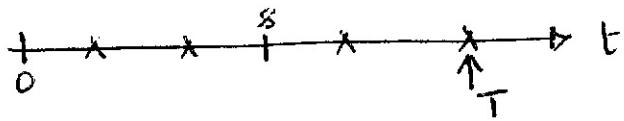


Poisson process of rate λ

FE 126 DISC

11/23

①



$X(t)$

$$X(0) = 0$$

$X(t)$ = number of arrivals upto time t

$$P(X(t) = R) = \frac{e^{-\lambda t} (\lambda t)^R}{R!}$$

$$P((X(t) - X(s)) = R) = \frac{e^{-\lambda(t-s)} (\lambda(t-s))^R}{R!}$$

Y_1 = time of first arrival.

Y_1 is exponentially distributed by λ .

$Y_2 - Y_1$

X

Pb Suppose customers arrive at a store according to a Poisson process having rate $\lambda = 2$

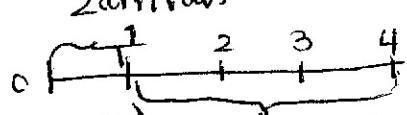
$X(t)$ is the number of customers that have arrived ~~at~~^{2 arrivals} up to time t ~~at~~^{4 arrivals}

② $P(X(1) = 2)$

⑥ $P(X(1) = 2 \text{ and } X(3) = 6)$ — (HINT: divide into 2 intervals)

⑦ $P(X(1) = 2 \mid X(3) = 6)$

⑧ $P(X(3) = 6 \mid X(1) = 2)$



Soln

a. $X(1) \sim P(2)$

$$P(X(1)=2) = \frac{e^{-2} \cdot 2^2}{2!} = 2e^{-2}.$$

b. $P(X(1)=2, X(3)=6) = P(X(1)=2, X(3)-X(1)=4)$

$X(1)$ and $X(3)-X(1)$ are independent [they are disjoint]

$$= P(X(1)=2) P(X(3)-X(1)=4) \quad X(3)-X(1) \sim P(2 \times 2)$$

$$= \frac{e^{-2} \cdot 2^2}{2!} \cdot \frac{e^{-4} \cdot 4^4}{4!}$$

$$= \frac{64}{3} e^{-6}.$$

c. $P(X(1)=2 | X(3)=6) = \frac{P(X(1)=2, X(3)=6)}{P(X(3)=6)}$

$$X(3) \sim P(3 \cdot 2)$$

$$\therefore P(X(1)=2 | X(3)=6) = \frac{\frac{64}{3} e^{-6}}{e^{-6} \frac{6^6}{6!}}$$

d. $P(X(3)=6 | X(1)=2) = P(X(3)-X(1)=4) = \frac{e^{-4} \cdot 4^4}{4!}$

↑ memoryless property

OR.

$$= \frac{P(X(3)=6, X(1)=2)}{P(X(1)=2)} = \frac{P(X(1)=2, X(3)-X(1)=4)}{P(X(1)=2)}$$

$$= \frac{\cancel{P(X(1)=2)} P(X(3)-X(1)=4)}{\cancel{P(X(1)=2)}}$$

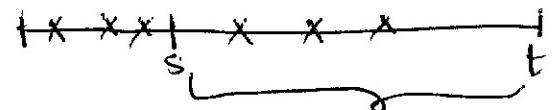
2. suppose customers arrive at a store according to a process having $\lambda = \lambda$.

a) $P(X(t) = R)$

b) Consider fixed times $0 < s < t$

$$P(X(t) = n+R | X(s) = n) = ?$$

$$\mathbb{E}[X(t), X(s)]$$



$X(t)$ is the number of customers that have arrived up to time t .

Ans. a. $X(t) \sim P(\lambda t)$

$$P(X(t) = R) = \frac{e^{-\lambda t} (\lambda t)^R}{R!}$$

$$\begin{aligned} b. P(X(t) = n+R | X(s) = n) &= P(X(t) - X(s) = R) \\ &= \frac{e^{-\lambda(t-s)} [\lambda(t-s)]^R}{R!} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[X(t) X(s)] &= \mathbb{E}[(X(t) - X(s)) + X(s)] X(s) \\ &= \mathbb{E}[(X(t) - X(s)) X(s)] + \mathbb{E}[X(s)^2] \\ &= \mathbb{E}[X(t) - X(s)] \cdot \mathbb{E}[X(s)] + \mathbb{E}[X(s)^2] \end{aligned}$$

$$\mathbb{E}[X(t) X(s)] = \lambda(t-s) \lambda(s) + \lambda s + \lambda^2 s^2$$

$$\therefore \mathbb{E}[X(t) X(s)] = \lambda^2 ts + \lambda s.$$

Remember:

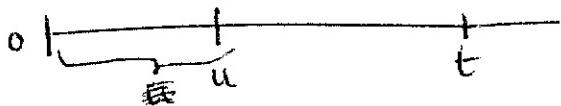
If $Z \sim P(\mu)$

$$\mathbb{E}[Z] = \mu$$

$$\text{var}(Z) = \mu = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$

$$\therefore \mathbb{E}[Z^2] = \mu + \mu^2$$

$$P(X(u) = k \mid X(t) = n)$$



$$= \frac{P(X(u) = k, X(t) = n)}{P(X(t) = n)}$$

$$= \frac{P(X(u) = k, X(t) - X(u) = n-k)}{P(X(t) = n)}$$

$$= \frac{P(X(u) = k) P(X(t) - X(u) = n-k)}{P(X(t) = n)}$$

$$= \frac{e^{-\lambda u} \frac{(\lambda u)^k}{k!} \left[\frac{\lambda^{(t-u)}}{(n-k)!} \right]}{e^{-\lambda t} \frac{(\lambda t)^n}{n!}}$$

$$= \binom{n}{k} \frac{u^k (t-u)^{n-k}}{t^n} = \binom{n}{k} \frac{u^k}{t^n} \frac{(t-u)^{n-k}}{t^{n-k}}$$

$$= \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k}$$