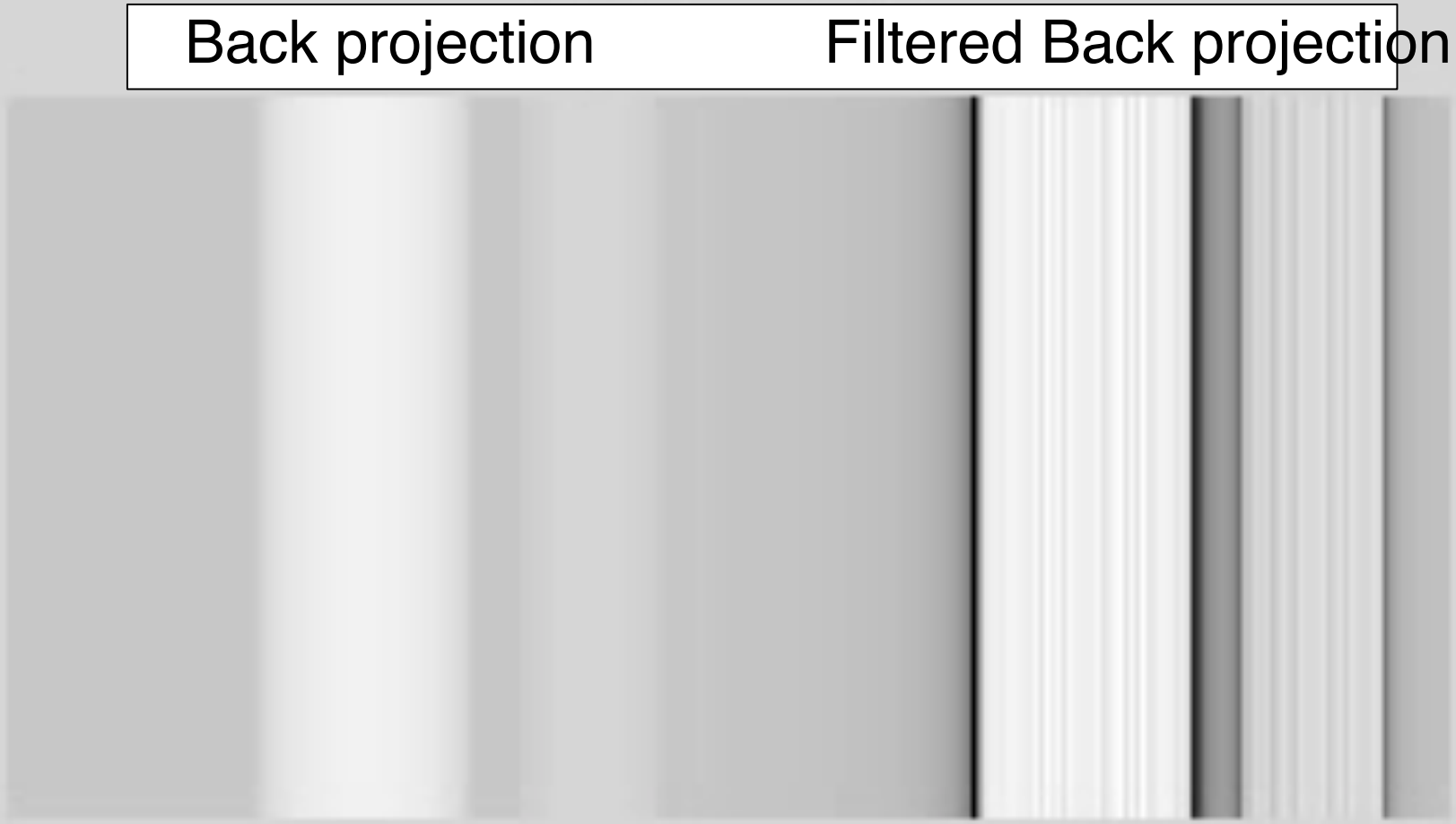


Convolution Back Projection

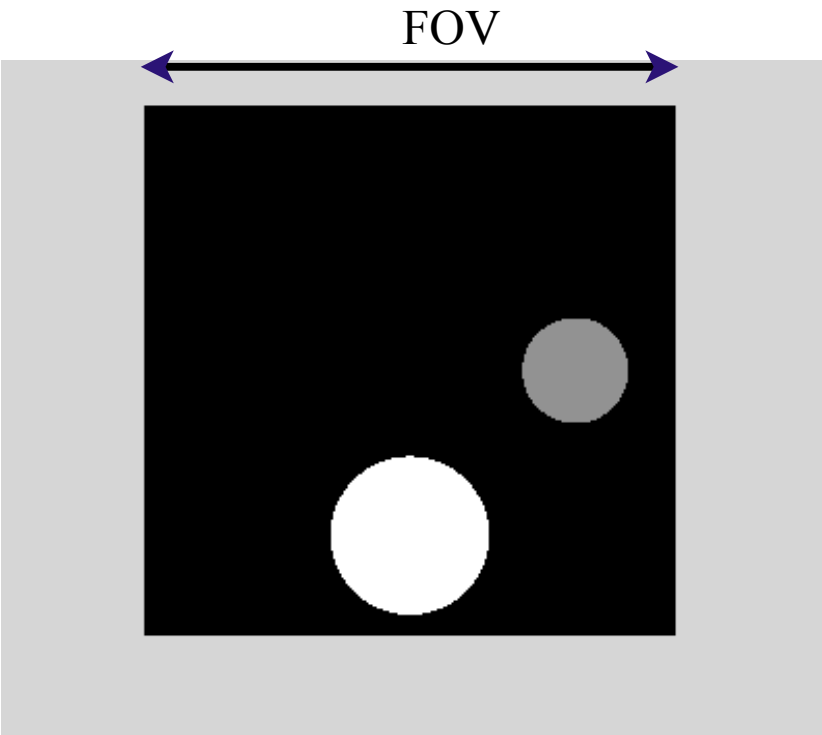


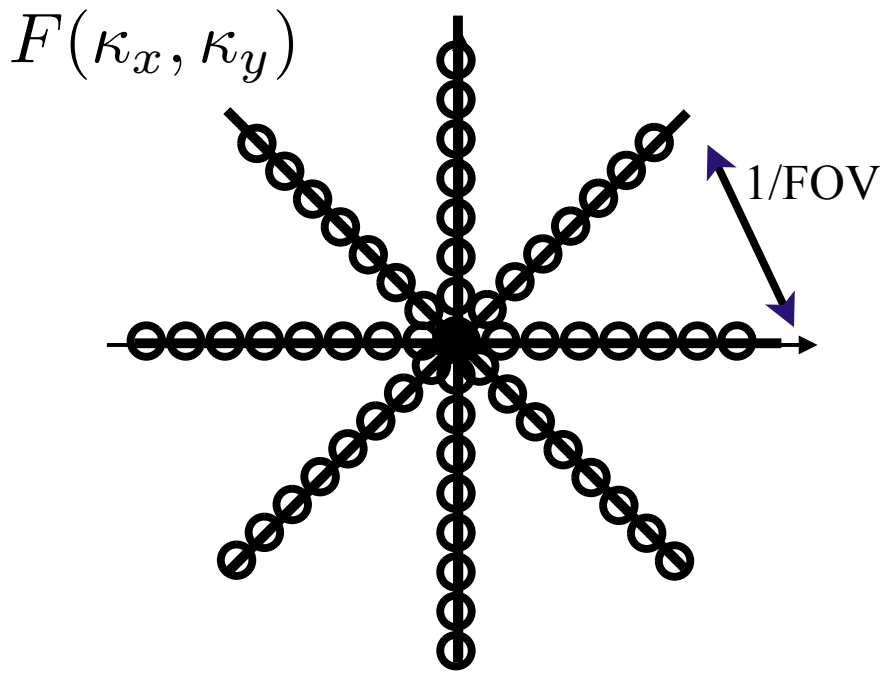


Filtered Back Projection



How Many Projections?

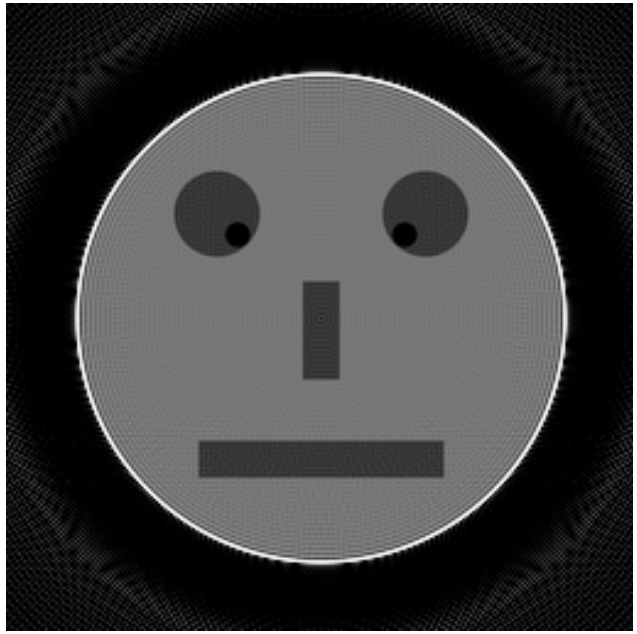




How Many Projections?

256 Proj.

128 Proj

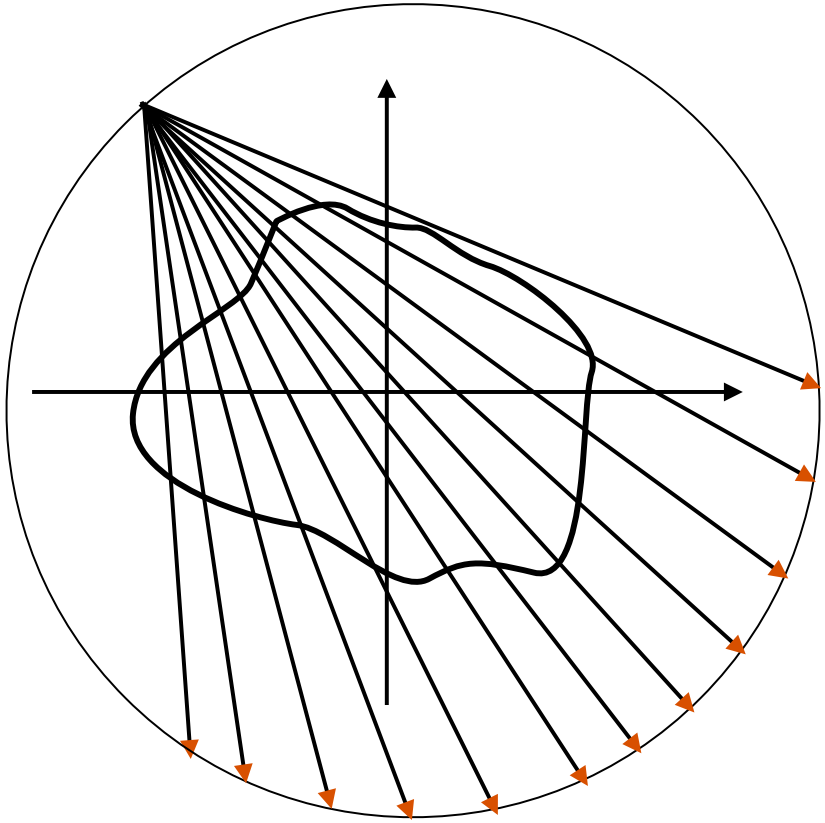


64 Proj

Fan Beam CT

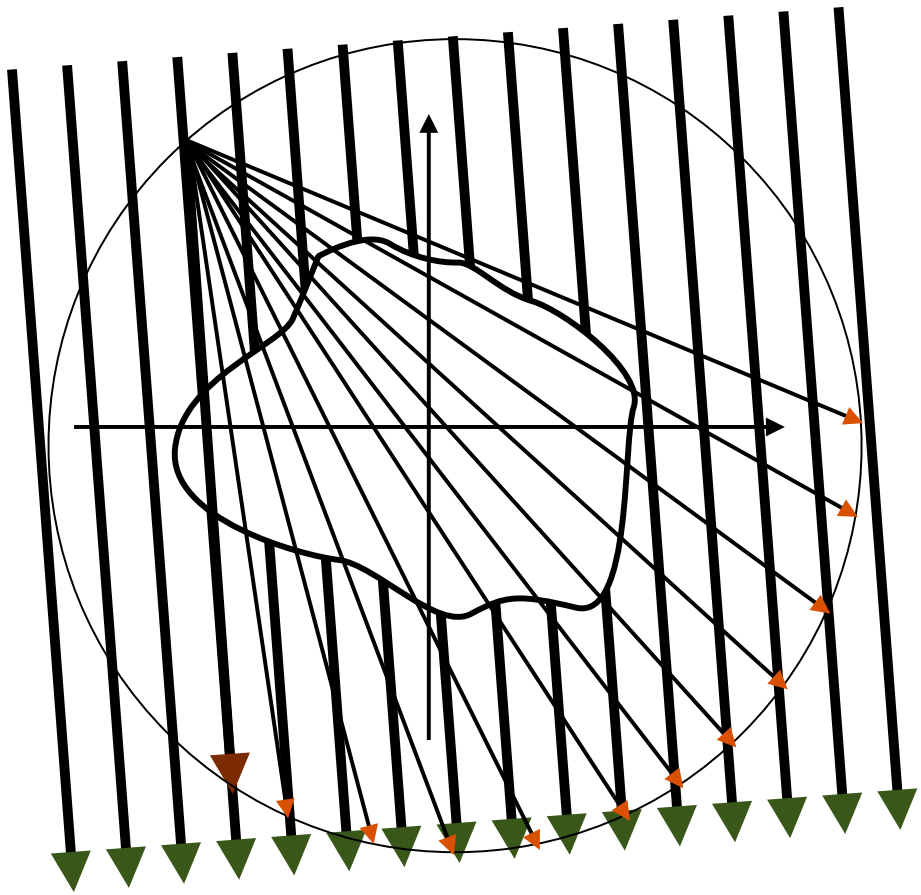
- Single Source
- Many detectors

- How to reconstruct?



Fan Beam CT

- Single Source
- Many detectors
- How to reconstruct?
- Re-binning!



Fan Beam CT

- Single Source
- Many detectors
- How to reconstruct?
- Re-binning!