Downsampling

Section 6, Nick Antipa, 3/9/2018

Based on slides by Jon Tamir

Some notes from Giulia Fanti, Frank Ong, Michael Lustig,

Midterm 1

- $X[k] = \sum_{n=0}^{3N-1} x[n]e^{-j\pi nk/N}$

$$= \sum_{n=0}^{N-1} x[n]e^{-j\pi nk/N} + \sum_{n=N}^{2N-1} x[n]e^{-j\pi nk/N} + \sum_{n=2N}^{3N-1} x[n]e^{-j\pi nk/N}$$

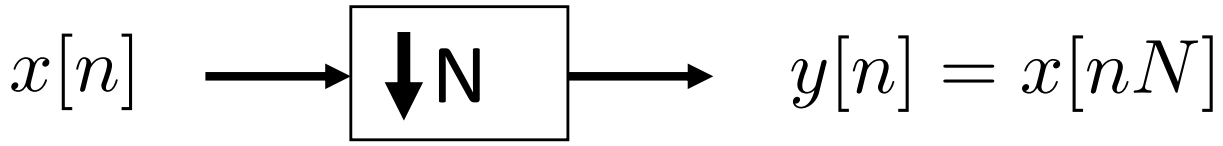
- What is the problem with this?

My favorite example of aliasing

And

- http://i.imgur.com/8X8Fcoy.gifv

Downsampling



- **Compresses** in the time domain
- **Expands** in the frequency domain

$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})$$

Alternative derivation of downsampling DTFT

- Recall: A N-periodic sequence has a discrete Fourier series (DFS):

$$\tilde{s}(m) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{S}_k e^{j2\pi mk/N}$$

DFS Representation

$$\tilde{S}_k = \sum_{m=0}^{N-1} \tilde{s}(m) e^{-j2\pi km/N}$$

DFS Coefficients

Alternative derivation of downsampling DTFT

- Impulse train:

$$\tilde{s}(m) = \sum_{k \in \mathbb{Z}} \delta(m - kN)$$

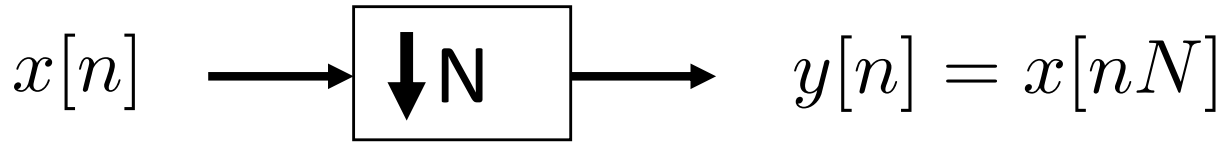
- DFS Coefficients (check):

$$\tilde{S}_k = 1$$

- DFS Representation:

$$\tilde{s}(m) = \frac{1}{N} \sum_{k=0}^{N-1} e^{j2\pi mk/N}$$

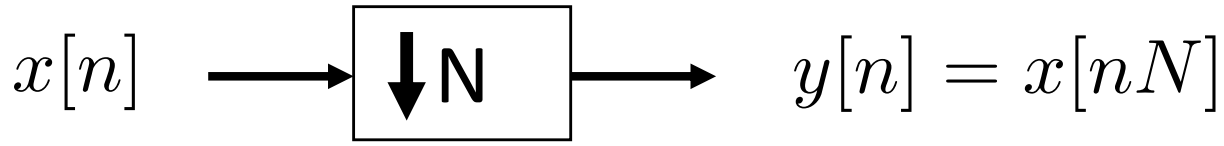
Alternative derivation of downsampling DTFT



$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} y[m]e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]s[m]e^{-j\omega m/N},$$

$$s[m] = \begin{cases} 1 & \text{if } m = kN \\ 0 & \text{o.w.} \end{cases} = \sum_{k=-\infty}^{\infty} \delta(m - kN) = \frac{1}{N}\sum_{k=0}^{N-1} e^{j2\pi km/N}$$

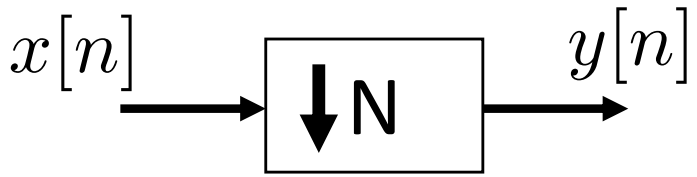
Alternative derivation of downsampling DTFT



$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]e^{-j(\omega-2\pi k)m/N)}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})$$

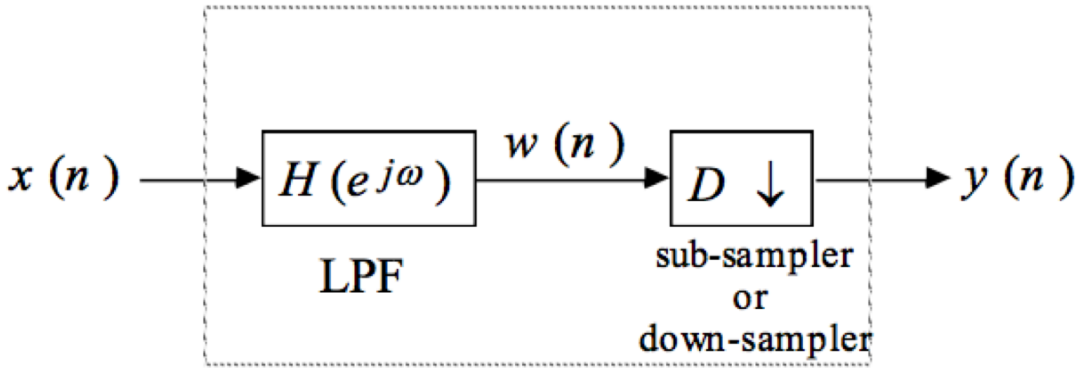
Downsampling



$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-2\pi k)/N})$$

1. **Stretch** $X\left(e^{j\omega}\right)$ to $X(e^{\frac{j\omega}{N}})$
2. Create (N-1) copies of the stretched versions
3. Shift each copy by successive multiples of 2π and add
4. Divide by N

Decimation (filtering and downsampling)



$$Y(e^{j\omega}) = \frac{1}{D} \sum_{p=0}^{D-1} H\left[e^{j\left(\frac{\omega - 2\pi p}{D}\right)}\right] X\left[e^{j\left(\frac{\omega - 2\pi p}{D}\right)}\right]$$

Chirp

Q: for a chirp of length L seconds, ranging from f_0 to f_m, what is the expression for $x_c(t)$, the continuous-time signal?

A: $x_c(t) = \sin\left(2\pi\left[f_0 + \frac{f_m - f_0}{\tau}t\right]t\right)$

Q: How fast do we need to sample this signal?

A: $f_s > 2f_m$

Q: What is $|X_d(j\Omega)|$? (ignore effects of windowing)

A: $rect\left(\frac{\Omega}{2\Omega_0}\right)$

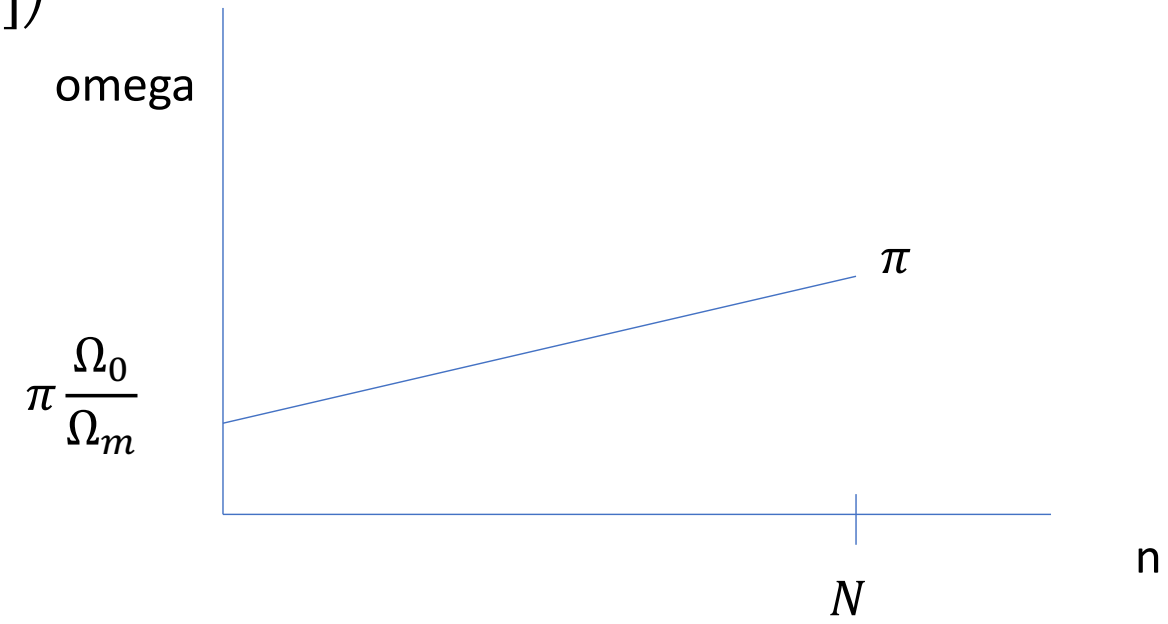
Chirp

$$x_c(t) = \sin\left(2\pi\left[f_0 + \frac{f_m - f_0}{\tau}t\right]t\right)$$

Q: What is x[n] if we sample exactly at Nyquist?

$$x[n] = \sin\left(2\pi\left[f_0 + \frac{f_m - f_0}{\tau}Tn\right]Tn\right)$$
$$= \sin\left(n\pi\left[\frac{\Omega_0}{\Omega_m}\left(1 - \frac{n}{N}\right) + \frac{n}{N}\right]\right)$$

Q: Plot the frequency vs n



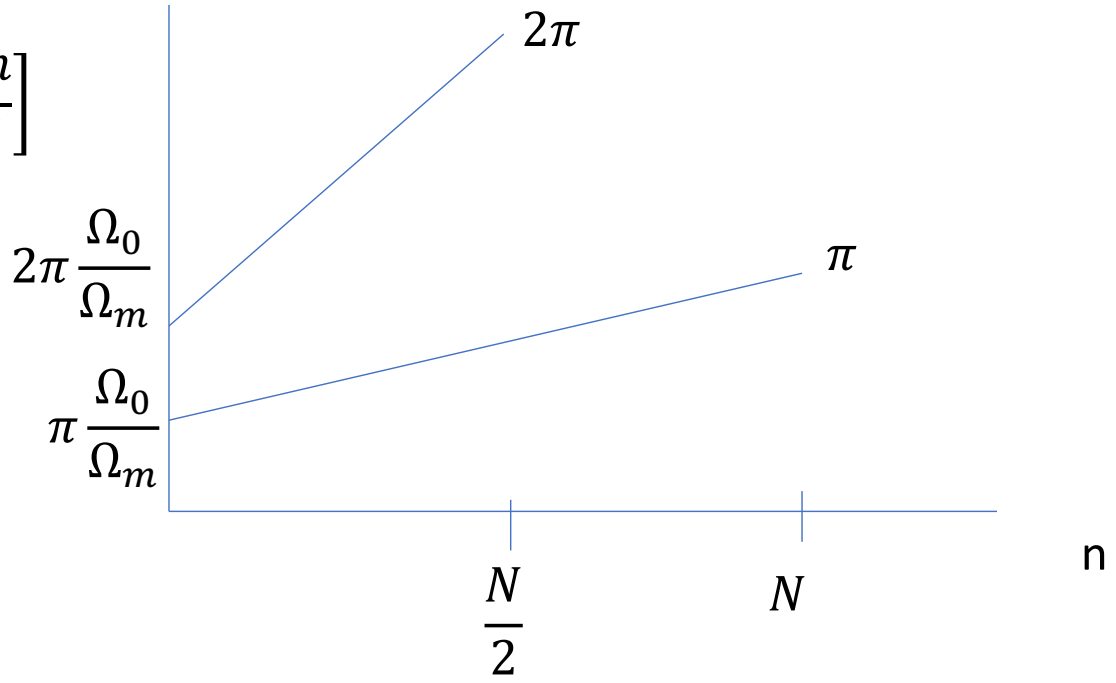
Downsample by 2 (without AA filter)

Recall: $x[n] = \sin\left(n\pi\left[\frac{\Omega_0}{\Omega_m}\left(1-\frac{n}{N}\right)+\frac{n}{N}\right]\right)$

A: $x[2n] = \sin\left(2n\pi\left[\frac{\Omega_0}{\Omega_m}\left(1-\frac{2n}{N}\right)+\frac{2n}{N}\right]\right)$

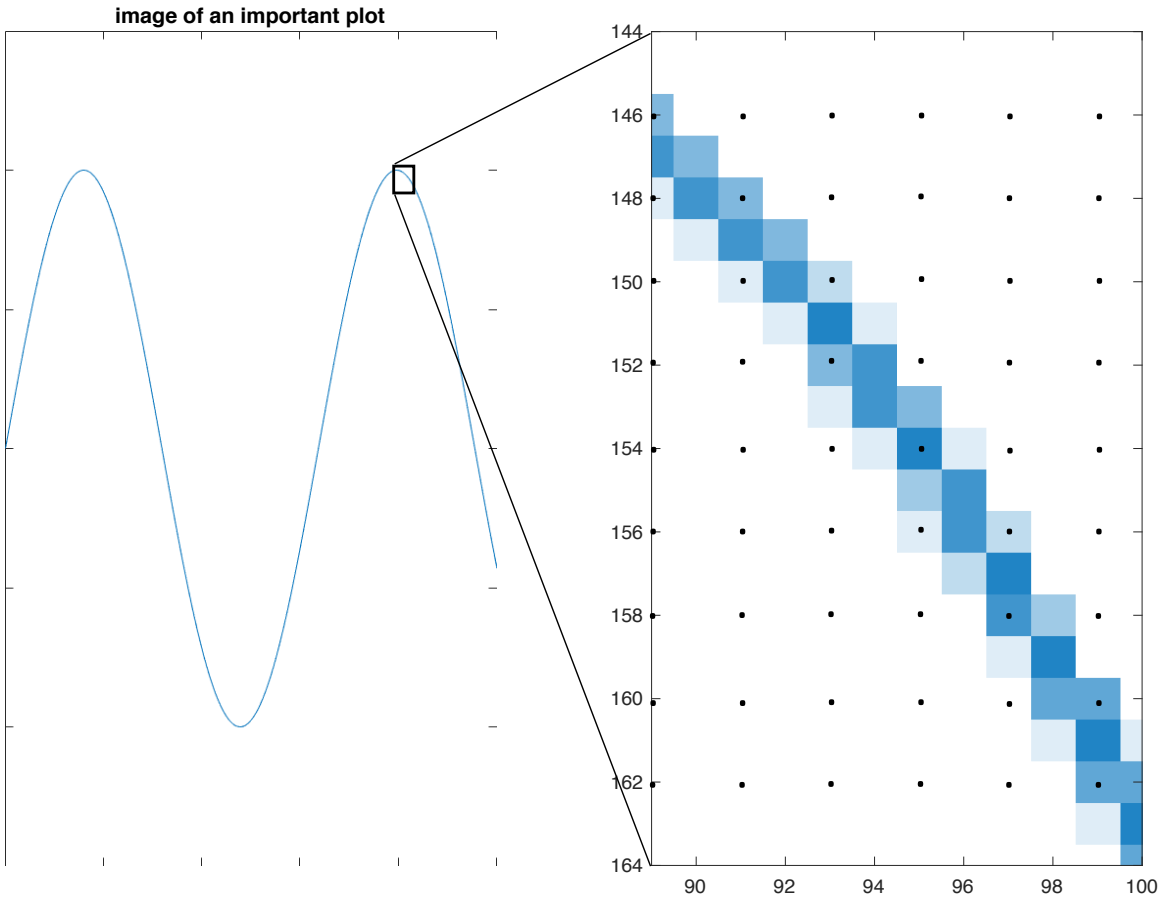
Q: What is the frequency vs n now?

$$2\pi\left[\frac{\Omega_0}{\Omega_m}\left(1-\frac{2n}{N}\right)+\frac{2n}{N}\right]$$

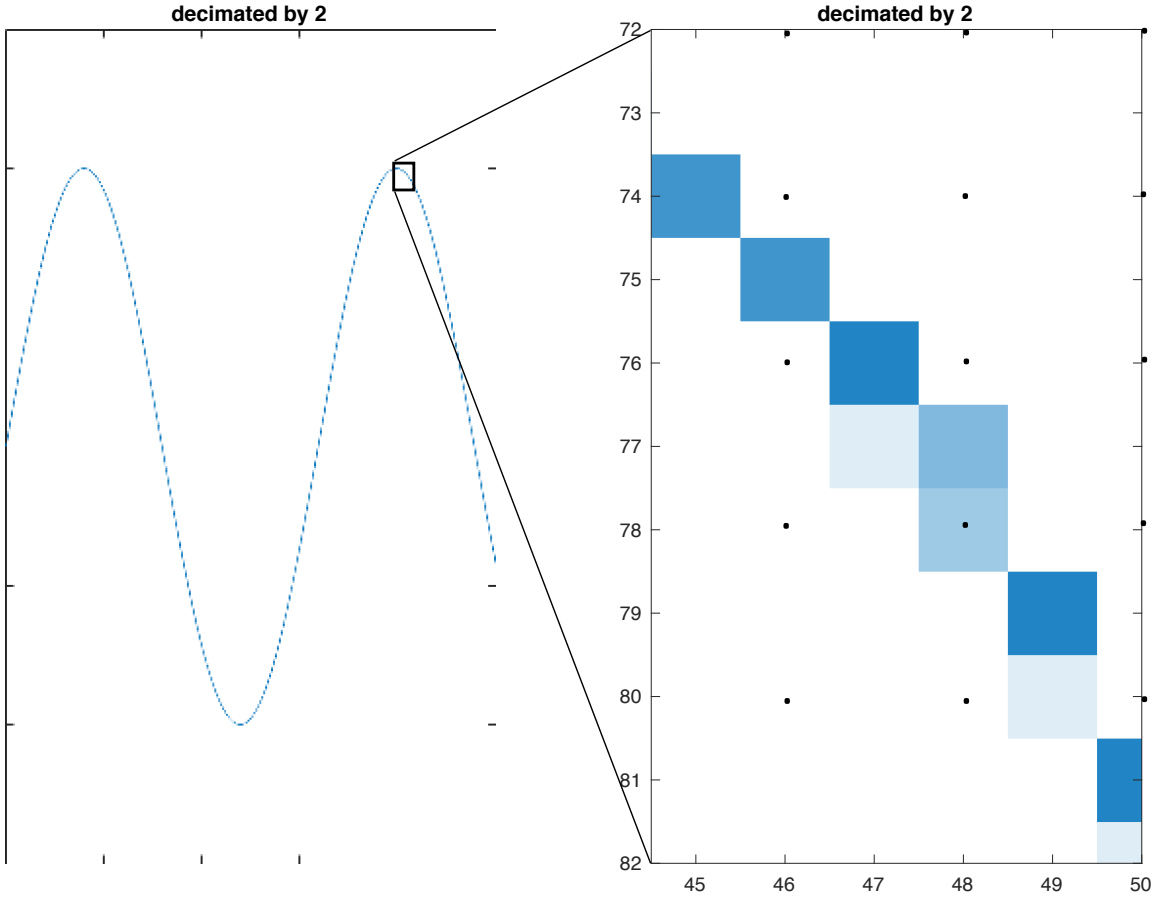


Chirp demo

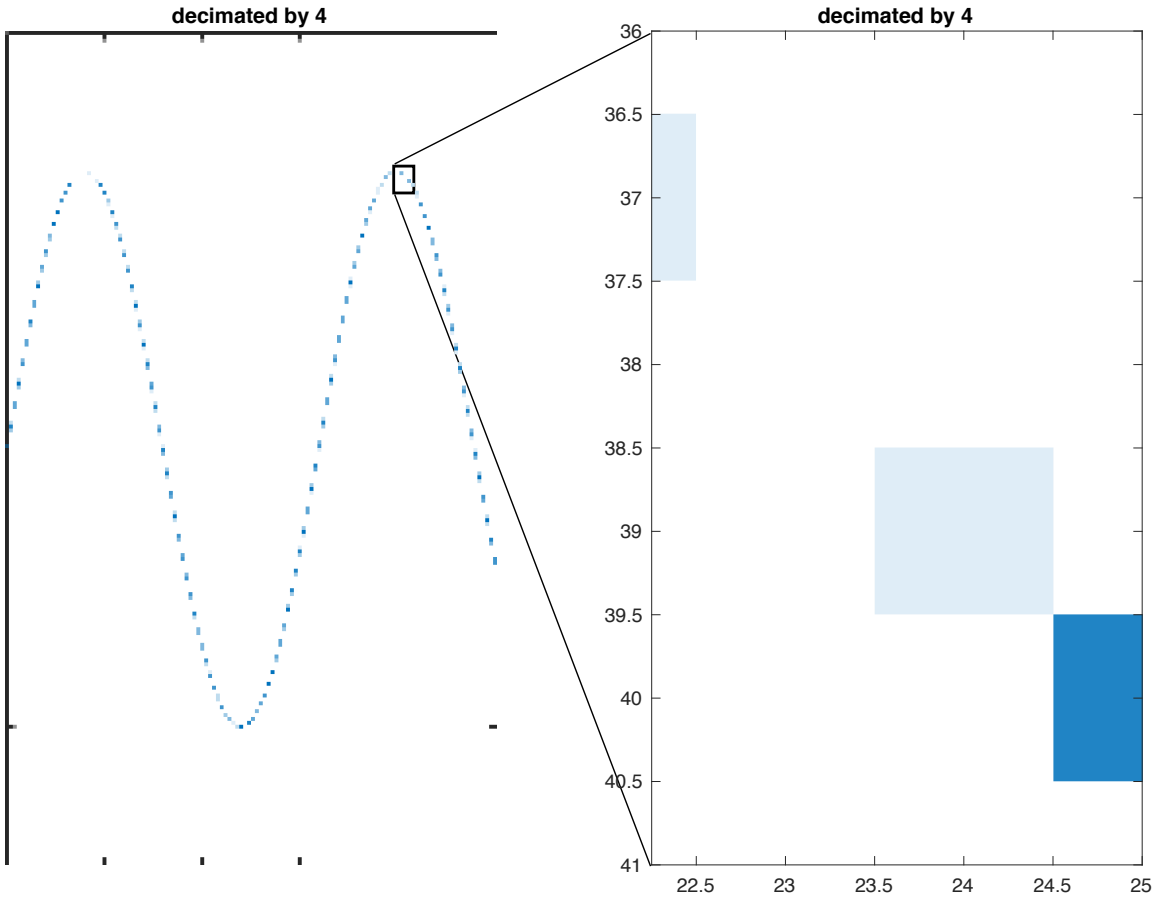
Imaging example



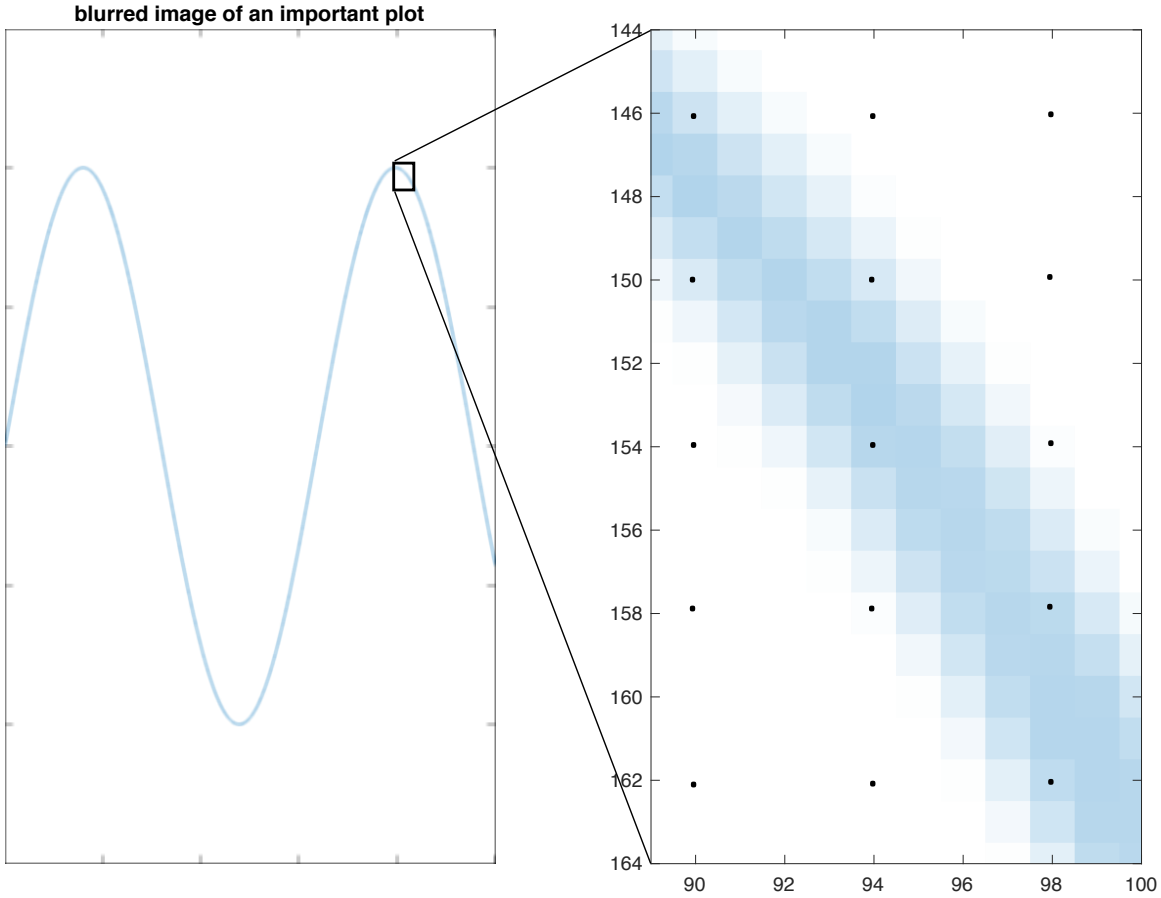
Decimated by 2



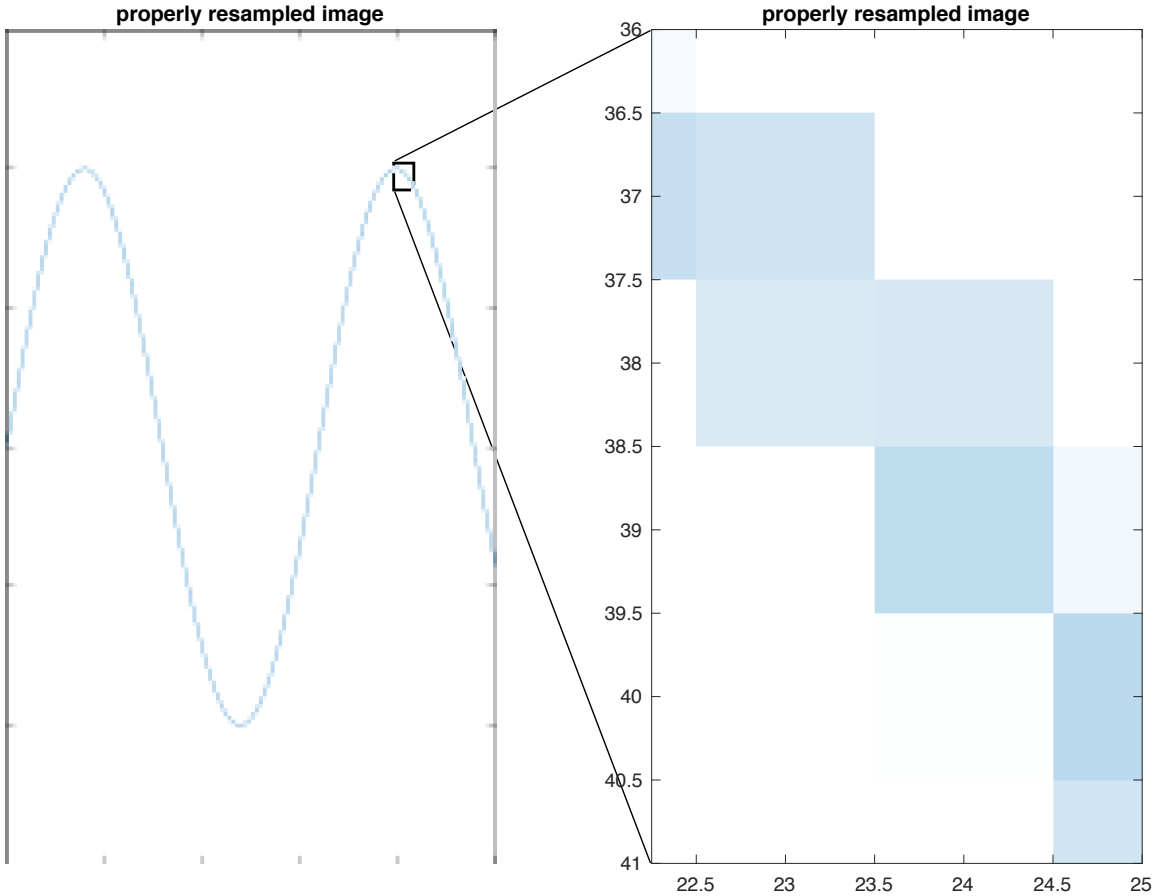
Decimated by 2 again (4 total)



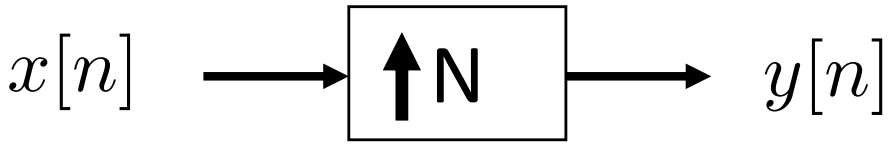
Lowpass filter first



Resampling looks fine!



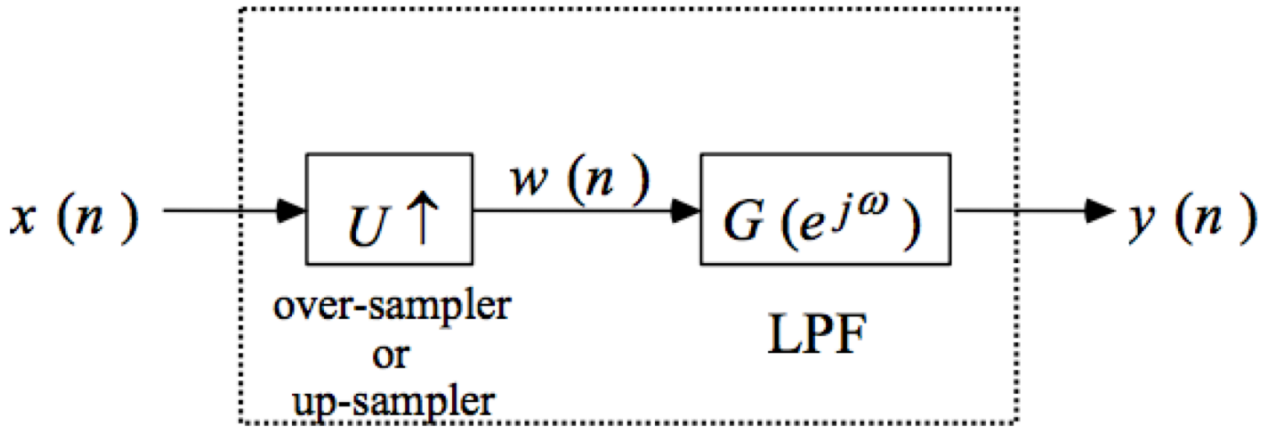
Upsampling



- **Expands** in the time domain
- **Compresses** in the frequency domain

$$Y(e^{j\omega}) = X(e^{j\omega N})$$

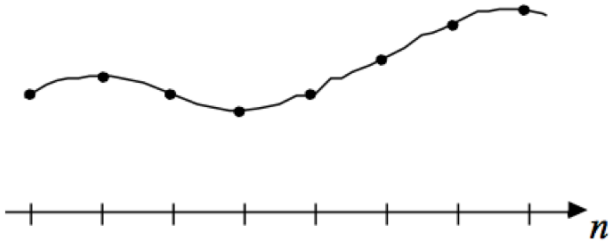
Interpolation (Upsampling and filtering)



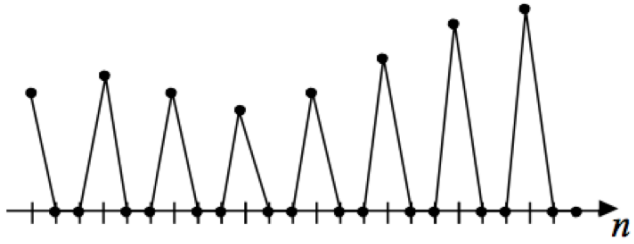
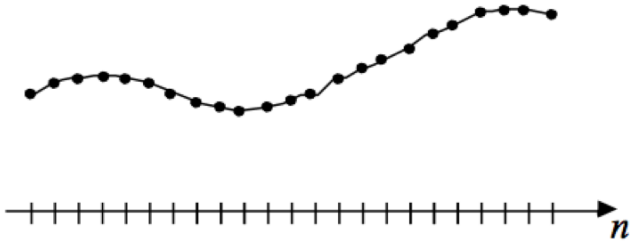
$$Y(e^{j\omega}) = G(e^{j\omega})X(e^{j\omega U})$$

Interpolation

1. Smooth discrete-time signal
   - Low frequency content



2. Upsample by 3
   - No longer smooth! Contains high fr



3. LPF
   - Removes high frequencies