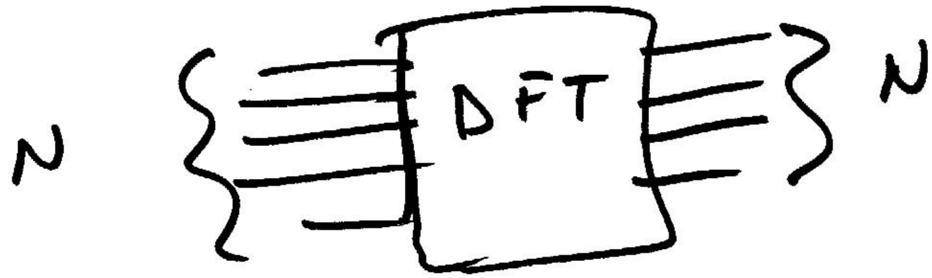
Nov 12, 03

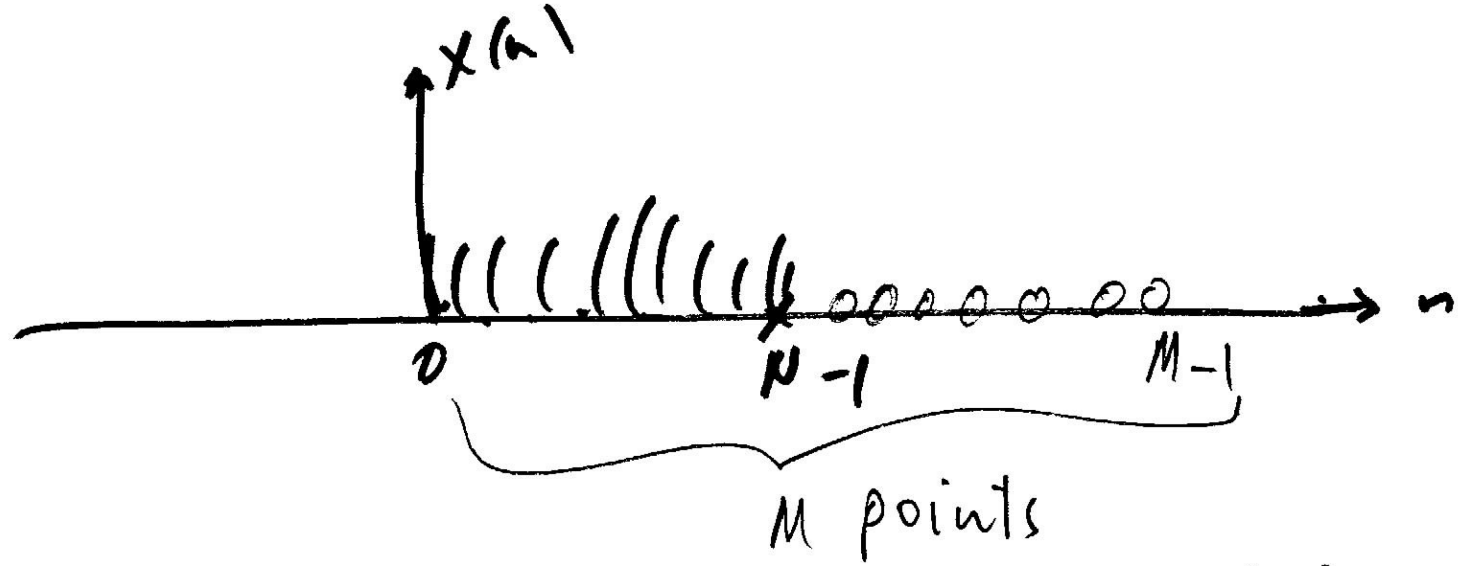
DFT = Discrete Fourier Transform

- finite length N pt seq $x(n)$, N non zero values

$$DTF\{x(n)\} = X(k) = \begin{cases} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi n k}{N}} & 0 \le k < N \\ 0 & \text{otherwise} \end{cases}$$

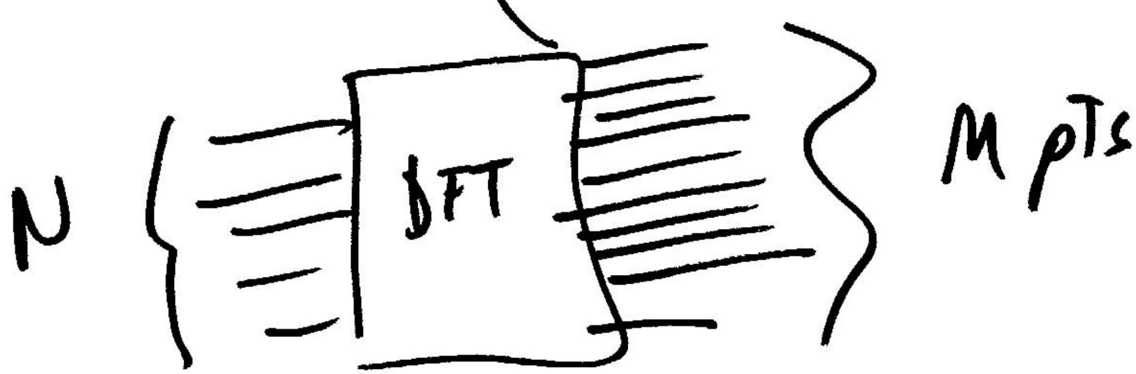


- Possible to pad an N pt seq. with N nonzero values to M points, ie pad it with $M-N$ additional zero's
- Possible to talk about DFT of M pt seq.

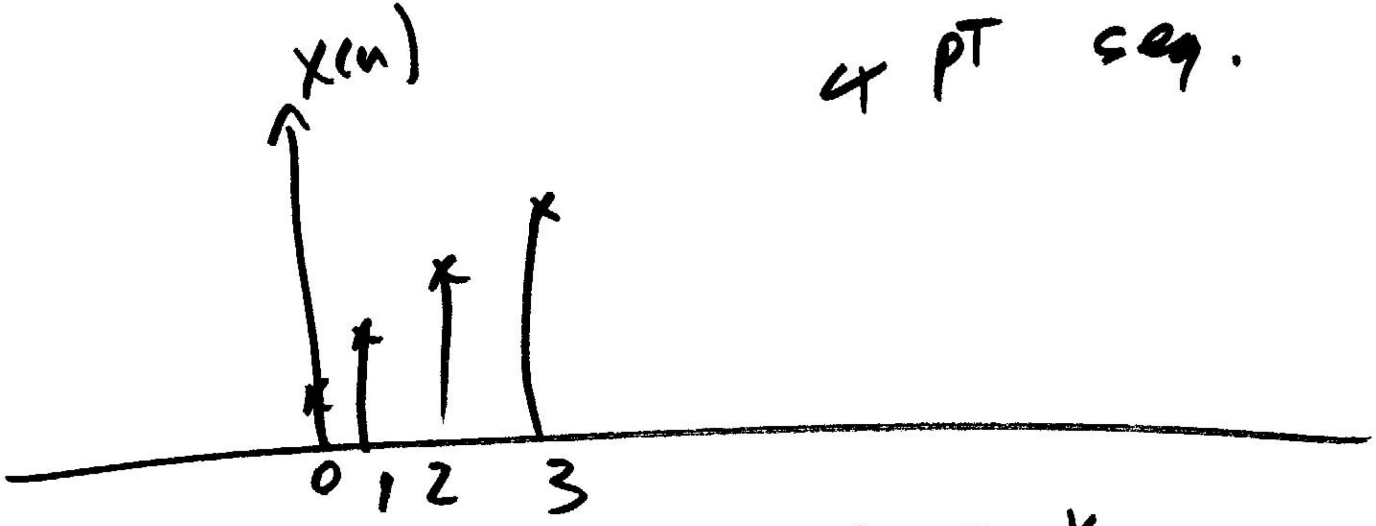


- Can take M pt DFT of $x(n)$:

$$X(k) = \begin{cases} \sum_{n=0}^{M-1} x(n)\, e^{-j\frac{2\pi n k}{M}} & 0 \le k < M \\ 0 & \text{otherwise.} \end{cases}$$



Ex



4 pT seq.

- 4 pT DFT x(n): $\sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi n k}{4}}$ 4 #s.

- 5 pt DFT x(n): $\sum_{n=0}^{4} x(n)\, e^{-j\frac{2\pi n k}{5}}$ 5 #s

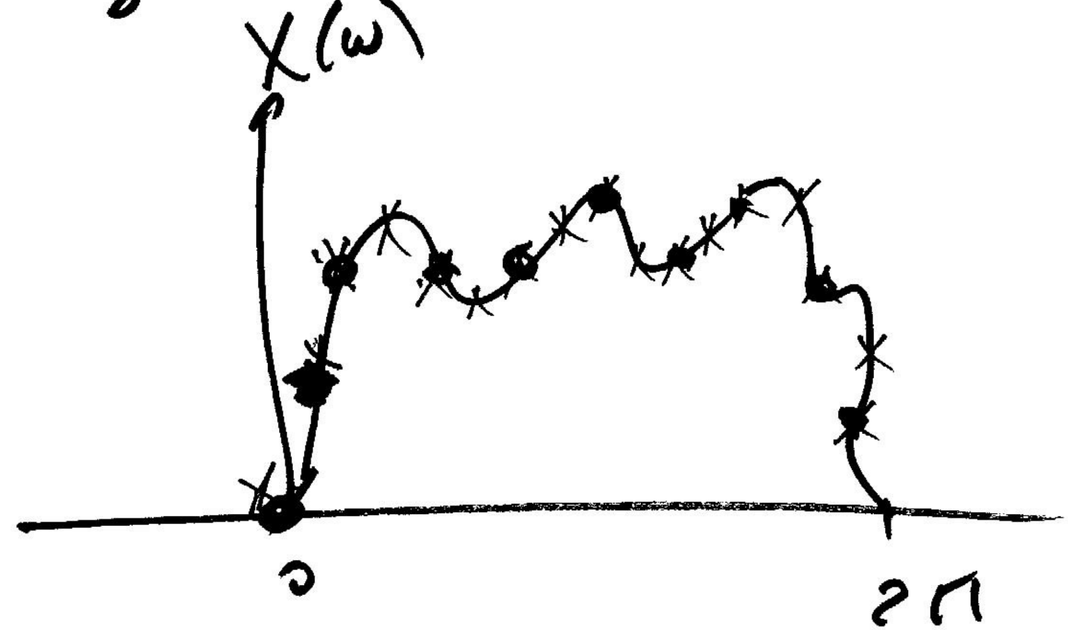
- 1000 pt DFT of x(n) $\sum_{n=0}^{999} x(n)\, e^{-j\frac{2\pi n k}{1000}}$ 1000 #s

- 8 pt DFT of x(n) $\sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi n k}{8}}$ 8 #s

$x(n)$, pad it $\dot{x}(n) \rightarrow X(k)$

N pt , $M-N$ zeros , M pt , M pt

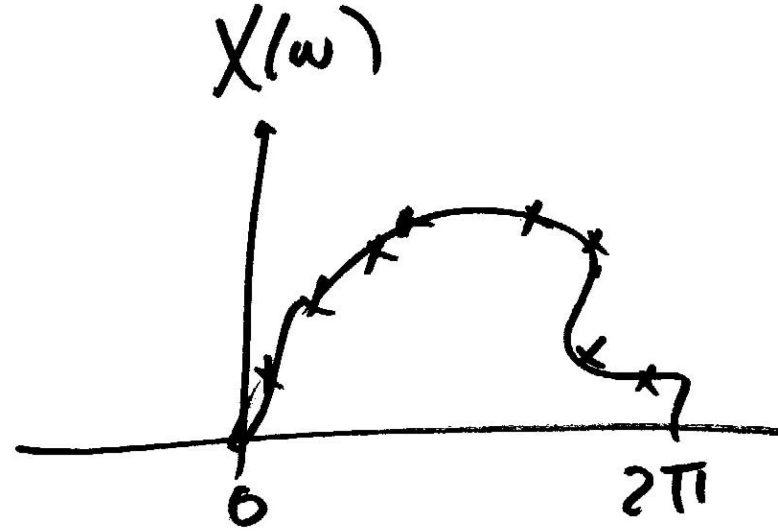
$x(n) \longrightarrow$



$x(n) \rightarrow \boxed{\text{N pt DFT}} \rightarrow$ N #s out

$x(n) \rightarrow \boxed{\text{2N pt DFT}} \rightarrow$ 2N #s

N point seq. Padded it N zeros.



4

Suppose we have nonuniform samples of $X(\omega)$ at N points. We know $X(\omega)$ is DTFT of an N pt seq..

Can I recover $x(n)$?

<u>Yes</u> solving a linear system of eqns.

$$X(\omega) = \sum x(n) e^{-j\omega n}$$

$$\left[X(\omega_i)\right]_{i=0,\dots N-1} = \sum_{n=0}^{N-1} x(n) e^{-j\omega_i n}.$$

$$\begin{bmatrix} X(w_0) \\ X(w_1) \\ \vdots \\ X(w_{N-1}) \end{bmatrix} = \begin{bmatrix} e^{-jw_0 \cdot 0} & e^{-jw_0 \cdot 1} & \cdots & e^{-jw_0 (N-1)} \\ \vdots & & \ddots & \vdots \\ e^{-jw_{N-1} 0} & e^{-jw_{N-1} 1} & \cdots & e^{-jw_{N-1} (N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Known.
non uniform
samples

$N \times N$

A

Unknown

N pt.

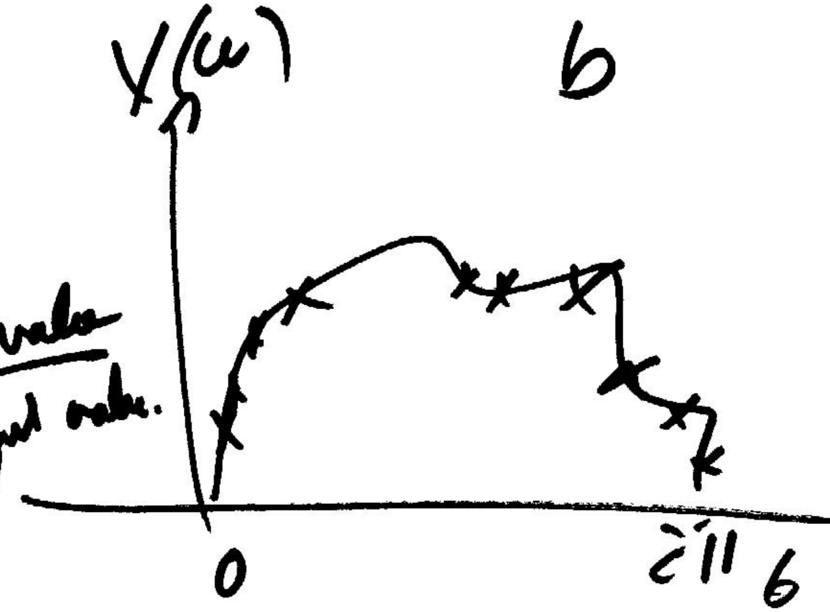
N pt

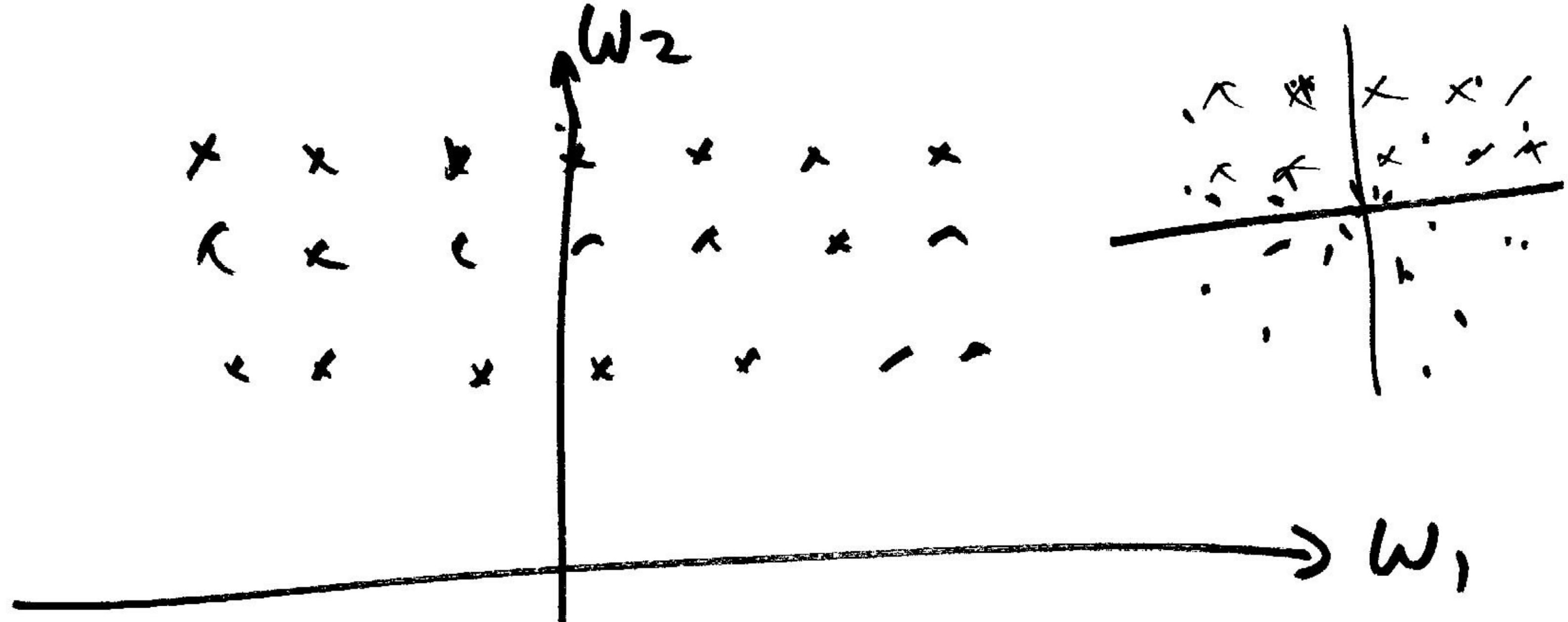
$\vec{x}$

b

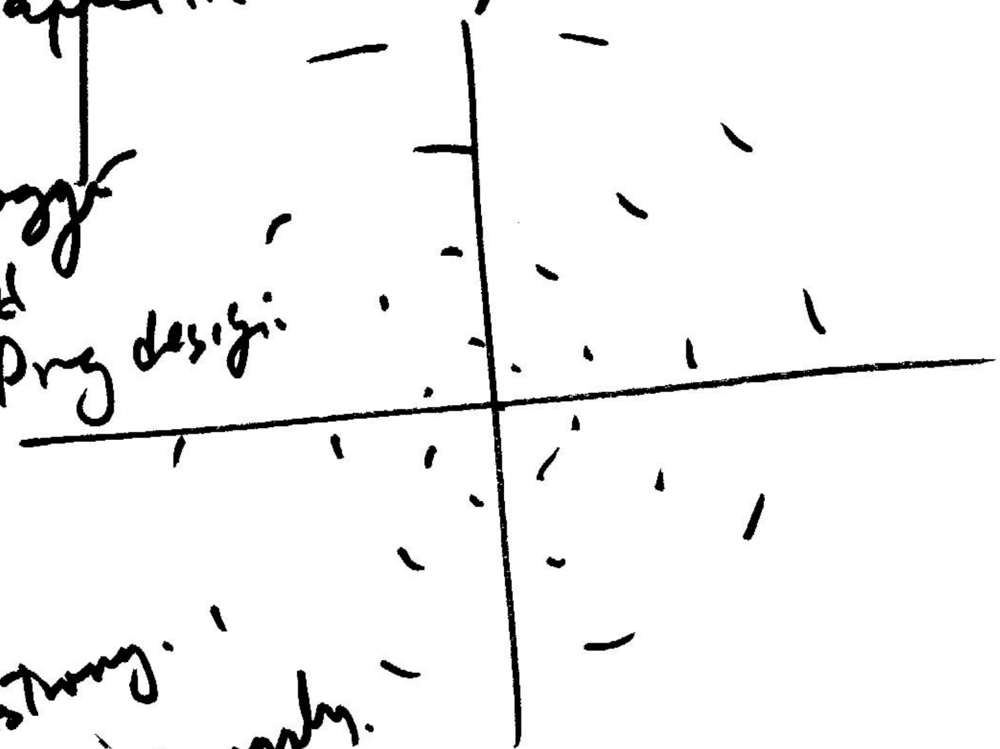
$A\vec{x} = b$

Condition # $= \dfrac{\text{largest Singular value}}{\text{smallest singular value.}}$





Graph non uniform samples in Freq.



1. X Ray Crystallography
   widely used biology / Drug design.
2. MRI
3. Radar
3. Radio Astronomy.
4. Tomography.

Thought Exp:

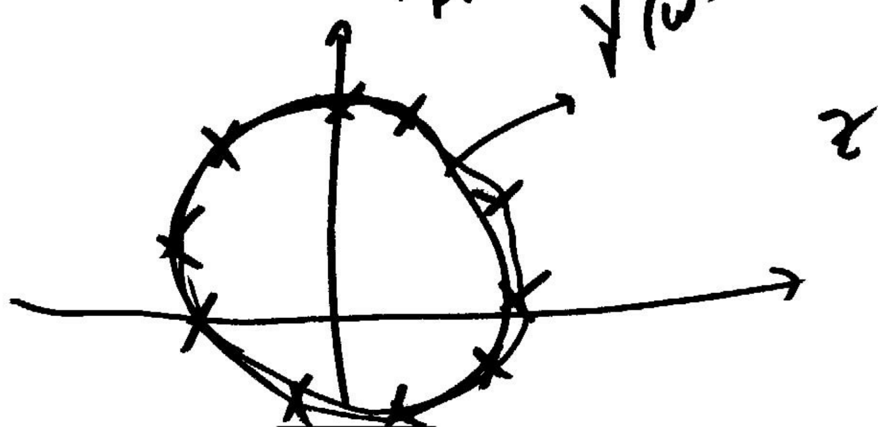
1. $y(n)$: either ∞ extent or finite extent.
2. $Y(\omega) = \text{DTFT}\{y(n)\}$.
3. Sample $Y(\omega)$ at N equally spaced points.

$$\left[Y(\omega)\right]_{\omega = \frac{2\pi k}{N}} = \tilde{X}(k)$$

N pt seq.

4. $\text{IDTFT}\{\underbrace{\tilde{X}(k)}_{N\,pT}\} = x(n)$

N pT seq.



Answer: periodicize $y(n)$ with period N.

$$\tilde{w}(n) = \sum_{k=-\infty}^{+\infty} y(n+kN)$$

extract one period:

$$x(n) = w(n)\, R_N(n).$$

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proof

$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{+j\frac{2\pi nk}{N}} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_m y(m)\, e^{-j\frac{2\pi km}{N}} \right) e^{+j\frac{2\pi nk}{N}} & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \displaystyle\sum_{m=-\infty}^{+\infty} y(m) \underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{-j2\pi k\frac{(m-n)}{N}}} & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

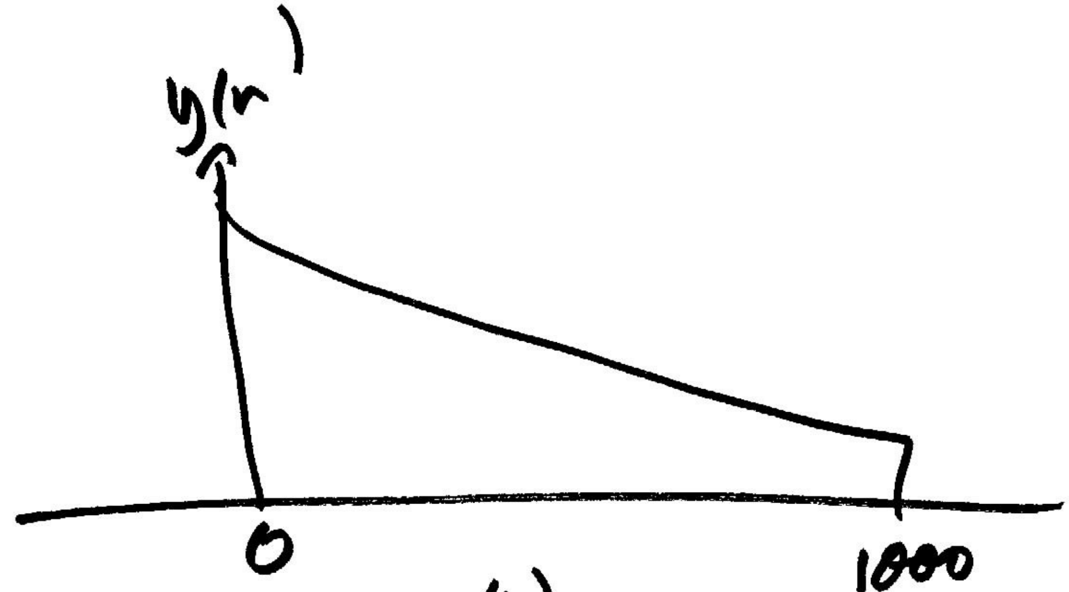
$$f(n-m) = \sum_{r=-\infty}^{+\infty} \delta(n-m+rN)$$

Call $f(n) = \displaystyle\sum_{r=-\infty}^{+\infty} \delta(n+rN)$

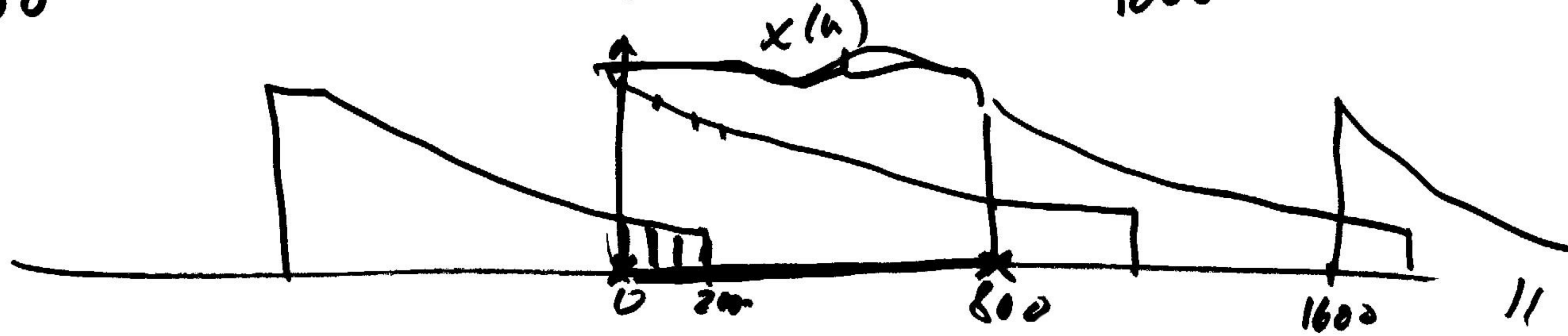
$$x(n) = \begin{cases} \displaystyle\sum_{m=-\infty}^{+\infty} y(m)\, f(n-m) & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$

$=$

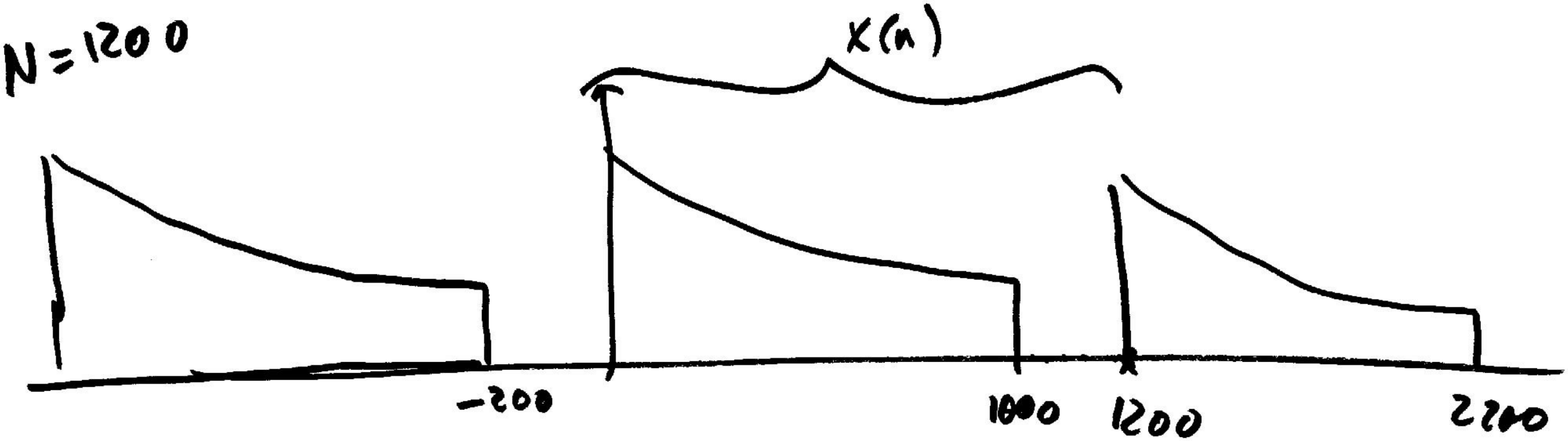
$$x(n) = \begin{cases} \sum_{r=-\infty}^{+\infty} y(n+rN) & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$



$N = 800$



$N = 1200$



To get $Y(n)$ back precisely, we need to sample its DTFT, $Y(\omega)$ at a higher rate than # of nonzero points in $G(n)$ containing.

Properties of DFT.

① Shift property.

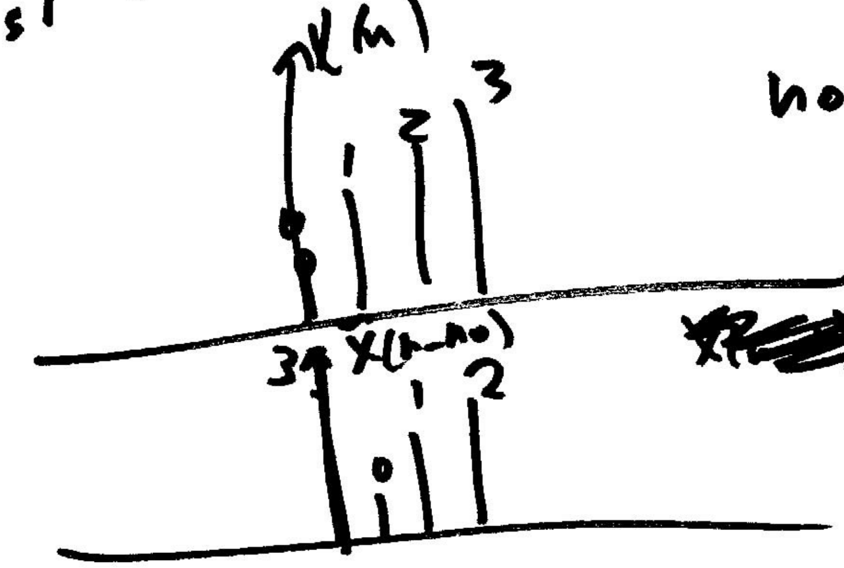
$x(n) \xleftrightarrow{\text{DFT}} X(k)$ N pt, $0 \le k < N$

N pt

$\cancel{x(n-n_0)} \xleftrightarrow{\text{IDFT}} X(k)\, e^{-j\frac{2\pi n_0 k}{N}}$ N pt, $0 \le k < N$

Should only be nonzero for first N points

$n_0 = 1$



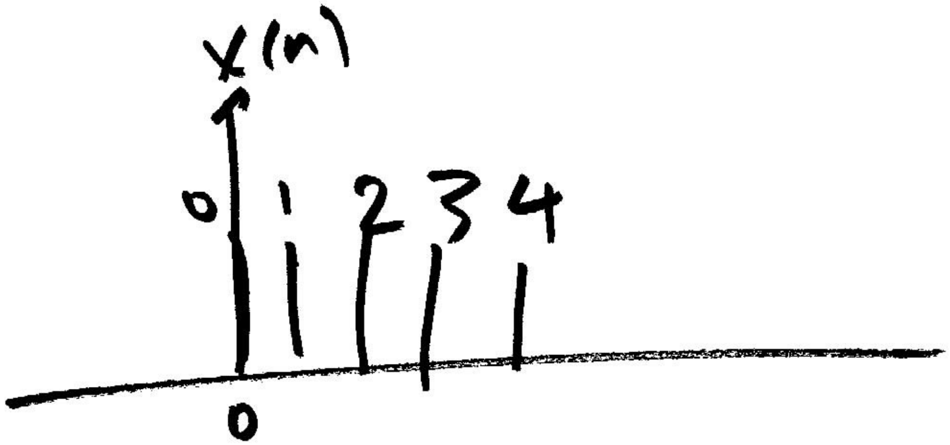
Recall

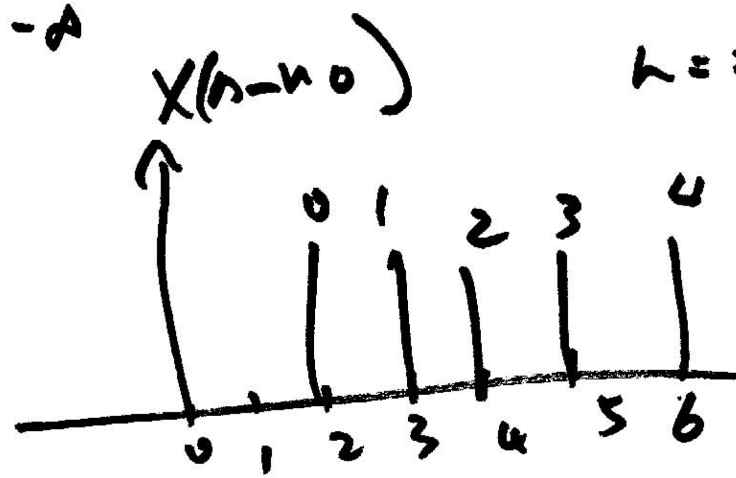
$$\tilde{x}(n-n_0)\, R_N(n) \xleftrightarrow{\text{IDFT}} X(k)\, e^{-j\frac{2\pi n_0 k}{N}}$$

<u>Why?</u>

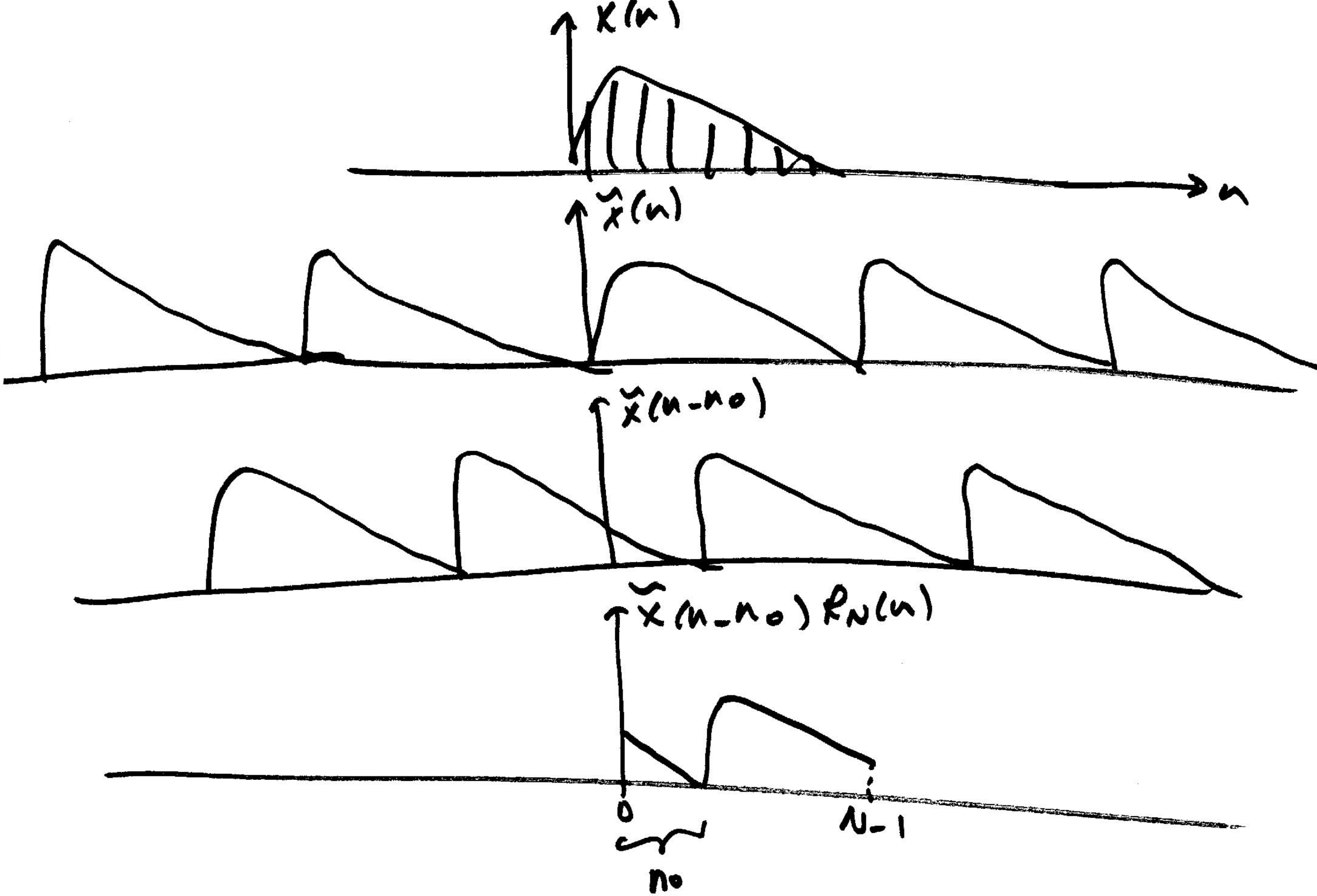
$$x(n) \xleftrightarrow{\text{periodize}} \tilde{x}(n) = \sum_{k=-\infty}^{+\infty} x(n+kN) \xleftrightarrow{\text{DFS}} \underset{\text{DFS}}{\tilde{X}(k)} \longrightarrow \underset{\text{DFT}}{X(k)}$$

$n_0 = 2$





$x(n-n_0)$ is not valid IDFT.



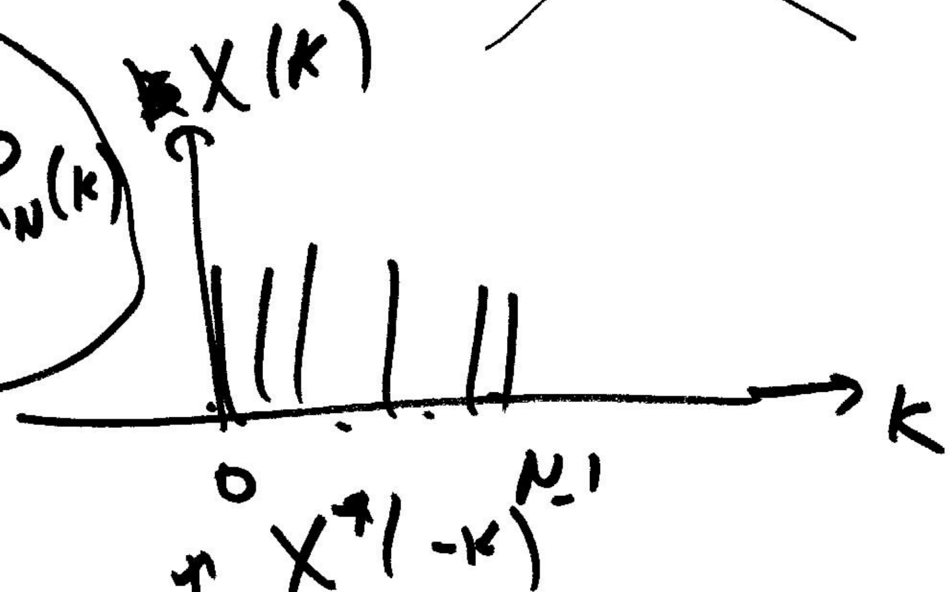
$x(n)$
n
$\tilde{x}(n)$
$\tilde{x}(n-n_0)$
$\tilde{x}(n-n_0)R_N(n)$
0
$N-1$
n_0

Another property of DFT:

$$x(n) \longleftrightarrow X(k) \rightarrow N\text{pt.}$$

$$y(n) = x^*(n) \xrightarrow{?} \cancel{X^*(-k)}$$

$$\mathrm{DFT}\{y(n)\} = \underbrace{X^*(-k)}\; R_N(k)$$



Symmetry Prop

$$x(n) \longleftrightarrow X(k)$$

$$\mathrm{Re}\{x(n)\} \longleftrightarrow R_N(k)\,\widetilde{X_e(k)}$$

$$j\,\mathrm{Im}\{x(n)\} \longleftrightarrow R_N(k)\,\widetilde{X_o(k)}$$

$$R_N(n)\,\widetilde{x_e(n)} \longleftrightarrow \mathrm{Re}\{X(k)\}$$

$$R_N(n)\,\widetilde{x_o(n)} \longleftrightarrow j\,\mathrm{Im}\{X(k)\}$$