UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Problem Set 6
Fall 2006

Issued: Thursday, October 12, 2006 **Due:** Friday, October 20, 2006

Reading: For this problem set: §3.4, 3.5, 3.6

Problem 6.1
Suppose X is a uniformly distributed random variable between -1 and 2. Find the density functions for the random variable Y where:

(a) $Y = X^2$.

(b) $Y = \sqrt{|X|}$.

(c) $Y = -\ln(|X|)$

Problem 6.2
Suppose n runners run a race. At one point, all the runners are uniformly distributed on a stretch of one mile that starts at point A and ends at point B. Number the runners from 1 to n, according to the order in which they are at this particular moment. Let X_i denote the distance between point A and the ith runner and let $X_0 = 0$ and $X_{n+1} = 1$.

(a) For $n = 2$, find $\mathbb{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, 3$.

(b) For $n \geq 1$, find $\mathbb{P}(X_i > X_{i-1} + t)$ for $i = 1, 2, \ldots, n+1$.
(*Hint:* The answer does not depend on i.)

Problem 6.3
Random variables X, Y describe the cartesian coordinates of a random point on a unit circle centered at the origin. Let $W = \max\{X, Y\}$. Find the distribution of W. (*Note:* You need not evaluate the integrals.)

Problem 6.4
On any given day, and independent of its performance on any other day, a particular machine will be functional with probability p, where p is a random variable, uniformly distributed between 0 and 1.

(a) Find the probability that $0.2 \leq p \leq 0.6$.

(b) Find the probability that tomorrow, the machine will be functional.

(c) Suppose you are told that the machine was functional on m out of the last n days. Find the a posteriori density function for p, i.e. find $f_{P|A}(p|A)$ where A is the event that the machine functioned on m out of the last n days.
Hint: You can use the identity:

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}.$$

Problem 6.5
Let X be uniformly distributed on $[0, 1]$, and let Y be exponentially distributed with parameter λ.

(a) Find $\mathbb{P}(X \geq \mathbb{E}[X])$.

(b) Given $X \geq t$, determine the conditional PDF of $X - t$.

(c) Find $\mathbb{P}(Y \geq \mathbb{E}[Y])$.

(d) Given $Y \geq t$, determine the conditional PDF of $Y - t$.

Problem 6.6
Alice and Bob work independently on a problem set. The time for Alice to complete the set is exponentially distributed with mean 4 hours. The time for Bob to complete the set is exponentially distributed with mean 6 hours.

(a) What is the probability that Alice finishes the problem set before Bob?

(b) Given that Alice requires more than 4 hours, what is the probability that she finishes the problem set before Bob?

(c) What is the probability that one of them finishes the problem set an hour or more before the other one?

Problem 6.7
In this problem, we establish that the conditional expectation $g(y) = \mathbb{E}[X \mid Y = y]$ is the optimal predictor of X based on Y under the mean-squared error (MSE) criterion. That is, g is the function that minimizes the MSE

$$\text{MSE}(g) \quad = \quad \mathbb{E}_{X,Y}\left[(g(Y) - X)^2\right] \tag{1}$$

over all functions g.

(a) Given any random variable Z, show that $\mathbb{E}[Z]$ minimizes the mean-squared error $\mathbb{E}[(c - Z)^2]$ where c ranges over all real numbers constants $c \in \mathbb{R}$.

(b) Show that the MSE can be re-written as

$$\text{MSE}(g) \quad = \quad \mathbb{E}_Y\left\{\mathbb{E}_{X|Y}\left[(g(y) - X)^2 \mid Y = y\right]\right\}.$$

(c) Use part (a) to argue that for each *fixed* y, the real number $h(y) = \mathbb{E}[X \mid Y = y]$ minimizes the quantity $\mathbb{E}_{X|Y}\left[(c - X)^2 \mid Y = y\right]$. Conclude from (b) that the function $h(y) = \mathbb{E}[X \mid Y = y]$ minimizes the MSE defined in equation (1) over all functions h.