Nov 26, 03

Fast Fourier Tranform

Decimation in Time.

" " Frequency.

Decimation in Time

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} \qquad 0 \le k < N$$

$$X(k) = \sum_{\substack{n \text{ even} \\ 0 \to N-1}} x(n)\, e^{-j\frac{2\pi n k}{N}} + \sum_{\substack{n\; 0 \to N-1 \\ \text{odd}}} x(n)\, e^{-j\frac{2\pi n k}{N}}$$

$n = 2r \quad r: 0 \to \frac{N}{2} - 1$

$n = 2r + 1$

$r: 0 \to \frac{N}{2} - 1$,

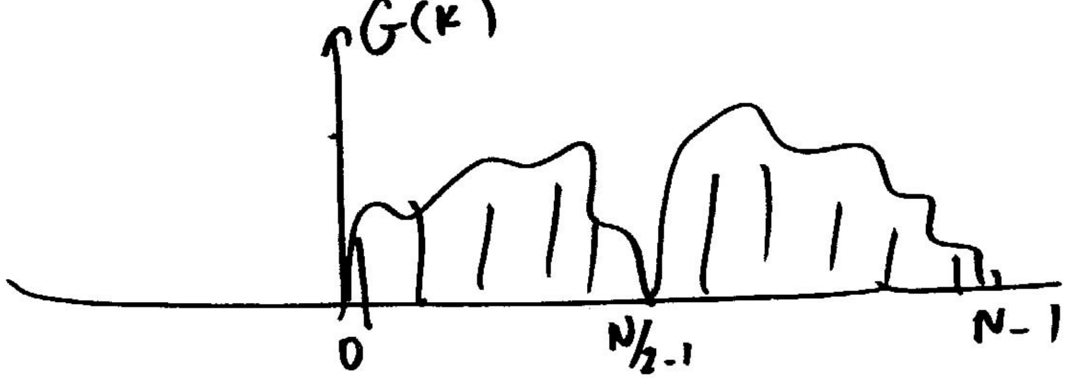
$$X(k) = \sum_{r=0}^{N/2-1} \underbrace{x(2r)}^{g(r)} e^{-j\frac{2\pi\, 2r\, k}{N}} + \sum_{r=0}^{N/2-1} \overbrace{x(2r+1)}^{h(r)} e^{-j\frac{2\pi k}{N}(2r+1)}$$

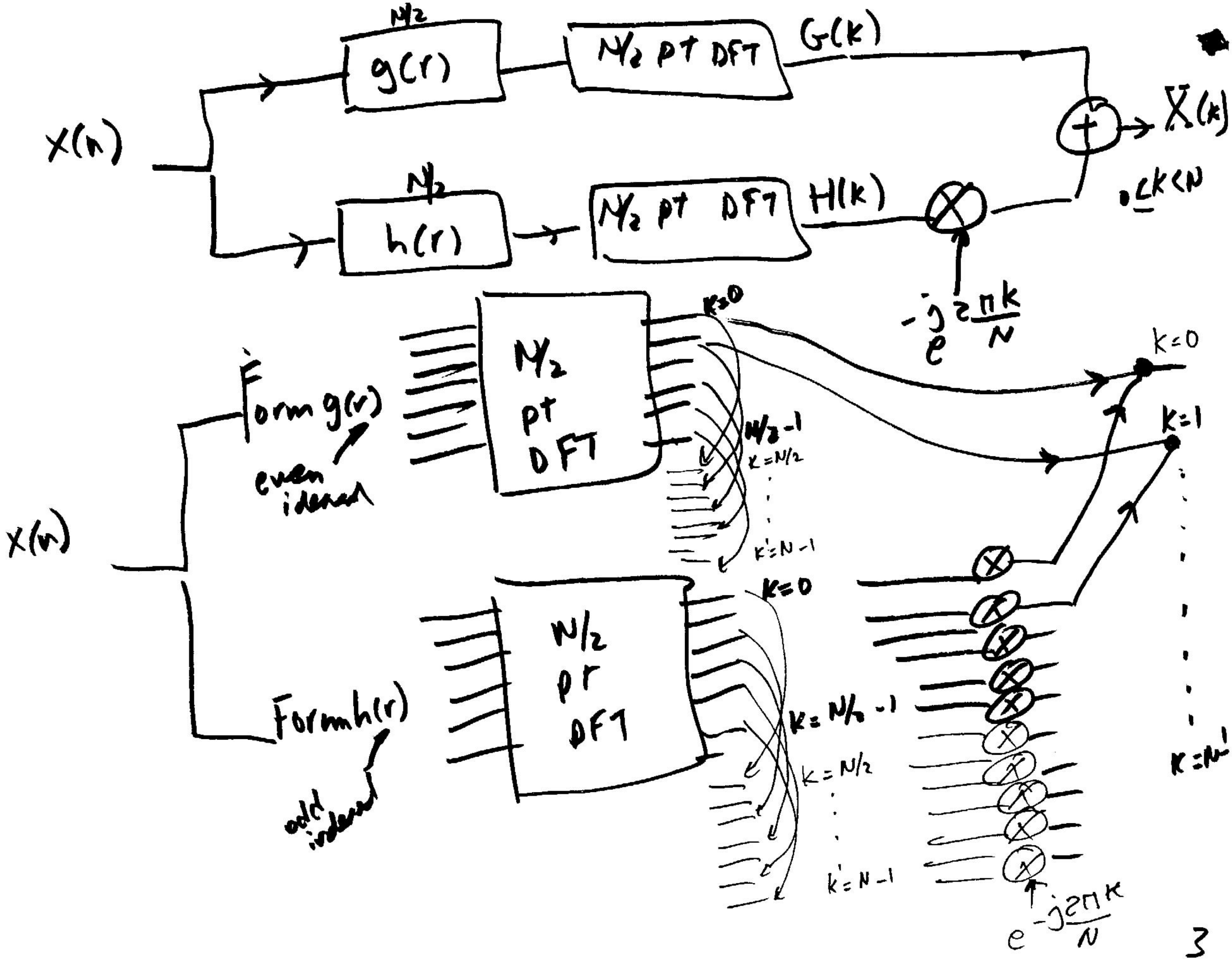
$$X(k) = \underbrace{\sum_{r=0}^{N/2-1} g(r)\, e^{-j\frac{2\pi k r}{N/2}}}_{N/2 \text{ pt DFT of } N/2 \text{ pT seq. } g(r)} + e^{-j\frac{2\pi k}{N}} \underbrace{\sum_{r=0}^{N/2-1} h(r)\, e^{-j\frac{2\pi k r}{N/2}}}_{N/2 \text{ pT DFT of } N/2 \text{ pt seq. } h(r)}$$

Also DFS of $g(r)$

Also DFS of $h(r)$

$K: 0 \longrightarrow N-1$




x(n)
N/2
g(r)
N/2 pt DFT
G(k)
N/2
h(r)
N/2 pt DFT
H(k)
X(k)
0≤k<N
e^(-j2πk/N)
Form g(r)
even indexed
Form h(r)
odd indexed
x(n)
N/2 pt DFT
N/2 pt DFT
k=0
N/2-1
k=N/2
k=N-1
k=0
k=N/2-1
k=N/2
k=N-1
k=0
k=1
k=N-1
e^(-j2πk/N)

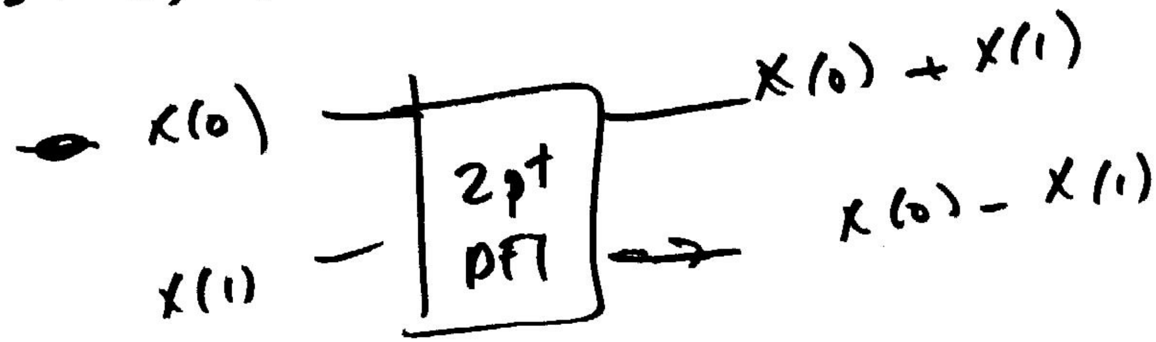
2 pt DFT $\quad x(n) \quad n = 0, 1$

$$\sum_{n=0}^{1} x(n)\, e^{-j\frac{2\pi n k}{2}} = x(0)\, e^{-j(0)} + x(1)\, e^{-j\frac{2\pi k}{2}}$$

$$= x(0) + x(1)\, e^{-j\pi k}$$

$$k=0 \rightarrow e^{-j\pi k} = 1 \Longrightarrow \quad X(0) = x(0) + x(1)$$

$$k=1 \rightarrow e^{-j\pi k} = -1 \longrightarrow \quad X(1) = x(0) - x(1)$$

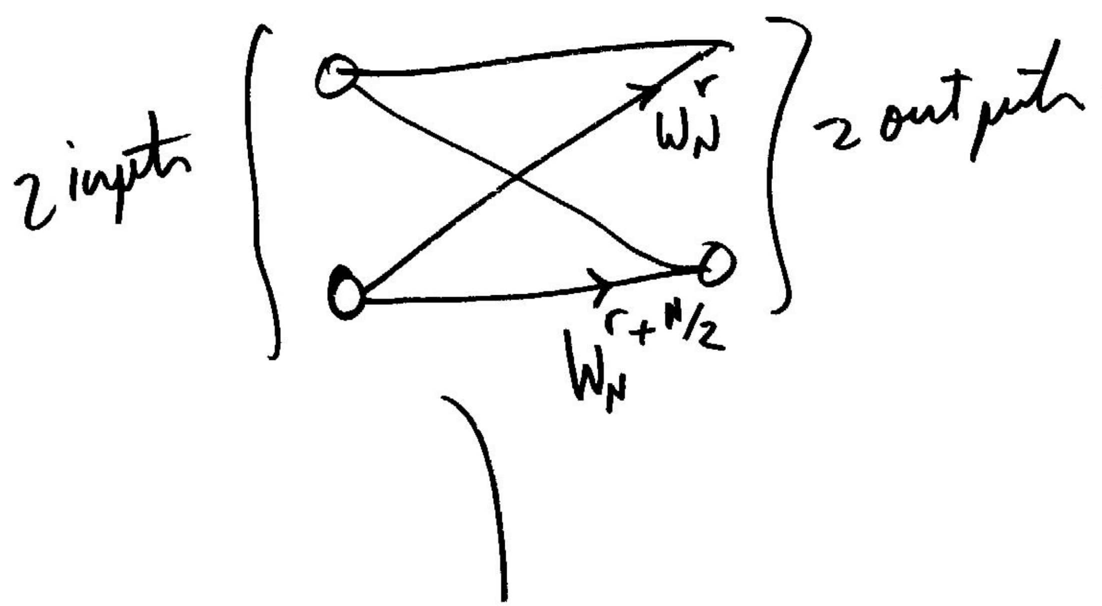


Show pics from O&S 9.3–9.8

Define Twiddle factor

$$W_N^K \triangleq e^{-j\frac{2\pi K}{N}}$$

General form of a butterfly.

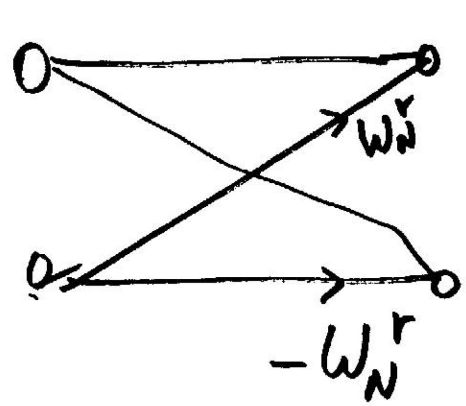


$$W_N^{r+N/2} \triangleq e^{-j\frac{2\pi}{N}\left(r+\frac{N}{2}\right)}$$

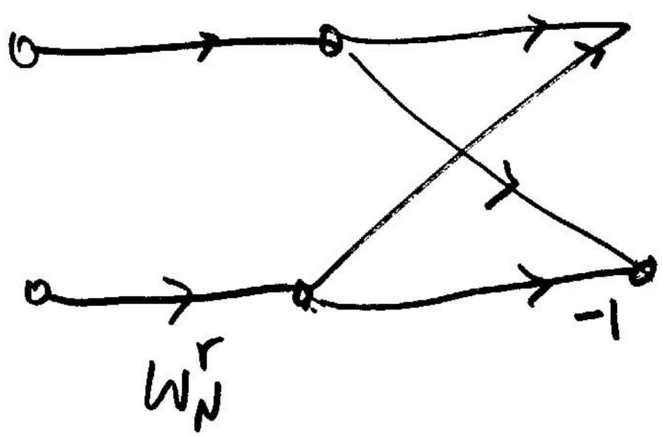
$$= e^{-j\frac{2\pi r}{N}} e^{-j\frac{2\pi N}{2N}}$$

$$= -e^{-j\frac{2\pi r}{N}}$$

$$= -W_N^r$$



2 mults
2 adds.



1 mult
2 adds

Show 9.10

Operation count

- $\log_2 N$ stages.
- Each stage. N input $\longrightarrow$ N outputs
- each butterfly has 2 input $\longrightarrow$ 2 outputs.

  $N/2$ butterflies per stage.

- Each butterfly has 2 adds, 1 mult.

→ Total

$$\frac{N}{2} \times 2 \log_2 N. \quad \text{adds} \approx N \log_2 N$$

$$\frac{N}{2} \times 1 \times \log_2 N \quad \text{mults} \quad \frac{N}{2} \log_2 N$$

Fig 9.10

Bit Reversing
input/output order

x(0)	0	0 0 0 →	0 0 0	0
x(4)	1	0 0 1 →	1 0 0	4
x(2)	2	0 1 0 →	0 1 0	2
x(6)	3	0 1 1 →	1 1 0	6
x(1)	4	1 0 0 →	0 0 1	1
x(5)	5	1 0 1 →	1 0 1	5
x(3)	6	1 1 0 →	0 1 1	3
x(7)	7	1 1 1 →	1 1 1	7

Decimation in Frequency

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}}$$

Divide Computation 2 parts: even idices of k $\quad k = 2r$
odd indices of k. $\quad k = 2r+1$

1 k even $\quad k = 2r$

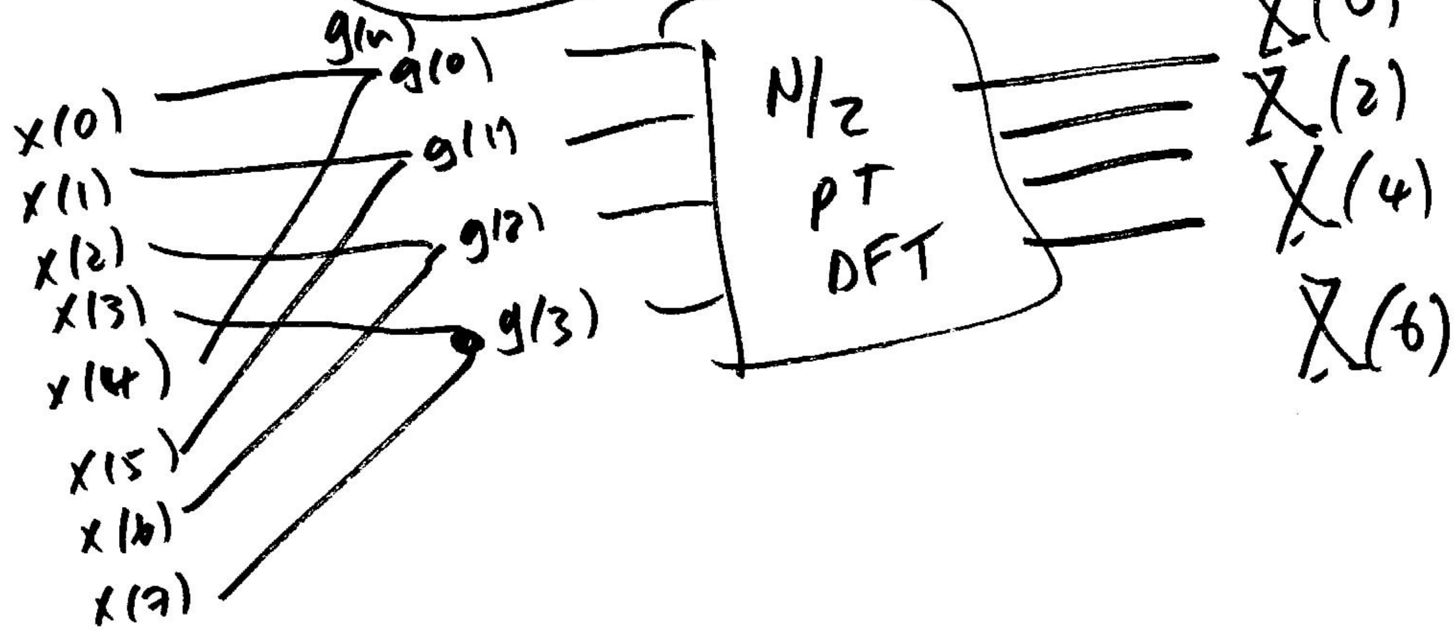
$$X(2r) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n r}{N/2}} \qquad 0 \leq r < N/2$$

$$X(2r) = \underbrace{\sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi r n}{N/2}}} + \underbrace{\sum_{n=N/2}^{N-1} x(n)\, e^{-j\frac{2\pi r n}{N/2}}}_{n = m + \frac{N}{2}}$$

$$X(2r) = \Big(\qquad \Big) + \sum_{m=0}^{N/2-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi r}{N/2}\left(m+\frac{N}{2}\right)}$$

$$X(2r) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi r n}{N/2}} + \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi r n}{N/2}}$$

$$X(2r) = \sum_{n=0}^{N/2-1} \Big[\underbrace{x(n) + x\left(n+\frac{N}{2}\right)}_{g(n)}\Big]\, e^{-j\frac{2\pi r n}{N/2}} \qquad 0 \le r < N/2$$



How about $K = 2r+1$, i.e. odd indices of k.

$$X(2r+1) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n(2r+1)}{N}} \qquad 0 \leq r < N/2$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi n(2r+1)}{N}} + \underbrace{\sum_{n=N/2}^{N-1} x(n)\, e^{-j\frac{2\pi n}{N}(2r+1)}}_{n = m + N/2}$$

$$X(2r+1) = \qquad + \underbrace{\sum_{m=0}^{N/2-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi}{N}\left(m+\frac{N}{2}\right)(2r+1)}}$$

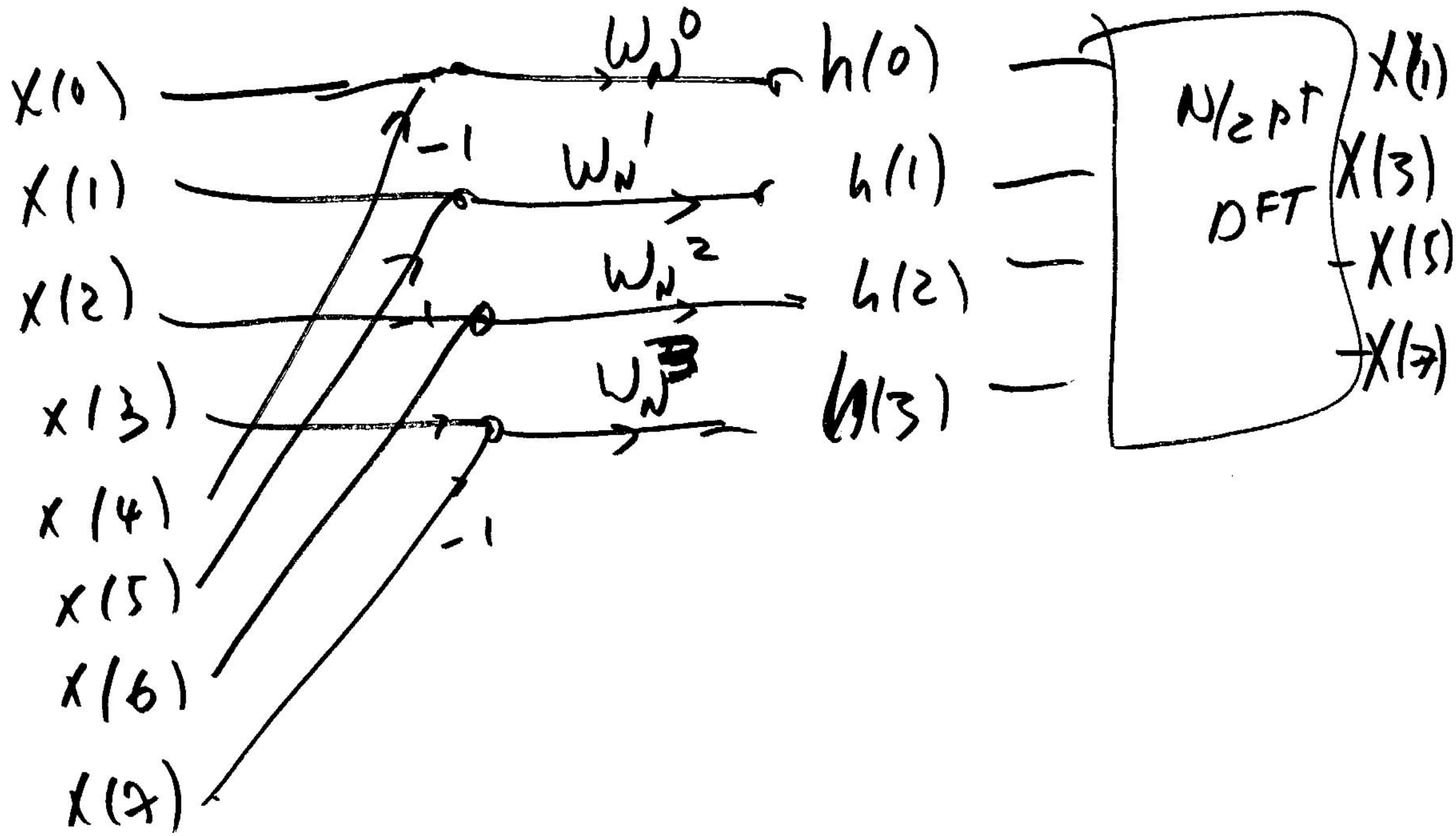
$$e^{-j\frac{2\pi}{N}\frac{N}{2}(2r+1)} \sum_{n=0}^{N/2-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi m}{N}(2r+1)}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi n}{N}(2r+1)} - \sum_{n=0}^{N/2-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi n}{N}(2r+1)}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] e^{-j\frac{2\pi n}{N}(2r+1)}$$

$$X(2r+1) = \sum_{n=0}^{N/2-1} \underbrace{\left(x(n) - x\left(n+\frac{N}{2}\right)\right) e^{-j\frac{2\pi n}{N}}}_{h(n)} e^{-j\frac{2\pi n r}{N/2}}$$

$N/2$ pt DFT $h(n)$



x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
-1
-1
W_N^0
W_N^1
W_N^2
W_N^3
h(0)
h(1)
h(2)
h(3)
N/2 pt DFT
X(1)
X(3)
X(5)
X(7)